Bounded induction without parameters

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Parameters in induction axioms

In arithmetic, induction (and other) schemata usually allow formulas with free parameters:

$$\varphi(0, y) \land \forall x \left(\varphi(x, y) \to \varphi(x+1, y)\right) \to \forall x\, \varphi(x, y)$$

Examples: $I\Sigma_i$, S^i_2, T^i_2, ...

- for full induction, parameters make no difference
- fragments become genuinely weaker without parameters
- strong theories: many intriguing results about $I\Sigma^-_n$, $I\Pi^-_n$
- closely related to induction rules
- characterizations using reflection principles
- this talk: parameter-free versions of Buss's theories

Parameter-free bounded arithmetic

- $T^i_2 = \hat{\Sigma}^b_i\text{-}IND = \hat{\Pi}^b_i\text{-}IND$, $S^i_2 = \hat{\Sigma}^b_i\text{-}PIND = \hat{\Pi}^b_i\text{-}PIND$
- $\hat{\Sigma}^b_i\text{-}(P)IND^-$, $\hat{\Pi}^b_i\text{-}(P)IND^-$:

$$\varphi(0) \land \forall x \, \big(\varphi(x) \to \varphi(x+1)\big) \to \forall x \, \varphi(x)$$
$$\varphi(0) \land \forall x \, \big(\varphi(\lfloor x/2 \rfloor) \to \varphi(x)\big) \to \forall x \, \varphi(x)$$

- $\hat{\Sigma}^b_i\text{-}(P)IND^R$, $\hat{\Pi}^b_i\text{-}(P)IND^R$:

$$\frac{\varphi(0) \quad \varphi(x) \to \varphi(x+1)}{\varphi(x)} \qquad \frac{\varphi(0) \quad \varphi(\lfloor x/2 \rfloor) \to \varphi(x)}{\varphi(x)}$$

  - parameters do not matter for $(P)IND^R$
  - this is not a sequent calculus (no “side formulas”)

Theories and rules

Theories axiomatized not just by axioms, but by more general rules

$$\frac{\varphi_1, \dots, \varphi_k}{\varphi}$$

T an ordinary FO theory, R a set of rules:

- $[T, R] =$ closure of T under unnested R-rules (axiomatized by $T +$ those φ s.t. $T \vdash \varphi_1 \wedge \cdots \wedge \varphi_k$)
- $[T, R]_0 := T$, $[T, R]_{n+1} := [[T, R]_n, R]$
  $T + R := \bigcup_n [T, R]_n$
- R is reducible to R' ($R \leq R'$) if $[T, R] \subseteq [T, R']$ for all T
- R and R' are equivalent ($R \equiv R'$) if $R \leq R' \leq R$

Parameter-free axioms vs. rules

$\Gamma = \hat{\Sigma}^b_i$ or $\hat{\Pi}^b_i$:

- variants of Γ-$(P)IND^R$ with and without parameters equivalent
- Γ-$(P)IND^-$ is the least theory whose all extensions are closed under Γ-$(P)IND^R$
  - conservation results over Γ-$(P)IND^-$ follow from conservation results over Γ-$(P)IND^R$
  - a converse also holds

Previous work

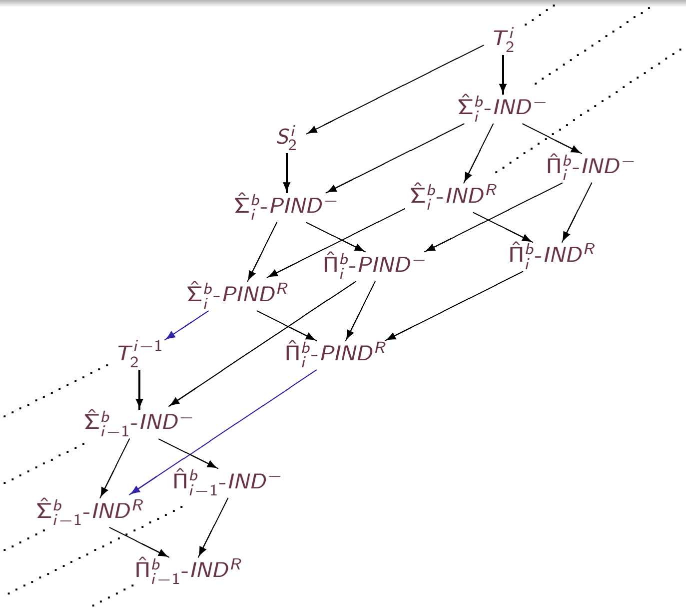
Parameter-free induction in bounded arithmetic:

- [K'90] IE_i is $\exists\forall E_i$-conservative over IE_i^-
- [BI'92] studied Σ^b_i parameter-free rules
- [CFL'09] proved conservation results for $\hat{\Sigma}^b_i$ rules and parameter-free schemata

Many questions left unanswered:

- $\hat{\Pi}^b_i$ rules and parameter-free schemata?
- nesting (number of instances)?
- reflection principles?

