A simplified lower bound on intuitionistic implicational proofs

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Outline

1. Classical proof complexity
2. Non-classical proof complexity
3. Lower bound for implicational logic

Classical proof complexity

1. **Classical proof complexity**
2. Non-classical proof complexity
3. Lower bound for implicational logic

Propositional proof systems

Proof system (pps): relation $P \subseteq \mathsf{Form} \times \Sigma^*$ s.t.

- P is decidable in polynomial time
- φ is a tautology $\iff \exists \pi\, P(\varphi, \pi)$

Main measure: length (=size) of proofs

- P polynomially bounded if all tautologies φ have P-proofs of size $\le |\varphi|^c$
- P p-simulates Q ($P \ge_p Q$):
  polynomial-time translation of Q-proofs to P-proofs
- P and Q are p-equivalent ($P \equiv_p Q$): $P \ge_p Q$ & $Q \ge_p P$

Theorem (Cook & Reckhow '79):
NP = **coNP** $\iff$ $\exists$ polynomially bounded pps

Frege (aka Hilbert-style) systems

R: finite set of schematic Frege rules $\alpha_1, \dots, \alpha_k \vdash \alpha_0$

R-derivation of φ from Γ: $\varphi_0, \dots, \varphi_t = \varphi$ where each φ_i derived from φ_j, $j < i$ by an instance of an R-rule, or $\varphi_i \in \Gamma$

If $\Gamma \vdash_R \varphi \iff \Gamma \vDash \varphi$: Frege system F_R

- typically: modus ponens + axiom schemata
- all Frege systems p-equivalent (Reckhow '76)
  $\implies$ write $F = F_R$
- p-equivalent to tree-like Frege F^* (Krajíček '94)
- p-equivalent to sequent calculus and natural deduction (Reckhow '76)
- known lower bounds: number of lines $\Omega(n)$, size $\Omega(n^2)$ (Krajíček '95)

Boolean circuits

Formulas: trees

Circuits: directed acyclic graphs (dag)

- finite dag labelled with variables and connectives
- each node appropriate number of incoming edges
- one node designated as output

As a model of computation:

- $L \subseteq \{0,1\}^*$ is in $\mathbf{P}/\mathbf{poly}$ if for each n,
  $L_n = L \cap \{0,1\}^n$ is computable by circuits C_n, $|C_n| \leq n^c$
- nonuniform version of **P**

Monotone circuits:

- only connectives $\wedge$, $\vee$ (possibly 0, 1)

Feasible interpolation

General lower bound method for weak pps (Krajíček '97):

P has feasible interpolation if for every P-proof Π of

$$\beta(\vec{p}, \vec{r}) \to \alpha(\vec{p}, \vec{q})$$

there exists a Boolean circuit $C(\vec{p})$, $|C| \le |\Pi|^c$, s.t.

$$\vDash \beta(\vec{p}, \vec{r}) \to C(\vec{p}), \qquad \vDash C(\vec{p}) \to \alpha(\vec{p}, \vec{q})$$

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$$C(\vec{p}) \vDash \alpha(\vec{p}, \vec{q}), \qquad \neg C(\vec{p}) \vDash \beta(\vec{p}, \vec{r})$$

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$$C(\vec{p}) \vDash \alpha(\vec{p},\vec{q}), \qquad \neg C(\vec{p}) \vDash \beta(\vec{p},\vec{r})$$

Theorem: If P has f.i., and $\exists$ a disjoint **NP**-pair $\langle A, B\rangle$ not separable in **P**/**poly**, then P is not polynomially bounded

Proof idea: express A_n by $\exists\vec{q}\,\neg\alpha_n(\vec{p},\vec{q})$ and B_n by $\exists\vec{r}\,\neg\beta_n(\vec{p},\vec{r})$

Circuit lower bounds

Lower bounds on the size of general circuits:

- random functions $\{0,1\}^n \to \{0,1\}$: size $\gtrsim 2^n/n$ whp
- explicit functions: size $\geq 5n$ or so
  $\Longrightarrow$ f.i. only yields conditional lower bounds

