Exercises for Mathematical Logic (1 Oct 2024)

We have seen in the lecture that the De Morgan language $\{\wedge, \vee, \neg, \top, \bot\}$ is functionally complete, and specifically, that every Boolean function can be represented by a CNF or DNF of size $O(2^n n)$.

1. Prove that $\{\vee, \neg\}$, $\{\to, \bot\}$, and $\{\uparrow\}$ are functionally complete, where $x \uparrow y$ denotes the Sheffer stroke $\neg(x \wedge y)$.

2. Prove that $\{\to\}$, $\{\wedge, \vee, \top, \bot\}$, and $\{\leftrightarrow, \top, \bot\}$ are not functionally complete.
[Hint: Find a nontrivial property of Boolean functions which is preserved by composition, and holds for functions in the given basis.]

3. For any Boolean function $f\colon \{0,1\}^n \to \{0,1\}$, the following are equivalent:

(i) $\{f\}$ is functionally complete.

(ii) $f(0,\dots,0) = 1$, $f(1,\dots,1) = 0$, and there exists an assignment $\langle e_0, \dots, e_{n-1}\rangle \in \{0,1\}^n$ such that $f(e_0,\dots,e_{n-1}) = f(\neg e_0, \dots, \neg e_{n-1})$.

[Hint: For (ii) $\to$ (i), look at functions obtained from f by identifying some of the variables.]

4. For any $n \in \mathbb{N}$, the *parity* function $\bigoplus_{i<n} x_i \colon \{0,1\}^n \to \{0,1\}$ is defined as $\left(\sum_{i<n} x_i\right) \bmod 2$. Show that any DNF or CNF representing $\bigoplus_{i<n} x_i$ has size $\Omega(2^n n)$. [Hint: What terms of the form $\bigwedge_{i\in I} x_i^{e_i}$ can imply one of $\bigoplus_{i<n} x_i = 0$ or $\bigoplus_{i<n} x_i = 1$? Here, $I \subseteq [n]$, $e_i \in \{0,1\}$, $x^1 = x$, $x^0 = \neg x$.]

5. There are formulas representing $\bigoplus_{i<n} x_i$ of size $O(n^c)$ for some constant c.
[Hint: Consider a balanced tree of binary parities. You may get it down to $c = 2$.]

6. Any DNF equivalent to the CNF $\bigwedge_{i<n}(x_i \vee y_i)$ has size $\Omega(2^n n)$.