Exercises for Mathematical Logic (17 Oct 2025)

In the lecture, we have proved completeness of a proof system using connectives $\{\to, \bot\}$. A complete system using the De Morgan language $\{\wedge, \vee, \neg, \bot, \top\}$ is given in the van den Dries lecture notes, but the next exercise shows how to construct one mechanically.

12. For any $\{\to, \bot\}$-formula φ, let φ^* denote the De Morgan formula such that $p^* = p$ for atoms p, $\bot^* = \bot$, and $(\varphi \to \psi)^* = (\neg\varphi^* \vee \psi^*)$. Similarly, given a De Morgan formula ψ, let $\psi^{\#}$ be its translation to a $\{\to, \bot\}$-formula using fixed $\{\to, \bot\}$-translations of all De Morgan connectives. Let $\vdash_0$ denote a sound and complete Hilbert-style proof system for $\{\to, \bot\}$-formulas such as the one given in the lecture, and let $\vdash_1$ be the Hilbert-style proof system in the De Morgan language that has inference rule schemata $\varphi_1^*, \dots, \varphi_k^* \;/\; \varphi_0^*$ for each rule schema $\varphi_1, \dots, \varphi_k \;/\; \varphi_0$ of $\vdash_0$ (where axioms are treated as rules with $k = 0$), and axiom schemata $\neg c(\varphi_0, \dots, \varphi_{k-1}) \vee c^{\#*}(\varphi_0, \dots, \varphi_{k-1})$, $\neg c^{\#*}(\varphi_0, \dots, \varphi_{k-1}) \vee c(\varphi_0, \dots, \varphi_{k-1})$ for each k-ary De Morgan connective c. Then $\vdash_1$ is a sound and complete proof system in the De Morgan language. [Hint: You will need to show $\vdash_1 \neg\psi \vee \psi^{\#*}$, $\vdash_1 \neg\psi^{\#*} \vee \psi$ for all De Morgan formulas ψ.]

13. Show that $\Gamma \subseteq \mathrm{Prop}_A$ is a maximal consistent set iff Γ is a complete theory, i.e., a consistent deductively closed set such that $\Gamma \vdash \varphi$ or $\Gamma \vdash \varphi \to \bot$ for every $\varphi \in \mathrm{Prop}_A$.

14. (If you are familiar with topology.) Give a direct proof of the propositional compactness theorem, not using the completeness theorem.
[Hint: Consider the product topology on the set $\{0,1\}^A$ of all assignments.]

15. A set of formulas S is *independent* if S is not equivalent to S' for any proper subset $S' \subsetneq S$.

(i) S is independent iff $S \smallsetminus \{\varphi\} \nvDash \varphi$ for all $\varphi \in S$.

(ii) Show that every countable set of formulas T has an independent axiomatization, i.e., an independent set of formulas S equivalent to T. [Hint: Generalize the fact that $\{\varphi, \psi\} \equiv \{\varphi, \psi \vee \neg\varphi\}$.]

(This works for first-order theories just the same. Uncountable theories have independent axiomatizations, too, by a theorem of I. Reznikoff, but this is more difficult to prove.)