Hereditarily bounded sets

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Essentially undecidable theories

T essentially undecidable
$\iff$ all consistent extensions of T are undecidable
$\iff$ no r.e. extension of T is complete and consistent

Typically: we verify T is ess. und. (Gödelian) by checking that it includes (or interprets) one of known ess. und. theories

Convenient weak ess. und. theories for the purpose:

- Robinson's arithmetic Q
- Robinson's theory R
- adjunctive set theory AS
- Vaught's set theory VS

Vaught's set theory

Weak set theory *VS* introduced in [Vau'67]

Language: $\in$

Axioms:

$$(\mathrm{V}_n) \qquad \forall x_0, \dots, x_{n-1}\, \exists y\, \forall t \left(t \in y \leftrightarrow \bigvee_{i<n} t = x_i \right)$$

for each standard $n \in \omega$

NB: (V_n) implies (V_m) for $n \geq m > 0$

- *VS* is ess. und.
- finite fragments $VS_n = (\mathrm{V}_0) + (\mathrm{V}_n)$ not ess. und.
  - VS_n interpretable in any theory with pairing

Theories with pairing

Assume $T \vdash \exists x \, \exists y \; x \neq y$

Pairing function in T: definable function $p(x, y)$ s.t. T proves

$$p(x, y) = p(x', y') \to x = x' \wedge y = y'$$

Non-functional pairing: a formula $\pi(x, y, p)$ s.t. T proves

$$\forall x \, \forall y \, \exists p \, \pi(x, y, p)$$
$$\pi(x, y, p) \wedge \pi(x', y', p) \to x = x' \wedge y = y'$$

Example: VS_2 has non-functional pairing $\{\{x\}, \{x, y\}\}$

See [Vis'08] for more background

Decidable theories with pairing

Theories with variable-length sequence encoding (sequential theories [Pud'85]) interpret $Q \implies$ ess. und.

In contrast: there are decidable theories with pairing

- [Mal'61,'62] theories of locally free algebras ($\approx$ term algebras, also with "commutativity" constraints) incl. acyclic pairing functions: $\langle \mathbb{N}, 2^x 3^y \rangle$
- [Ten'72] p.f. acyclic up to a few exceptions e.g.: $2^x(2y+1)-1$, $\max\{x^2, y^2+x\}+y$, $\binom{x+y+1}{2}+x$

Even with more arithmetical structure:

- [Sem'83] $\langle \mathbb{N}, +, 2^x \rangle$ (has p.f. $2^x + 2^{x+y}$ [CR'99])
- [CR'01] $\langle \mathbb{N}, S, \binom{x+y+1}{2}+x \rangle$

Pairing and k-sets

Let $\langle x, y\rangle$ be a pairing function, $k \geq 2$

- encode k-tuples by pairs:
  $\langle x_0, \dots, x_{k-1}\rangle = \langle \cdots \langle\langle x_0, x_1\rangle, x_2\rangle, \cdots, x_{k-1}\rangle$
- encode k-element sets by k-tuples:

$$x \in y \iff \exists x_0, \dots, x_{k-1} \left(y = \langle x_0, \dots, x_{k-1}\rangle \wedge \bigvee_{i<k} x = x_i\right)$$

Satisfies VS_k if $\langle x, y\rangle$ non-surjective (easily fixable)

Also works for non-functional pairing

Lemma

Any theory with pairing interprets VS_k for each k

Decidable extensions of VS_k

Corollary

For any k, VS_k has a decidable completion

The extensions of VS_k we get from theories of pairing are quite unnatural as theories of sets

- Extensionality fails:
  $\langle x, y\rangle$ and $\langle y, x\rangle$ represent the same set

Problem (informal)

Find a natural decidable extension of VS_k with a transparent meaning

Hereditarily finite sets

Work in ZF(C)

The set H_ω of hereditarily finite sets:

- The smallest set s.t. $\forall x\,(x \subseteq H_\omega \land x \text{ finite} \implies x \in H_\omega)$
- The unique set s.t. $\forall x\,(x \subseteq H_\omega \land x \text{ finite} \iff x \in H_\omega)$
- $x \in H_\omega \iff \mathrm{tc}(x) \text{ finite} \iff \forall y \in \mathrm{tc}(\{x\})\, y \text{ finite}$
- $H_\omega = V_\omega = \bigcup_{n\in\omega} V_n$, where $V_0 = \varnothing$, $V_{n+1} = \mathcal{P}(V_n) \supseteq V_n$

Transitive closure $\mathrm{tc}(x)$: smallest transitive set that includes x

$$\mathrm{tc}(x) = \bigcup_n \mathrm{tc}_n(x), \text{ where } \mathrm{tc}_0(x) = x,\ \mathrm{tc}_{n+1}(x) = \mathrm{tc}_n(x) \cup \bigcup_{y\in\mathrm{tc}_n(x)} y$$

$\mathbf{H}_\omega = \langle H_\omega, \in\rangle$ is bi-interpretable with $\langle \mathbb{N}, +, \cdot\rangle$

Hereditarily bounded sets

The set H_k of sets hereditarily of size $\leq k$:

- The smallest set s.t. $\forall x\,(x \subseteq H_k \land |x| \leq k \implies x \in H_k)$
- The unique set s.t. $\forall x\,(x \subseteq H_k \land |x| \leq k \iff x \in H_k)$
- $x \in H_k \iff \forall y \in \mathrm{tc}(\{x\})\,|y| \leq k$
- $H_k = \bigcup_n V_{n,\leq k}$, where $V_{0,\leq k} = \varnothing$, $V_{n+1,\leq k} = [V_{n,\leq k}]^{\leq k}$

NB: $H_\omega = \bigcup_{k\in\omega} H_k$

$\mathbf{H}_k = \langle H_k, \in\rangle$ is a natural model of VS_k

Minimality: $\mathbf{H}_k$ embeds (transitively) in any model of VS_k

Problem

What is $\mathrm{Th}(\mathbf{H}_k)$? Is it decidable?

Easy cases

- $k = 0$: $\mathbf{H}_0$ is a one-element structure
- $k = 1$: $H_1 = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\{\{\varnothing\}\}\}, \dots\}$
  $\implies \mathbf{H}_1 \simeq \langle \mathbb{N}, S(x) = y \rangle$
  - decidable, PSPACE-complete
  - quantifier elimination in language $\langle \varnothing, \{x\} \rangle$
  - strongly minimal, uncountably categorical, ...

Not really easy, but already known:

- $k = 2$: $\mathbf{H}_2$ is definitionally equivalent to $\langle H_2, \varnothing, \{x, y\} \rangle$
  $\{x, y\}$ free commutative operation [Mal'62]
  - decidable, some form of quantifier elimination, stable

NB: For $k \geq 3$, Malcev's results do not apply
$\{x, x, y\} = \{x, y, y\}$

The general case

The rest of this talk:

- an explicit axiomatization S_k for $\mathrm{Th}(\mathbf{H}_k)$
- characterization of elementary equivalence of tuples
- S_k is decidable, with iterated exponential complexity
- quantifier elimination

The theory S_k

S_k is axiomatized by:

- the axioms (V_0) and (V_k) of VS_k
- extensionality

$$(\mathrm{E}) \qquad \forall x, y \, \big(\forall t \, (t \in x \leftrightarrow t \in y) \to x = y\big)$$

- boundedness (all sets have $\leq k$ elements)

$$(\mathrm{B}_k) \qquad \forall x, u_0, \dots, u_k \left(\bigwedge_{i \leq k} u_i \in x \to \bigvee_{i < j \leq k} u_i = u_j \right)$$

- acyclicity: for each $n \in \omega$,

$$(\mathrm{C}_n) \qquad \forall x_0, \dots, x_n \, \neg\Big(\bigwedge_{i<n} x_i \in x_{i+1} \wedge x_n \in x_0 \Big)$$

Basic strategy

Main goal: prove S_k is complete

$$\implies S_k = \mathrm{Th}(\mathbf{H}_k)$$
$$\implies S_k \text{ is decidable}$$

We use an Ehrenfeucht–Fraïssé argument:

- combinatorial description of $\mathbf{A}, \overline{a} \equiv \mathbf{B}, \overline{b}$ for $\mathbf{A}, \mathbf{B} \vDash S_k$
- for empty $\overline{a}, \overline{b}$, it gives $\mathbf{A} \equiv \mathbf{B}$

Bounded elementary equivalence

Quantifier rank:

$$\begin{aligned} \mathrm{rk}(\varphi) &= 0 && \varphi \text{ atomic} \\ \mathrm{rk}(c(\varphi_0, \varphi_1, \dots)) &= \max\{\mathrm{rk}(\varphi_0), \mathrm{rk}(\varphi_1), \dots\} && c \text{ connective} \\ \mathrm{rk}(Qx\,\varphi) &= \mathrm{rk}(\varphi) + 1 && Q \in \{\exists, \forall\} \end{aligned}$$

$\mathbf{A} = \langle A, \in^{\mathbf{A}} \rangle$, $\mathbf{B} = \langle B, \in^{\mathbf{B}} \rangle$, $\overline{a} \in A$, $\overline{b} \in B$:

$$\mathbf{A}, \overline{a} \equiv \mathbf{B}, \overline{b} \iff \forall \varphi\, (\mathbf{A} \vDash \varphi(\overline{a}) \iff \mathbf{B} \vDash \varphi(\overline{b}))$$

$$\mathbf{A}, \overline{a} \equiv_n \mathbf{B}, \overline{b} \iff \text{the same for } \varphi \text{ s.t. } \mathrm{rk}(\varphi) \le n$$

Ehrenfeucht–Fraïssé games

$\mathrm{EF}_n(\mathbf{A}; \mathbf{B})$:

- players Spoiler, Duplicator
- n rounds, in round i:
  - S chooses an element of one of A, B
  - D responds by an element of the other one
  - $\implies \alpha_i \in A$, $\beta_i \in B$
- D wins iff $\alpha_i \mapsto \beta_i$ is a partial isomorphism (= preserves atomic predicates both ways)

$\mathrm{EF}_n(\mathbf{A}, \overline{a}; \mathbf{B}, \overline{b})$:

- D wins iff $\alpha_i \mapsto \beta_i$, $a_j \mapsto b_j$ is a partial isomorphism

EF games vs. elementary equivalence

Theorem (Fraïssé, Ehrenfeucht)

$\mathbf{A}, \overline{a} \equiv_n \mathbf{B}, \overline{b}$ iff D has a winning strategy in $\mathrm{EF}_n(\mathbf{A}, \overline{a}; \mathbf{B}, \overline{b})$

Graded back-and-forth system for **A**, **B**: relations E_n s.t.

- $\overline{a}\ E_n\ \overline{b} \implies a_i \mapsto b_i$ is a partial isomorphism
- $\overline{a}\ E_{n+1}\ \overline{b} \implies \forall c \in A\, \exists d \in B\, (\overline{a}, c\ E_n\ \overline{b}, d)$ and v.v.

Corollary

If $\{E_n : n < \omega\}$ is a graded back-and-forth system, then

$$\overline{a}\ E_n\ \overline{b} \implies \mathbf{A}, \overline{a} \equiv_n \mathbf{B}, \overline{b}$$

Transitive closures

$\mathbf{A} \vDash S_k$, $\overline{a} \in A$, $l = \mathrm{lh}(\overline{a})$: define $\mathrm{tc}^{\mathbf{A}}_n(\overline{a}) \subseteq A$

$$\mathrm{tc}^{\mathbf{A}}_0(\overline{a}) = \{a_i : i < l\}$$

$$\mathrm{tc}^{\mathbf{A}}_{n+1}(\overline{a}) = \mathrm{tc}^{\mathbf{A}}_n(\overline{a}) \cup \bigcup_{u \in \mathrm{tc}^{\mathbf{A}}_n(\overline{a})} \{v \in A : v \in^{\mathbf{A}} u\}$$

$$\mathrm{tc}^{\mathbf{A}}(\overline{a}) = \bigcup_{n \in \omega} \mathrm{tc}^{\mathbf{A}}_n(\overline{a})$$