At a glance



T^i_2
$\hat\Sigma^b_i$-IND^-
S^i_2
$\hat\Pi^b_i$-IND^-
$\hat\Sigma^b_i$-$PIND^-$
$\hat\Sigma^b_i$-IND^R
$\hat\Pi^b_i$-$PIND^-$
$\hat\Pi^b_i$-IND^R
$\hat\Sigma^b_i$-$PIND^R$
T^{i-1}_2
$\hat\Pi^b_i$-$PIND^R$
$\hat\Sigma^b_{i-1}$-IND^-
$\hat\Pi^b_{i-1}$-IND^-
$\hat\Sigma^b_{i-1}$-IND^R
$\hat\Pi^b_{i-1}$-IND^R

Conservation over $\hat{\Sigma}^b_i$ rules

The following was proved by [CFL'09], based on [K'90,BI'92]:

Theorem

If T is $\forall\exists\hat{\Sigma}^b_{i+1}$, then $T + T^i_2$ ($T + S^i_2$) is $\forall\hat{\Sigma}^b_i$-conservative over $T + \hat{\Sigma}^b_i\text{-}(P)IND^R$

Corollary

- T^i_2 (S^i_2) is $\exists\forall\hat{\Pi}^b_{i+1}$-conservative over $\hat{\Sigma}^b_i\text{-}(P)IND^-$
- If T is $\forall\hat{\Sigma}^b_i$, $T + \hat{\Pi}^b_{i+1}\text{-}PIND^R = T + \hat{\Sigma}^b_i\text{-}IND^R$
- [Buss]: … and $T + \hat{\Sigma}^b_{i+1}\text{-}PIND^R = T + T^i_2$

Conservativity over $\hat{\Pi}^b_i$ rules

Theorem

If T is $\forall\hat{\Sigma}^b_i$, then $T + T^i_2$ ($T + S^i_2$) is $\forall\exists\hat{\Sigma}^b_{i-1}$-conservative over $T + \hat{\Pi}^b_i$-$(P)IND^R$

Corollary

T^i_2 (S^i_2) is $\exists\hat{\Sigma}^b_{i+1} \vee \forall\exists\hat{\Sigma}^b_{i-1}$ conservative over $\hat{\Pi}^b_i$-$(P)IND^-$

Nesting of rules

For $\Gamma = \hat{\Sigma}^b_i, \hat{\Pi}^b_i$, every $\varphi \in [T, \Gamma\text{-}(P)IND^R]_k$ can be proved using k instances of $\Gamma\text{-}(P)IND^R$

Theorem

- If T is $\forall\hat{\Sigma}^b_\infty$: $T + \hat{\Pi}^b_i\text{-}(P)IND^R = [T, \hat{\Pi}^b_i\text{-}(P)IND^R]$
- If T is $\forall\hat{\Sigma}^b_i$: $T + \hat{\Sigma}^b_i\text{-}(P)IND^R = [T, \hat{\Sigma}^b_i\text{-}(P)IND^R]$

The instance only depends on axioms of T used:

- If $T = BTC^0 + \forall x\, \xi(x)$ with $\xi \in \hat{\Sigma}^b_i$:

$$T + \hat{\Sigma}^b_i\text{-}(P)IND^R = BTC^0 + \mathrm{RFN}_i(G_i^{(*)} + \xi)$$

$$T + \hat{\Pi}^b_i\text{-}(P)IND^R = T + \mathrm{RFN}_{i-1}(G_i^{(*)} + \xi)$$

  - closure finitely axiomatizable

Parameter-free conservation

Generalizing a result of [K'88]:

Theorem

Let $\Gamma = \hat{\Sigma}^b_i, \hat{\Pi}^b_i$, and T be of any complexity:

- $T + \Gamma\text{-}(P)IND^-$ is $\forall\Gamma$-conservative over $T + \Gamma\text{-}(P)IND^R$
- All $\forall\Gamma$ consequences of T + arbitrary k instances of $\Gamma\text{-}(P)IND^-$ are in $[T, \Gamma\text{-}(P)IND^R]_k$

If $\Gamma\text{-}(P)IND^-$ is finitely axiomatizable, there is k s.t. $T + \Gamma\text{-}(P)IND^R = [T, \Gamma\text{-}(P)IND^R]_k$ for every T

Propositional proof systems

$G_i = \Sigma^q_i$-fragment of quantified propositional sequent calculus

$\mathrm{RFN}_j(P)$ = "every P-provable Σ^q_j sequent is valid"

$\varphi(x) \in \hat\Sigma^b_\infty \implies$ propositional translations $[\![\varphi]\!]_n(p_0, \dots, p_{n-1})$

Definition

Let $\xi \in \hat\Sigma^b_i$:

- $G_i + \xi = G_i$ with extra initial sequents

$$\implies [\![\xi]\!]_n(A_0, \dots, A_{n-1})$$

  with $A_0, \dots, A_{n-1}$ quantifier-free
- $G^*_i + \xi$ is its tree-like version

