Exercises for Mathematical Logic (17 Oct 2023)

11. (If you are familiar with topology.) Give a direct proof of the propositional compactness theorem, not using the completeness theorem.
[Hint: Consider the product topology on the set $\{0,1\}^A$ of all assignments.]

In the lecture, we have proved completeness of a proof system using connectives $\{\to,\bot\}$. A complete system using the De Morgan language $\{\wedge,\vee,\neg,\bot,\top\}$ is given in the van den Dries lecture notes, but the next exercise shows how to construct one mechanically.

12. For any $\{\to,\bot\}$-formula φ, let φ^* denote the De Morgan formula such that $p^* = p$ for atoms p, $\bot^* = \bot$, and $(\varphi \to \psi)^* = (\neg\varphi^* \vee \psi^*)$. Similarly, given a De Morgan formula ψ, let $\psi^\#$ be its translation to a $\{\to,\bot\}$-formula using fixed $\{\to,\bot\}$-translations of all De Morgan connectives. Let $\vdash_0$ denote a sound and complete Hilbert-style proof system for $\{\to,\bot\}$-formulas such as the one given in the lecture, and let $\vdash_1$ be the Hilbert-style proof system in the De Morgan language that has inference rule schemata $\varphi_1^*,\dots,\varphi_k^* \;/\; \varphi_0^*$ for each rule schema $\varphi_1,\dots,\varphi_k \;/\; \varphi_0$ of $\vdash_0$ (where axioms are treated as rules with $k=0$), and axiom schemata $\neg c(\varphi_0,\dots,\varphi_{k-1}) \vee c^{\#*}(\varphi_0,\dots,\varphi_{k-1})$, $\neg c^{\#*}(\varphi_0,\dots,\varphi_{k-1}) \vee c(\varphi_0,\dots,\varphi_{k-1})$ for each k-ary De Morgan connective c. Then $\vdash_1$ is a sound and complete proof system in the De Morgan language. [Hint: You will need to show $\vdash_1 \neg\psi \vee \psi^{\#*}$, $\vdash_1 \neg\psi^{\#*} \vee \psi$ for all De Morgan formulas ψ.]

13. A set of propositional or first-order sentences S is *independent* if S is not equivalent to S' for any proper subset $S' \subsetneq S$.

(i) S is independent iff $S \smallsetminus \{\varphi\} \nvDash \varphi$ for all $\varphi \in S$.

(ii) Show that every countable theory T has an independent axiomatization, i.e., an independent set of sentences S equivalent to T. [Hint: Try to generalize the fact that $\{\varphi,\psi\} \equiv \{\varphi, \psi \vee \neg\varphi\}$.]

(Uncountable theories have independent axiomatizations as well by a theorem of I. Reznikoff, but this is more difficult to prove.)