NB: $\mathrm{tc}^{\mathbf{A}}_n(\overline{a})$ finite

$$|\mathrm{tc}^{\mathbf{A}}_n(\overline{a})| \le l \cdot k^{\le n}, \qquad k^{\le n} = \sum_{i=0}^{n} k^i = \frac{k^{n+1} - 1}{k - 1} \qquad (k \ne 1)$$

Similarity relations

When considered as structures:

$$\mathbf{tc}_n^{\mathbf{A}}(\overline{a}) = \langle \mathrm{tc}_n^{\mathbf{A}}(\overline{a}), \in^{\mathbf{A}}, \overline{a} \rangle, \qquad \mathbf{tc}^{\mathbf{A}}(\overline{a}) = \langle \mathrm{tc}^{\mathbf{A}}(\overline{a}), \in^{\mathbf{A}}, \overline{a} \rangle$$

We define

$$\mathbf{A}, \overline{a} \sim \mathbf{B}, \overline{b} \iff \mathbf{tc}^{\mathbf{A}}(\overline{a}) \simeq \mathbf{tc}^{\mathbf{B}}(\overline{b})$$
$$\mathbf{A}, \overline{a} \sim_n \mathbf{B}, \overline{b} \iff \mathbf{tc}_n^{\mathbf{A}}(\overline{a}) \simeq \mathbf{tc}_n^{\mathbf{B}}(\overline{b})$$

NB: Using the finiteness of tc_n, Kőnig's lemma implies

$$\mathbf{A}, \overline{a} \sim \mathbf{B}, \overline{b} \iff \forall n \left(\mathbf{A}, \overline{a} \sim_n \mathbf{B}, \overline{b} \right)$$

Definability of tc_n

The finiteness of tc_n easily implies:

Lemma

$\mathbf{A} \vDash S_k$, $\overline{a} \in A$, $l = \mathrm{lh}(\overline{a})$, $n < \omega$
$\Longrightarrow$ $\exists$ formula $\varphi_{\overline{a},n}(\overline{x})$ s.t. $\forall$ $\mathbf{B} \vDash S_k$, $\overline{b} \in B$:

$$\mathbf{B} \vDash \varphi_{\overline{a},n}(\overline{b}) \iff \mathbf{A}, \overline{a} \sim_n \mathbf{B}, \overline{b}$$

We may take $\varphi_{n,\overline{a}}$ as a Boolean combination of bounded existential formulas of rank $l(k^{\leq n} - 1)$

Bounded quantifiers: $\exists y \in x\, \varphi \equiv \exists y\, (y \in x \wedge \varphi)$

Elementary equivalence implies similarity

Corollary

If $\mathbf{A}, \mathbf{B} \vDash S_k$, $\overline{a} \in A$, $\overline{b} \in B$, $l = \mathrm{lh}(\overline{a}) = \mathrm{lh}(\overline{b})$, $n < \omega$:

$$\mathbf{A}, \overline{a} \equiv \mathbf{B}, \overline{b} \implies \mathbf{A}, \overline{a} \sim \mathbf{B}, \overline{b}$$

$$\mathbf{A}, \overline{a} \equiv_{l(k^{\leq n}-1)} \mathbf{B}, \overline{b} \implies \mathbf{A}, \overline{a} \sim_n \mathbf{B}, \overline{b}$$

The converse is more difficult, and will require an Ehrenfeucht–Fraïssé argument

Extending tc_n isomorphisms

The crux of the argument:

Lemma

Let $\mathbf{A}, \mathbf{B} \vDash S_k$, $\overline{a} \in A$, $\overline{b} \in B$, $l = \mathrm{lh}(\overline{a}) = \mathrm{lh}(\overline{b})$, $n > 0$.

If

$$\mathbf{A}, \overline{a} \sim_{k^{\leq n}+n} \mathbf{B}, \overline{b}$$

then

$$\forall c \in A \quad \exists d \in B \quad \mathbf{A}, \overline{a}, c \sim_{n-1} \mathbf{B}, \overline{b}, d$$

This gives a graded back-and-forth system ...