Monotone circuits:

- Razborov '85: superpolynomial lower bound for Clique
- Alon & Boppana '87: improved to exponential lower bound
- also applies to the Clique–Colouring **NP**-pair (Tardos '87)
- variant: Colouring–Cocolouring (Hrubeš & Pudlák '17)

Monotone feasible interpolation

P has monotone feasible interpolation if for every P-proof Π of

$$\alpha(\vec{p}, \vec{q}) \lor \beta(\vec{p}, \vec{r})$$

where $\vec{p}$ only occur positively in α,
there exists a monotone circuit $C(\vec{p})$, $|C| \le |\Pi|^c$, s.t.

$$C(\vec{p}) \vDash \alpha(\vec{p}, \vec{q}), \qquad \neg C(\vec{p}) \vDash \beta(\vec{p}, \vec{r})$$

Theorem: If P has m.f.i. then P is not polynomially bounded

Example:
Resolution has f.i. and m.f.i.
Frege likely does not

Non-classical proof complexity

1. Classical proof complexity
2. **Non-classical proof complexity**
3. Lower bound for implicational logic

Non-classical Frege systems

L finitely axiomatizable propositional logic $\Longrightarrow$ Frege system L-F

Unconditional exponential lower bounds for many logics L:

- Hrubeš '07,'09: some modal logics, intuitionistic logic (Frege, Extended Frege)
- J. '09: extensions of K4 or IPC with unbounded branching
- Jalali '21: extensions of FL included in ...

Further strengthening:

- exponential separation between Extended Frege and Substitution Frege (J. '09)
- purely implicational tautologies (J. '17)

Feasible disjunction property

P proof system for $L \supseteq \mathrm{IPC}$:

P has the feasible disjunction property if given a P-proof of $\varphi_0 \vee \varphi_1$, we can compute in polynomial time $i \in \{0, 1\}$ such that $\vdash_L \varphi_i$

Modal logics: the same with $\Box\varphi_0 \vee \Box\varphi_1$

Example: IPC-F has f.d.p. (Buss & Mints '99, ...)

(Pudlák '99) f.d.p. can serve the role of f.i.
$\Longrightarrow$ conditional lower bounds

(Hrubeš '07) analogue of monotone f.i.
$\Longrightarrow$ unconditional lower bounds

f.d.p. serving as f.i.

$P \geq_p$ IPC-F closed under substitution of $0, 1$:

- $\alpha(\vec{p}, \vec{q}) \lor \beta(\vec{p}, \vec{r})$ classical tautology $\implies$ IPC proves

$$(*) \qquad \bigwedge_{i<n} (p_i \lor \neg p_i) \to \neg\neg\alpha(\vec{p}, \vec{q}) \lor \neg\neg\beta(\vec{p}, \vec{r})$$

- if P has f.d.p. and $(*)$ has a short P-proof: small circuit C such that for all $\vec{a} \in \{0, 1\}^n$,

$$C(\vec{a}) = 1 \implies \vdash \neg\neg\alpha(\vec{a}, \vec{q})$$
$$C(\vec{a}) = 0 \implies \vdash \neg\neg\beta(\vec{a}, \vec{r})$$

f.d.p. serving as f.i.

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- if P has f.d.p. and $(*)$ has a short P-proof:
  small circuit C such that

$$C(\vec{p}) \vDash \alpha(\vec{p}, \vec{q}), \qquad \neg C(\vec{p}) \vDash \beta(\vec{p}, \vec{r})$$

In a galaxy far, far away

Persistent claims (2016–) by L. Gordeev and E. H. Haeusler:

- implicational IPC tautologies have polynomial-size proofs in dag-like natural deduction
- **NP** = **PSPACE**
- published ('19,'20), some people seem to take it seriously

Flatly contradicts known lower bounds, but this requires a complex argument, hard to track down by non-specialists:

- IPC-F lower bounds (Hrubeš '07)
- monotone circuit lower bounds (Alon–Boppana '87, . . .)
- reduction to implicational logic (J. '17)
- simulation of natural deduction by Frege (idea Reckhow '76, Cook–Reckhow '79, but for a different system)

