Canonical rules

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Derivable and admissible rules

Consider any (propositional) logic L, defined by a set of finitary inference rules closed under substitution.
A rule

$$\varrho = \frac{\varphi_1, \ldots, \varphi_k}{\psi}$$

is

- derivable in L, if there exists a proof of ψ using the postulated rules of L, and the axioms $\varphi_1, \ldots, \varphi_k$,
- admissible in L, if the set of theorems of L is closed under ϱ: for every substitution σ, if L proves all $\sigma\varphi_i$, then it proves $\sigma\psi$.

Most non-classical logics admit some nonderivable rules.

Sets of admissible rules

Questions about admissibility:

- decidability
- semantic characterization
- description of a basis
- preservation
- …

Common approaches:

- Rybakov: combinatorics of universal frames
- Ghilardi, Iemhoff: projective formulas, extension properties

Canonical formulas and rules

Zakharyschev's canonical formulas

- axiomatize all logics extending $K4$ or IPC
- syntactical objects with built-in semantics
- powerful tool for certain types of problems

This talk: we introduce canonical rules

- "axiomatize" all systems of (multiple-conclusion) rules over $K4$ or IPC
- properties similar to canonical formulas
- can be used to analyze admissible rules

Systems of rules

Multiple-conclusion rules

We generalize the concept of a rule to allow more (or less) formulas in the conclusion.

Multiple-conclusion rule: Γ/Δ, where Γ and Δ are finite sets of formulas.

Rule system: a set of multiple-conclusion rules which

- contains φ/φ
- is closed under substitution, cut, weakening

Example: let A be the set of all single-conlusion rules derivable from a set X of (single-conclusion) rules. Then the closure of A under weakening on right is a rule system.

In particular, any modal or s.i. logic defines a rule system.

Frame semantics (modal)

Frame $\langle W, R, V \rangle$:

- $<$ binary relation on a set W
- $V \subseteq \mathcal{P}(W)$ is closed under Boolean operations and $\Box$, where

$$\Box X = \{u \in W;\ \forall v \in W\ (u\ R\ v \Rightarrow v \in X)\}$$

An admissible valuation $\Vdash$ is a homomorphism of the free algebra of formulas into $\langle V, \cap, \cup, -, \Box \rangle$.

A Kripke frame $\langle W, R \rangle$ is identified with the frame $\langle W, R, \mathcal{P}(W) \rangle$.

Frame semantics (modal)

A frame is

- refined, if for every $u, v \in W$,

$$\forall X \in W\, (u \in X \Leftrightarrow v \in X) \Rightarrow u = v,$$

$$\forall X \in W\, (u \in \Box X \Rightarrow v \in X) \Rightarrow u\, R\, v,$$

- compact, if every $S \subseteq V$ with fip has nonempty intersection,
- descriptive, if it is refined and compact.

Descriptive frames are categorically dual to modal algebras.

Kripke frames are refined, but infinite Kripke frames are not compact. Finite refined frames are Kripke.

Frame semantics (intuitionistic)

Intuitionistic frame $\langle W, \leq, V \rangle$:

- $\leq$ partial order
- V set of upwards closed subsets of W, closed under monotone Boolean operations, and the operation

$$X \to Y = \Box(-X \cup Y)$$

- definition of refined and compact frames accordingly modified

Descriptive intuitionistic frames are dual to Heyting algebras.

Semantics for rules

Let $\langle W, R, V\rangle$ be a modal (intuitionistic) frame, and $\varrho = \Gamma/\Delta$ a modal (intuitionistic) rule.

A formula φ is satisfied by an admissible valuation $\Vdash$, if $\forall u \in W\ u \Vdash \varphi$, otherwise it is refuted by $\Vdash$.

The rule ϱ is satisfied by $\Vdash$, if some $\varphi \in \Gamma$ is refuted by $\Vdash$, or some $\psi \in \Delta$ is satisfied by $\Vdash$.

The rule ϱ is valid in $\langle W, R, V\rangle$, if it is satisfied by every admissible valuation.

Caveat: frames may be empty.

Semantics for rules

Soundness: the set of rules valid in a class of modal (intuitionistic) frames is a rule system extending K (IPC).

Completeness: let A be a rule system extending K (IPC), and $\varrho \notin A$. There exists a descriptive modal (intuitionistic) frame which validates A, and refutes ϱ.

The validity of a rule system A is preserved by

- p-morphic images,
- disjoint unions iff A is equivalent to a set of single-conclusion rules,
- generated subframes iff A is equivalent to a set of assumption-free rules.