Characterization of elementary equivalence

Theorem

Let $\mathbf{A}, \mathbf{B} \vDash S_k$, $\overline{a} \in A$, $\overline{b} \in B$, $l = \mathrm{lh}(\overline{a}) = \mathrm{lh}(\overline{b})$, $n < \omega$. Then

$$\mathbf{A}, \overline{a} \equiv \mathbf{B}, \overline{b} \iff \mathbf{A}, \overline{a} \sim \mathbf{B}, \overline{b}.$$

More precisely, for all $n \in \omega$,

$$\mathbf{A}, \overline{a} \equiv_{l(k^{\leq n}-1)} \mathbf{B}, \overline{b} \implies \mathbf{A}, \overline{a} \sim_n \mathbf{B}, \overline{b},$$
$$\mathbf{A}, \overline{a} \sim_{t_k(n)} \mathbf{B}, \overline{b} \implies \mathbf{A}, \overline{a} \equiv_n \mathbf{B}, \overline{b},$$

where $t_k(0) = 0$, $t_k(n+1) = k^{\leq t_k(n)+1} + t_k(n) + 1$.

Completeness and decidability

Since $\mathrm{tc}^{\mathbf{A}}(\langle\rangle) = \varnothing$, we have $\mathbf{A}, \langle\rangle \sim \mathbf{B}, \langle\rangle$ for any $\mathbf{A}, \mathbf{B} \vDash S_k$:

Corollary

S_k is a complete theory, thus $S_k = \mathrm{Th}(\mathbf{H}_k)$

Any recursively axiomatizable complete theory is decidable:

Corollary

$S_k = \mathrm{Th}(\mathbf{H}_k)$ is decidable

In particular, S_k is a decidable extension of VS_k

Quantifier elimination

Corollary

In S_k, any formula is equivalent to a Boolean combination of bounded existential formulas.

If we expand the language with the predicates $y = \varnothing$ and $y = \{x_0, \dots, x_{k-1}\}$, every formula is equivalent to a bounded existential and a bounded universal formula.

NB: $y = \varnothing$ and $y = \{x_0, \dots, x_{k-1}\}$ have bounded universal definitions in the original language

Further properties

Proposition

S_k is a stable theory

Proposition

$k \geq 1 \implies S_k$ is not finitely axiomatizable

Problem (A. Visser)

Is there a consistent finitely axiomatized decidable theory with pairing?

Complexity: lower bound

Superexponential function: $2^x_0 = x$, $2^x_{n+1} = 2^{2^x_n}$

Theorem [FR'79]

T consistent theory with pairing $\implies \exists\gamma > 0$ s.t. any decision procedure for T has complexity $\geq 2^0_{\gamma n}$

- complexity measure: take your pick
- theories of [Mal'62], [Ten'72] meet the bound

Corollary

$\exists\gamma > 0$ s.t. any decision procedure for a consistent extension of VS_2 has complexity $\geq 2^0_{\gamma n}$

Complexity: upper bound

$t_k(n) \le 2^{c_k}_n$ for some constant c_k

Turning the Ehrenfeucht–Fraïssé argument into an algorithm:

Theorem

S_k is decidable in time $2^{c_k}_{n/4}$

- matches the [FR'79] lower bound for $k \ge 2$
- S_1 is PSPACE-complete, S_0 is NC^1-complete
- overestimates the complexity for formulas with a small number of quantifier alternations

Improved algorithm

Handle blocks of quantifiers in one go:

Theorem

Given a sentence φ with

- $\varphi \in \exists_r$
- n: number of symbols
- q: max length of quantifier blocks

we can decide whether $S_k \vdash \varphi$ in

$$\begin{cases} \mathrm{NTIME}(n^{O(1)}) & r = 1 \\ \mathrm{NTIME}\big((kq)^{O(kq)} n^{O(1)}\big) & r = 2 \\ \mathrm{NTIME}\big(2_{r-1}^{O(qk \log k)} n^{O(1)}\big) & r \geq 3 \end{cases}$$

Summary

We identified $\mathrm{Th}(\mathbf{H}_k)$ as a natural extension of VS_k:

- decidable (of lowest possible complexity)
- transparent explicit axiomatization
- combinatorial characterization of elementary equivalence
- quantifier elimination

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