$\Longrightarrow$ desire for something simpler/more direct

Lower bound for implicational logic

Intuitionistic/minimal implicational logic

Language: $\to$, atoms $p_0, p_1, p_2, \dots$

the set of formulas: Form

Notation: $\varphi \to \psi \to \chi \to \omega = (\varphi \to (\psi \to (\chi \to \omega)))$

Frege system $F_\to$:

$$\vdash (\varphi \to \psi \to \chi) \to (\varphi \to \psi) \to (\varphi \to \chi)$$

$$\vdash \varphi \to \psi \to \varphi$$

$$\varphi, \varphi \to \psi \vdash \psi$$

Sequent calculus $LJ_\to$: structural rules (incl. cut) +

$$\frac{}{\varphi \Longrightarrow \varphi} \qquad \frac{\Gamma \Longrightarrow \varphi \quad \Gamma, \psi \Longrightarrow \alpha}{\Gamma, \varphi \to \psi \Longrightarrow \alpha} \qquad \frac{\Gamma, \varphi \Longrightarrow \psi}{\Gamma \Longrightarrow \varphi \to \psi}$$

Natural deduction

Prawitz-style tree-like natural deduction:

$$[\varphi] \longleftarrow \text{discharged}$$

$$(\to\mathsf{E})\ \frac{\varphi \quad \varphi \to \psi}{\psi} \qquad\qquad (\to\mathsf{I})\ \frac{\begin{array}{c}[\varphi]\\ \vdots\\ \psi\end{array}}{\varphi \to \psi}$$

- every leaf of the proof tree must be discharged

Gordeev & Haeusler dag-like natural deduction $\mathrm{NM}_{\to}$:

- every leaf of the proof dag must be discharged on every path to the root
- checkable in polynomial-time: inductively compute for each node $v \in V$ the set

$$A_v = \{\gamma_u : u \text{ leaf, undischarged on some path to } v\}$$

Notation: $\langle V, E\rangle$ underlying dag, γ_v = formula label of node v

Efficient Kleene's slash

For $P \subseteq \mathrm{Form}$: a P-slash is a unary predicate $|\varphi$ on Form s.t.

$$|(\varphi \to \psi) \iff (\underbrace{|\varphi \text{ and } \varphi \in P}_{\|\varphi} \implies |\psi)$$

- free to choose $|p$ for atoms p
- Kleene's original $\Gamma \mid \varphi$ has $P = \{\varphi : \Gamma \vdash \varphi\}$,
  we take for P an efficiently computable finite set

$\mathrm{NM}_{\to}$-proof Π: P is Π-closed if $\forall v\,(A_v \subseteq P \implies \gamma_v \in P)$

Lemma: Π proof of φ, P is Π-closed, $|$ is a P-slash $\implies |\varphi$

- by induction on the length of the proof

Constructibility of Π-closure

$\mathrm{cl}_\Pi(X)$ = smallest Π-closed set $P \supseteq X$

Observation: $\varphi \in \mathrm{cl}_\Pi(X) \implies X \vdash \varphi$

$\mathrm{cl}_\Pi(X)$ is computable in polynomial time, moreover:

Lemma: Π $\mathrm{NM}_\to$-proof, $\{\varphi_i : i < n\} \subseteq \mathrm{Form}$, $\varphi \in \mathrm{Form}$
$\implies \exists$ monotone circuit C of size $|\Pi|^3$ s.t.

$$C(x_0, \dots, x_{n-1}) = 1 \iff \varphi \in \mathrm{cl}_\Pi(\{\varphi_i : x_i = 1\})$$

- describe inductive construction of closure
- only involves formulas from Π
- terminates in $|\Pi|$ steps