Canonical rules

Subreductions

Let $\langle W, <, V \rangle$ be a transitive frame, and $\langle F, < \rangle$ a finite transitive Kripke frame.

A subreduction of W to F is a partial mapping f from W to F such that for every $u, v \in W$, $i \in F$,

- $\mathrm{rng}(f) = F$
- if $u, v \in \mathrm{dom}(f)$ and $u < v$, then $f(u) < f(v)$
- if $f(u) < i$, there exists $w \in \mathrm{dom}(f)$ such that $u < w$ and $f(w) = i$
- $f^{-1}(i) \in V$ in the modal case, and $W \smallsetminus f^{-1}(i){\downarrow} \in V$ in the intuitionistic case

Here $X{\downarrow} = \{u; \exists v \in X\, u \leq v\}$.

Closed domain condition

Domain is an upwards closed subset $d \subseteq F$.

A subreduction f satisfies the global closed domain condition (GCD) for d, if there does not exist a $u \in W$ such that

- $u \notin \mathrm{dom}(f)$
- $f(u{\uparrow}) = d$

If there is no $u \in \mathrm{dom}(f){\uparrow}$ with these properties, f satisfies the (local) closed domain condition (CD).

If D is a set of domains, f satisfies GCD (CD) for D if it satisfies GCD (CD) for every $d \in D$.

Canonical rules (modal)

Let $\langle F, < \rangle$ be a finite transitive modal Kripke frame, and D a set of domains in F. The canonical rule $\gamma(F, D)$ uses variables p_i, $i \in F$, and it is defined as follows.

- assumptions of $\gamma(F, D)$ consist of:
  - $p_i \vee p_j$, for every $i \neq j$
  - $\Box p_j \rightarrow p_i$, for every $i < j$
  - $\Box p_j \vee p_i$, for every $i \not< j$
  - $\bigwedge\limits_{i} p_i \wedge \bigwedge\limits_{i \notin d} \Box p_i \rightarrow \bigvee\limits_{i \in d} \Box p_i$, for every $d \in D$
- conclusions of $\gamma(F, D)$ are the variables p_i

Canonical rules (intuitionistic)

If the frame F is intuitionistic, we also define the intuitionistic canonical rule $\delta(F, D)$.

- assumptions of $\delta(F, D)$ consist of:
  - $(\bigwedge_{j \not\geq i} p_j \to p_i) \to p_i$, for every $i \in F$
  - $p_j \to p_i$, for every $i \leq j$
  - $\bigwedge_{i \notin d} p_i \to \bigvee_{i \in d} p_i$, for every $d \in D$ which is not rooted
- conclusions of $\delta(F, D)$ are the variables p_i

Refutation conditions

Canonical rules are a syntactic counterpart of subreductions.

- A transitive modal frame $\langle W, <, V \rangle$ refutes $\gamma(F, D)$ iff there exists a subreduction of W to F with GCD on D.

$$f(u) = i \Leftrightarrow u \nVdash p_i$$

- An intuitionistic frame $\langle W, \leq, V \rangle$ refutes $\delta(F, D)$ iff there exists a subreduction of W to F with GCD on D.

$$f(u) = i \Leftrightarrow u \Vdash \bigwedge_{j \ngeq i} p_j, u \nVdash p_i$$

$$u \Vdash p_i \Leftrightarrow i \notin f(u\uparrow)$$

Canonical rules vs. formulas

Main differences of canonical rules $\gamma(F, D)$, $\delta(F, D)$ to Zakharyaschev's (normal) canonical formulas $\alpha(F, D)$, $\beta(F, D)$:

- rules correspond to global CD,
  formulas correspond to local CD
- F need not be rooted
  (rules may have multiple conclusions)
- F may be empty
  (rules may have zero conclusions)
- we may have $F \in D$

Notation

Let $\gamma(F, D, \bot) := \gamma(F, D \cup \{\emptyset\})$, $\delta(F, D, \bot) := \delta(F, D \cup \{\emptyset\})$
$F^\sharp$ is the set of all nonempty domains in F

Special cases of canonical rules:

- subframe rules $\gamma(F) := \gamma(F, \emptyset)$, $\delta(F) := \delta(F, \emptyset)$
- cofinal subframe rules $\gamma(F, \bot)$, $\delta(F, \bot)$
- dense subframe rules $\gamma^\sharp(F) := \gamma(F, F^\sharp)$, $\delta^\sharp(F) := \delta(F, F^\sharp)$
- frame rules $\gamma^\sharp(F, \bot)$, $\delta^\sharp(F, \bot)$

