Exercises for Mathematical Logic (21 Nov 2023)

(Exercises 18–20 were already included in the previous batch, but in the end we didn't get to the compactness theorem and Vaught's test in the November 7 lecture, hence they really belong here. Let me add one more application of Vaught's test for good measure.)

21. An *atom* in a Boolean algebra $\mathcal{A} = \langle A, 0, 1, \wedge, \vee, -, \leq \rangle$ is an element $a \in A$ such that $a > 0$, but $0 < x < a$ for no $x \in A$; $\mathcal{A}$ is *atomless* if $0 \neq 1$ and $\mathcal{A}$ has no atoms. Show that the theory of atomless Boolean algebras is $\aleph_0$-categorical, hence complete.

[Hint: Construct an isomorphism between two countable atomless Boolean algebras $\mathcal{A}$ and $\mathcal{B}$ by a back-and-forth argument, as a union of a sequence of isomorphisms between finite subalgebras. It might help to observe that if $\mathcal{A}_0$ is a finite subalgebra of $\mathcal{A}$, and $\mathcal{A}_1$ is the algebra generated by $A_0 \cup \{b\}$ for some $b \in A$, then each atom of $\mathcal{A}_0$ either remains an atom in $\mathcal{A}_1$, or splits into two atoms.]