Feasible disjunction property

Theorem: Given a $\mathrm{NM}_{\to}$-proof Π of

$$\varphi = \big(\alpha_0(\vec{p}) \to u\big) \to \big(\alpha_1(\vec{p}) \to u\big) \to u,$$

we can compute in polynomial time $i \in \{0,1\}$ s.t. $\vdash \alpha_i$

Proof: $P = \mathrm{cl}_\Pi(\alpha_0 \to u, \alpha_1 \to u)$, let $|$ be P-slash s.t. $\nmid u$

We have $|\varphi$, thus $\nparallel(\alpha_0 \to u)$ or $\nparallel(\alpha_1 \to u)$

$\nparallel(\alpha_i \to u) \implies \nmid(\alpha_i \to u) \implies \|\alpha_i \implies \alpha_i \in P$

We can compute i s.t. $\alpha_i \in P$

Then: $\alpha_0 \to u, \alpha_1 \to u \vdash \alpha_i$

Substitute $\top$ for $u \implies$ get $\vdash \alpha_i$

Monotone feasible interpolation

Theorem: Given a $\mathrm{NM}_{\to}$-proof Π of

$$
\begin{aligned}
&\big((p_0 \to u) \to (p'_0 \to u) \to u\big)\\
&\ \to \big((p_1 \to u) \to (p'_1 \to u) \to u\big)\\
&\quad \to \big((p_2 \to u) \to (p'_2 \to u) \to u\big)\\
&\qquad\qquad \ddots\\
&\qquad\quad \to \big((p_{n-1} \to u) \to (p'_{n-1} \to u) \to u\big)\\
&\qquad\qquad\qquad \to \big(\alpha_0(\vec{p}, \vec{q}) \to u\big) \to \big(\alpha_1(\vec{p}\,', \vec{r}) \to u\big) \to u,
\end{aligned}
$$

there is a monotone circuit C of size $|\Pi|^3$ such that

$$C(\vec{p}) \vDash \alpha_0(\vec{p}, \vec{q}), \qquad \neg C(\vec{p}) \vDash \alpha_1(\neg\vec{p}, \vec{r})$$

Monotone feasible interpolation

Notation: $\langle\varphi_i\rangle_{i<n} \to \psi = \varphi_{n-1} \to \dots \to \varphi_1 \to \varphi_0 \to \psi$

Theorem: Given a $\mathrm{NM}_{\to}$-proof Π of

$$\Big\langle (p_i \to u) \to (p'_i \to u) \to u \Big\rangle_{i<n} \to \big(\alpha_0(\vec{p},\vec{q}) \to u\big) \to \big(\alpha_1(\vec{p}\,',\vec{r}) \to u\big) \to u,$$

there is a monotone circuit C of size $|\Pi|^3$ such that

$$C(\vec{p}) \vDash \alpha_0(\vec{p},\vec{q}), \qquad \neg C(\vec{p}) \vDash \alpha_1(\neg\vec{p},\vec{r})$$

Colouring–Cocolouring disjoint NP pair

Observation: Any graph $G = \langle V, E \rangle$ with $V = [n]$ satisfies

$$\chi(G)\chi(\overline{G}) \geq n$$

$c\colon V \to [k]$ colouring of G,

$c'\colon V \to [k']$ colouring of $\overline{G}$

$\implies c \times c'\colon V \to [k] \times [k']$ is injective

Colouring–Cocolouring disjoint NP pair

Observation: Any graph $G = \langle V, E \rangle$ with $V = [n]$ satisfies

$$\chi(G)\chi(\overline{G}) \geq n$$

Colouring–Cocolouring disjoint **NP** pair: distinguish

- graphs G s.t. G is k-colourable from
- graphs G s.t. $\overline{G}$ is k-colourable

where $k = \lceil \sqrt{n} \rceil - 1$

Represent E by an $\binom{n}{2}$-tuple of Boolean variables

Theorem (Hrubeš & Pudlák '17):
Any monotone circuits separating the Colouring–Cocolouring pair must have size $2^{\Omega(n^{1/8})}$

Colouring–Cocolouring tautologies

p_{ij} $(i < j < n)$: represent E

q_{il}, r_{il} $(i < n, l < k)$: k-colouring of G and $\overline{G}$ (respectively)