Examples

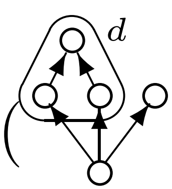
"consistency rule": $\dfrac{\bot}{} = \gamma(\emptyset, \bot)$

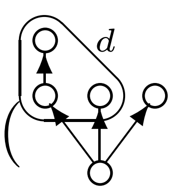
"unboxing rule": $\dfrac{\Box p}{p} = \gamma^\sharp(\bullet)$

disjunction property: $\dfrac{p \vee q}{p, q} = \delta^\sharp(\circ\ \circ)$

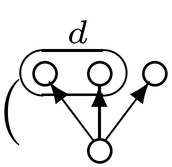
modal disjunction property: $\dfrac{\Box p \vee \Box q}{p, q} = \gamma^\sharp(\bullet) + \gamma(*\,*, \{\{*\,*\}\})$

Kreisel–Putnam rule:

$\dfrac{\neg p \to q \vee r}{(\neg p \to q) \vee (\neg p \to r)} = \delta($ 

$, \{\emptyset, d\}) + \delta($ 

$, \{\emptyset, d\})$

$+\, \delta($ 

$, \{\emptyset, d\})$

More examples

$/$	$\gamma(\emptyset)$
$\bot\ /$	$\gamma(\emptyset, \bot)$
$\Box p \to p\ /\ p$	$\gamma(\circ)$
$\Diamond\top\ /\ \bot$	$\gamma(\circ, \bot)$
$/\ \Box p \to p$	$\gamma(\bullet)$
$\odot\Box\bot\ /\ \bot$	$\gamma(\bullet, \bot)$
$\Box p\ /\ p$	$\gamma^\sharp(\bullet)$
$\Box\bot\ /\ \bot$	$\gamma^\sharp(\bullet, \bot)$
$\Box(p \to \Box p) \to p\ /\ p$	$\gamma(\circ\!\!\longleftrightarrow\!\!\circ)$
$\Diamond p, \Diamond\neg p\ /\ \bot$	$\gamma(\circ\!\!\longleftrightarrow\!\!\circ, \bot)$

$/$	$\delta(\emptyset)$
$\bot\ /$	$\delta(\emptyset, \bot)$
$/\ \bot$	$\delta(\circ) =$
	$\delta(\circ, \bot)$
$/\ p \to q,\ q \to p$	$\delta(\circ\ \circ)$
$/\ \neg p, \neg\neg p$	$\delta(\circ\ \circ, \bot)$
$p \vee q\ /\ p, q$	$\delta^\sharp(\circ\ \circ)$
$p \vee \neg p\ /\ p, \neg p$	$\delta^\sharp(\circ\ \circ, \bot)$
$/\ p \vee \neg p$	$\delta(\overset{\circ}{\underset{\circ}{\uparrow}}) =$
	$\delta(\overset{\circ}{\underset{\circ}{\uparrow}}, \bot)$

Completeness

Theorem. For every modal rule ϱ, there are efficiently computable canonical rules $\gamma(F_i, D_i)$ such that

$$K4 + \varrho = K4 + \{\gamma(F_i, D_i);\ i < n\}.$$

For every intuitionistic rule ϱ, there are efficiently computable sequence of canonical rules $\delta(F_i, D_i)$ such that

$$IPC + \varrho = IPC + \{\delta(F_i, D_i);\ i < n\}.$$

In both cases, we may further require

- $\emptyset \in D_i$
- no $d \in D_i$ is generated by a reflexive point

for every $i < n$.

Completeness

Proof sketch:

- reduce intuitionistic case to modal case by Gödel translation
- reduce modal rule systems to quasinormal modal logics using characteristic formulas

$$\chi(\Gamma/\Delta) = \bigwedge_{\varphi\in\Gamma} \Box\varphi \to \bigvee_{\psi\in\Delta} \Box\psi$$

- use Zakharyaschev's canonical formulas for quasinormal extensions of $K4$

Single-conclusion rule systems

The canonical rule $\gamma(F, D)$ or $\delta(F, D)$ is a single-conclusion rule iff F is rooted.