Correspondence

By extension of standard results, one can show easily

Theorem

Let $\xi, \varphi \in \hat{\Sigma}^b_i$:

- If $T^i_2 + \forall x\, \xi(x) \vdash \forall x\, \varphi(x)$, then ($BTC^0$-provably) there are poly-size $G_i + \xi$ proofs of $[\![\varphi]\!]_n$
- $T^i_2 + \forall x\, \xi(x)$ proves $\mathrm{RFN}_i(G_i + \xi)$
- $T^i_2 \rightsquigarrow S^i_2$: $G_i + \xi \rightsquigarrow G^*_i + \xi$

Induction rules vs. reflection principles

Theorem

The rules on the LHS are equivalent to the rules on the RHS for $\xi \in \hat{\Sigma}^b_i$:

$$
\begin{array}{ll}
\hat{\Sigma}^b_i\text{-}(P)IND^R & \forall x\,\xi(x) \;/\; \mathrm{RFN}_i(G_i^{(*)} + \xi) \\
\hat{\Sigma}^b_i\text{-}(P)IND^- & \forall x\,\xi(x) \to \mathrm{RFN}_i(G_i^{(*)} + \xi) \\
\hat{\Pi}^b_i\text{-}(P)IND^R & \forall x\,\xi(x) \;/\; \mathrm{RFN}_{i-1}(G_i^{(*)} + \xi) \\
\hat{\Pi}^b_i\text{-}(P)IND^- & \forall x\,\xi(x) \to \mathrm{RFN}_{i-1}(G_i^{(*)} + \xi)
\end{array}
$$

A witnessing theorem

Theorem

If T^i_2 (S^i_2) proves $\forall x\,\varphi(x)$, $\varphi \in \exists\forall\hat{\Pi}^b_i$, there are $\hat{\Pi}^b_{i-1}$ formulas $\theta_1(x_0,x_1),\dots,\theta_k(x_0,\dots,x_k)$ s.t.

$$\vdash \varphi(x_0) \lor \exists y\,\theta_j(x_0,\dots,x_{j-1},y) \qquad (j=1,\dots,k)$$

$$\vdash \bigwedge_{j=1}^{k}\Big[\theta_j(x_0,\dots,x_j) \land \bigwedge_{l=1}^{k}\neg\big(x_l \prec x_j \land \theta_j(x_0,\dots,x_{j-1},x_l)\big)\Big] \to \varphi(x_0)$$

where $y \prec x$ denotes $y < x$ ($|y| < |x|$)

A witnessing theorem (contd.)

- k-times iterated $\hat{\Pi}^b_{i-1}$-$(L)MIN$ ($\equiv \hat{\Pi}^b_i$-$(P)IND$)
  - restricted parameters
- Also works in the presence of a $\forall\exists\hat{\Sigma}^b_i$ ground theory
- Has an analogue one level higher (see next slide)
- If $\varphi \in \exists\hat{\Pi}^b_i$, can bound the quantifiers and take $k = 1$:

$$\vdash y \geq t(x) \to \theta(x, y)$$
$$\vdash \big(\theta(x, y) \wedge \neg\exists z \prec y\, \theta(x, z)\big) \to \varphi(x)$$

  - conservation over $[T, \hat{\Pi}^b_i\text{-}(P)IND^R]$, $[T, \hat{\Sigma}^b_i\text{-}(P)IND^R]$
- Can we reduce to $k = 1$ in other cases?
  - Is T^i_2 (S^i_2) $\exists\forall\hat{\Pi}^b_i$-conservative over $\hat{\Pi}^b_i$-$(P)IND^-$?
  - $\iff$ Is $T + T^i_2$ ($T + S^i_2$) $\forall\hat{\Pi}^b_i$-conservative over $T + \hat{\Pi}^b_i$-$(P)IND^R$ for any $T \subseteq \forall\exists\hat{\Sigma}^b_i$?

A predecessor variant

Theorem

If T^i_2 (S^i_2) proves $\forall x\,\varphi(x)$, $\varphi \in \exists\forall\hat{\Pi}^b_{i+1}$, there are $\hat{\Pi}^b_i$ formulas $\theta_1(x_0, x_1), \dots, \theta_k(x_0, \dots, x_k)$ s.t.

$$\vdash \varphi(x_0) \lor \exists y\,\theta_j(x_0, \dots, x_{j-1}, y) \qquad (j = 1, \dots, k)$$

$$\vdash \bigwedge_{j=1}^{k} \Big[\theta_j(x_0, \dots, x_j) \land \neg\big(x_j \neq 0 \land \theta_j(x_0, \dots, x_{j-1}, P(x_j))\big)\Big] \to \varphi(x_0)$$

where $P(x)$ denotes $x - 1$ ($\lfloor x/2 \rfloor$)

Thank you for attention!

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