Classical tautologies:

$$\neg\Big[\Big(\bigwedge_{i<n}\bigvee_{l<k} q_{il} \wedge \bigwedge_{\substack{i<j<n\\ l<k}} \neg(q_{il} \wedge q_{jl} \wedge p_{ij})\Big) \wedge \Big(\bigwedge_{i<n}\bigvee_{l<k} r_{il} \wedge \bigwedge_{\substack{i<j<n\\ l<k}} \neg(r_{il} \wedge r_{jl} \wedge \neg p_{ij})\Big)\Big]$$

Colouring–Cocolouring tautologies

p_{ij} $(i < j < n)$: represent E
q_{il}, r_{il} $(i < n,\ l < k)$: k-colouring of G and $\overline{G}$ (respectively)

Classical tautologies:

$$\Bigl(\bigwedge_{i<n}\bigvee_{l<k} q_{il} \to \bigvee_{\substack{i<j<n\\ l<k}}(q_{il} \wedge q_{jl} \wedge p_{ij})\Bigr)$$
$$\vee\ \Bigl(\bigwedge_{i<n}\bigvee_{l<k} r_{il} \to \bigvee_{\substack{i<j<n\\ l<k}}(r_{il} \wedge r_{jl} \wedge \neg p_{ij})\Bigr)$$

Colouring–Cocolouring tautologies

p_{ij}, p'_{ij} $(i < j < n)$: represent E and its complement
$q_{il},\ r_{il}$ $(i < n,\ l < k)$: k-colouring of G and $\overline{G}$ (respectively)

Classical tautologies:

$$\bigwedge_{i<j<n} (p_{ij} \lor p'_{ij}) \to \Big[\Big(\bigwedge_{i<n} \bigvee_{l<k} q_{il} \to \bigvee_{\substack{i<j<n\\ l<k}} (q_{il} \land q_{jl} \land p_{ij})\Big) \lor \Big(\bigwedge_{i<n} \bigvee_{l<k} r_{il} \to \bigvee_{\substack{i<j<n\\ l<k}} (r_{il} \land r_{jl} \land p'_{ij})\Big)\Big]$$

Colouring–Cocolouring tautologies

p_{ij}, p'_{ij} $(i < j < n)$: represent E and its complement
q_{il}, r_{il} $(i < n,\ l < k)$: k-colouring of G and $\overline{G}$ (respectively)

Intuitionistic tautologies:

$$\bigwedge_{i<j<n} (p_{ij} \vee p'_{ij}) \to \Bigg[\Bigg(\bigwedge_{i<n} \bigvee_{l<k} q_{il} \to \bigvee_{\substack{i<j<n\\ l<k}} (q_{il} \wedge q_{jl} \wedge p_{ij})\Bigg) \vee \Bigg(\bigwedge_{i<n} \bigvee_{l<k} r_{il} \to \bigvee_{\substack{i<j<n\\ l<k}} (r_{il} \wedge r_{jl} \wedge p'_{ij})\Bigg)\Bigg]$$

Colouring–Cocolouring tautologies

p_{ij}, p'_{ij} $(i < j < n)$: represent E and its complement
q_{il}, r_{il} $(i < n,\ l < k)$: k-colouring of G and $\overline{G}$ (respectively)
u: auxiliary

Intuitionistic tautologies:

$$\Big[\Big(\bigwedge_{i<n}\bigvee_{l<k} q_{il} \to \bigvee_{\substack{i<j<n\\ l<k}}(q_{il}\wedge q_{jl}\wedge p_{ij})\Big) \to u\Big]$$
$$\to \Big[\Big(\bigwedge_{i<n}\bigvee_{l<k} r_{il} \to \bigvee_{\substack{i<j<n\\ l<k}}(r_{il}\wedge r_{jl}\wedge p'_{ij})\Big) \to u\Big]$$
$$\to \bigwedge_{i<j<n}(p_{ij}\vee p'_{ij}) \to u$$