Bad news: some single-conclusion rules cannot be written as a combination of single-conclusion canonical rules

Example: $D4 = K4 + \Diamond\top$

- $D4 = K4 + \gamma(\bullet, \bot) + \gamma(\bullet \circ, \bot)$
- $\gamma(F, D) \in D4$ iff $\emptyset \in D$ and F contains a dead end
- if $\emptyset \in D$, F contains a dead end, and F is rooted, then

$$\gamma(F, D) \in K4 + \gamma(\bullet, \bot) + \gamma\left(\begin{tikzcd}[sep=small] \bullet & & \circ \\ & * \arrow[ul] \arrow[ur] & \end{tikzcd}, \bot\right)$$

- $\bullet \circ$ refutes $D4$, but validates $\gamma(\bullet, \bot) + \gamma\left(\begin{tikzcd}[sep=small] \bullet & & \circ \\ & * \arrow[ul] \arrow[ur] & \end{tikzcd}, \bot\right)$

Restricted canonical rules

restricted canonical rule $\gamma(F, D; X)$, where $X \subseteq F$

- assumptions: as $\gamma(F, D)$, conclusions: $\{p_i;\ i \in X\}$
- W refutes $\gamma(F, D; X)$ iff there is a subreduction of W onto a generated subframe $G \subseteq\cdot F$ such that $X \subseteq G$ with GCD on D
- $K4 + \gamma(F, D; X) = K4 + \{\gamma(G, D \upharpoonright G);\ X \subseteq G \subseteq\cdot F\}$
- a rule system $A \supseteq K4$ is single-conclusion iff A is axiomatizable by restricted canonical rules $\gamma(F, D, \{r\})$

Intuitionistic restricted canonical rules $\delta(F, D; X)$: analogous

Admissibility of canonical rules

Admissible rules revisited

Let L be a logic. A rule

$$\frac{\varphi_1, \ldots, \varphi_k}{\psi_1, \ldots, \psi_\ell}$$

is admissible in L, if the following holds for every substitution σ:
if L proves $\sigma\varphi_i$ for every i, then L proves $\sigma\psi_j$ for some j.

The set A_L of all rules admissible in L is a rule system.

Admissible rules of $K4$

Let W be a transitive frame, and $X \subseteq W$. An $x \in W$ is

- an irreflexive tight predecessor of X, if $x{\uparrow} = X{\Uparrow}$
- a reflexive tight predecessor of X, if $x{\uparrow} = \{x\} \cup X{\Uparrow}$

We define two sets of rules:

- $A^\bullet = \dfrac{\Box\varphi \to \bigvee_{i<n} \Box\psi_i}{\boxdot\varphi \to \psi_0, \dots, \boxdot\varphi \to \psi_{n-1}}$
- $A^\circ = \dfrac{\bigwedge_{j<m}(\varphi_j \equiv \Box\varphi_j) \to \bigvee_{i<n} \Box\psi_i}{\boxdot \bigwedge_{j<m} \varphi_j \to \psi_0, \dots, \boxdot \bigwedge_{j<m} \varphi_j \to \psi_{n-1}}$

Admissible rules of $K4$

Theorem. The following are equivalent for any canonical rule $\gamma(F, D)$.

- $K4$ admits $\gamma(F, D)$
- $\gamma(F, D) \in K4 + A^\bullet + A^\circ$
- some $d \in D$ lacks a reflexive or irreflexive tight predecessor in F
- $\gamma(F, D)$ is not equivalent to an assumption-free rule over $K4$

Admissible rules of $K4$

Proof sketch: we assume

- every $d \in D$ has reflexive and irreflexive tight predecessors in F,
- U is a generated submodel of W,
- U refutes $\gamma(F, D)$,

we show that W refutes $\gamma(F, D)$.

We fix a subreduction of U to F with GCD for D, and expand it to a subreduction of W to F. The only problem is with GCD.

We take $d \in D$ one by one, and fix f to satisfy GCD for d: points where it fails are mapped to a tight predecessor of d.

Admissibility in extensions

We have a dichotomy: every canonical rule $\gamma(F, D)$ is admissible in $K4$, or assumption-free over $K4$.

An assumption-free rule is admissible iff it is derivable.

Corollary: Admissible rules of any $L \supseteq K4$ have a basis consisting of rules admissible in both $K4$ and L.

Furthermore:

- the corollary holds also for single-conclusion rules
- aside from $K4$, a similar analysis works for IPC, GL, $S4$, $K4.3$, ...

Other logics

The “dichotomy” is far from universal.

Example: $\delta^\sharp(\circ\,\circ)$ is neither admissible nor assumption-free in KC

Problem: is there a more general criterion for admissibility of canonical rules?

Thank you for attention!