Colouring–Cocolouring tautologies

p_{ij}, p'_{ij} $(i < j < n)$: represent E and its complement
q_{il}, r_{il} $(i < n,\ l < k)$: k-colouring of G and $\overline{G}$ (respectively)
u, v, w: auxiliary

Intuitionistic tautologies:

$$\Bigg[\Bigg(\Bigg(\bigvee_{\substack{i<j<n\\ l<k}}(q_{il} \wedge q_{jl} \wedge p_{ij}) \to v\Bigg) \to \bigwedge_{i<n}\bigvee_{l<k} q_{il} \to v\Bigg) \to u\Bigg]$$
$$\to \Bigg[\Bigg(\Bigg(\bigvee_{\substack{i<j<n\\ l<k}}(r_{il} \wedge r_{jl} \wedge p'_{ij}) \to w\Bigg) \to \bigwedge_{i<n}\bigvee_{l<k} r_{il} \to w\Bigg) \to u\Bigg]$$
$$\to \bigwedge_{i<j<n}(p_{ij} \vee p'_{ij}) \to u$$

Colouring–Cocolouring tautologies

p_{ij}, p'_{ij} $(i < j < n)$: represent E and its complement
$q_{il},\ r_{il}$ $(i < n,\ l < k)$: k-colouring of G and $\overline{G}$ (respectively)
u, v, w: auxiliary

Intuitionistic implicational tautologies τ_n:

$$\Big\langle (p_{ij} \to u) \to (p'_{ij} \to u) \to u \Big\rangle_{i<j<n} \to (\alpha_n \to u) \to (\alpha'_n \to u) \to u$$

where

$$\alpha_n = \Big\langle \langle q_{il} \to v \rangle_{l<k} \to v \Big\rangle_{i<n} \to \langle q_{il} \to q_{jl} \to p_{ij} \to v \rangle_{\substack{i<j<n \\ l<k}} \to v$$
$$\alpha'_n = \Big\langle \langle r_{il} \to w \rangle_{l<k} \to w \Big\rangle_{i<n} \to \langle r_{il} \to r_{jl} \to p'_{ij} \to w \rangle_{\substack{i<j<n \\ l<k}} \to w$$

The lower bound

τ_n: $\mathrm{IPC}_{\to}$ tautologies of size $O(n^2 k) = O(n^{2.5})$

Monotone feasible interpolation $\implies$

Lemma: If τ_n has a proof of size s, then there is a monotone circuit of size s^3 separating the Colouring–Cocolouring **NP** pair

Hrubeš–Pudlák bound $\implies$

Theorem: Any $\mathrm{NM}_{\to}$-proof of τ_n has size $2^{\Omega(n^{1/8})}$

Corollary: There are infinitely many intuitionistic implicational tautologies φ that require $\mathrm{NM}_{\to}$-proofs of size $2^{\Omega(|\varphi|^{1/20})}$

(Clique–Colouring tautologies with $k \approx n^{2/3}$: $2^{\Omega(|\varphi|^{1/10-\varepsilon})}$)

Other calculi

The argument adapts to $\mathrm{F}_{\to}$ or $\mathrm{LJ}_{\to}$:

- adjust the definition of Π-closed sets

Actually: $\mathrm{F}_{\to} \equiv_p \mathrm{LJ}_{\to} \equiv_p \mathrm{NM}_{\to} \equiv_p \underbrace{\mathrm{F}^*_{\to} \equiv_p \mathrm{LJ}^*_{\to} \equiv_p \mathrm{NM}^*_{\to}}_{\text{tree-like versions}}$

- $\mathrm{F}_{\to} \equiv_p \mathrm{LJ}_{\to} \equiv_p \mathrm{NM}_{\to}$ go back to Reckhow '76
- $\mathrm{F}_{\to} \equiv_p \mathrm{F}^*_{\to}$ due to Krajíček '94, implicational version J. '17

Further extensions of the lower bound (as in J. '09, J. '17):

- full language of IPC
- superintuitionistic logics $\mathrm{IPC} \subseteq L \subseteq \mathrm{BD}_2$
- exponential separation between Extended Frege and Substitution Frege

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