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The physical nature of the event horizon in the Schwarzschild black hole solution

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Abstract This study explores the relationship between the Schwarzschild metric and alternative metrics used to describe the gravitational field of a black hole in free space. While it is well-established that an infinite number of coordinate systems can be employed in general relativity, we demonstrate that the black hole solution is unique when expressed in a physically meaningful (proper) coordinate system. Notably, this coordinate system differs from the Schwarzschild metric due to the distinction between the true physical distance R and the Schwarzschild coordinate distance r. Consequently, the event horizon, commonly associated with the Schwarzschild solution, is shown to be a coordinate artefact of the chosen covariant metric tensor rather than a coordinate-invariant physical feature. As a result, no boundary prevents outgoing photons from escaping the black hole's vicinity. This finding challenges the mainstream interpretation but remains fully consistent with general relativity. Moreover, it is supported by numerical modelling of light rays near a black hole. By reconsidering the existence of event horizons, this work offers potential resolutions to long-standing issues in black hole formation theories and the emission of electromagnetic and gravitational waves from black holes.

1 Introduction

It is commonly understood that the choice of coordinates in general relativity (GR) equations is irrelevant for solving physical problems. According to the principle of general covariance, coordinates are merely labels for spacetime events. Since Einstein's field equations are invariant under coordinate transformations, their physical solution should be independent of the chosen coordinates. Nevertheless, selecting an appropriate coordinate system is crucial for solving the equations for specific problems [1].

For example, the spacetime geometry produced by the static gravitational field of a point mass in an empty region can be studied using a spherically symmetric metric tensor g_{kl}, which is expressed in its general form as

$$ds^2 = -g_{tt}(r)c^2dt^2 + g_{rr}(r)dr^2 + g_{\omega\omega}(r)d\omega^2 \,. \tag{1}$$

However, choosing a simpler coordinate system can make solving Einstein's field equations more straightforward. A well-known example is the Schwarzschild metric, first derived by Droste [2] and Hilbert [3], being a modification of the metric proposed originally by Schwarzschild [4, 5]. The Schwarzschild metric is one of the most famous metrics in GR and is extensively discussed in classical relativity textbooks [6–9]. It is given as [7, their eq. 31.1]

$$ds^2 = -\left(1 - \frac{r_s}{r}\right)c^2dt^2 + \left(1 - \frac{r_s}{r}\right)^{-1}dr^2 + r^2d\omega^2 \,, \tag{2}$$

where $d\omega^2 = d\vartheta^2 + \sin^2\vartheta\, d\varphi^2$ is the element of the coordinate solid angle, $r_s = 2GM/c^2$ is the Schwarzschild radius, G is the gravitational constant, M is the mass of the central body, and r and t are the coordinate (contravariant) distance and time. The speed of light in a vacuum far from the source of gravity is denoted as c.

The metric in Eq. (2) describes the so-called Schwarzschild black hole solution, with a sphere of radius r_s defining the event horizon. The event horizon, at $r = r_s$, is characterized by $g_{tt} = 0$ and is commonly described as a boundary around the point mass, from which neither particles nor light can escape [10–12]. It is assumed that events within the horizon cannot influence external observers [7]. In addition, the radial term in the Schwarzschild metric exhibits a coordinate singularity at $r = r_s$, where $g_{rr} \to \infty$.

However, the Schwarzschild metric is not the only option for describing the Schwarzschild black hole solution and solving Einstein's field equations. The same problem can be addressed using alternative metrics and coordinate systems, such as isotropic or harmonic coordinates [8]. As shown by Painlevé [13], Fromholz et al. [1], and Crothers [14, 15], the number of possible alternative

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coordinate systems that satisfy Einstein's equations is, in fact, infinite. For instance, a particularly simple alternative is the so-called Brillouin metric, derived using Brillouin's coordinates [16]

$$ds^2 = -\left(1 + \frac{r_s}{r}\right)^{-1} c^2 dt^2 + \left(1 + \frac{r_s}{r}\right) dr^2 + \left(1 + \frac{r_s}{r}\right)^2 r^2 d\omega^2 , \tag{3}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, as in Eq. (2). Interestingly, the properties of the Brillouin metric are completely different from those of the Schwarzschild metric: the metric tensor is regular, with no zeros or singularities, for all $r \neq 0$.

Similarly, no zeros or singularities occur for all $r \neq 0$ in the original Schwarzschild metric [4, 5, 17, 18] that slightly differs from the standard Schwarzschild metric in Eq. (2). It is expressed as

$$ds^2 = -\left(1 - \frac{r_s}{r'}\right) c^2 dt^2 + \left(1 - \frac{r_s}{r'}\right)^{-1} dr^2 + r'^2 d\omega^2 , \tag{4}$$

where $r_s = 2GM/c^2$ is the Schwarzschild radius, and $r' = \left(r^3 + r_s^3\right)^{1/3}$.

The differences in the properties of the metrics in Eqs. (2)–(4) highlight the fact that these metrics depend on the chosen coordinate system, which is curvilinear. Consequently, quantities expressed in such systems are formal parameters and are not necessarily physically measurable. Thus, a true physical solution to the problem, which should be invariant under coordinate transformations, has yet to be found [19]. Ignoring this necessary step and directly interpreting metrics corresponding to various coordinate systems in physical terms can lead to confusion and misinterpretation of the gravitational field of a point mass [13, 20–24].

The issue of the physical meaning of the Schwarzschild and other coordinate-dependent metrics used to describe the Schwarzschild black hole spacetime involves two key difficulties:

- First, there is confusion regarding whether the Schwarzschild coordinates are inherently preferable to other coordinates for describing black hole spacetime. For example, Misner et al. [7] advocate for using Schwarzschild coordinates as a particularly simple coordinate system with exceptional intrinsic geometric properties. Moreover, the exclusivity of Schwarzschild coordinates seems to be reinforced by the Birkhoff theorem, which states that the geometry of a spherically symmetric spacetime satisfying Einstein's field equations in a vacuum must be static and asymptotically flat, and can always be written using the Schwarzschild metric ([7, p. 843]; [12, their Appendix B]).
  However, a closer examination reveals that other metrics, such as those mentioned earlier, are also static and asymptotically flat, and the spherically symmetric Schwarzschild spacetime can also be rewritten using these alternative coordinates. Moreover, Hawking and Ellis [12] consider the Schwarzschild coordinates a poor choice, and Misner et al. [7, §31.3] discuss the coordinate pathology of the Schwarzschild metric at $r < r_s$, where t and r reverse their roles as timelike and spacelike coordinates (i.e., signs of g_{tt} and g_{rr} are reversed).
  Hawking and Ellis [12] argue that this coordinate singularity, where $g_{rr} \to \infty$ at $r = r_s$, is not a true physical singularity but rather an apparent one, as it can be removed by using other coordinate systems. Clearly, this argument can be applied not only to the coordinate singularity, where $g_{rr} \to \infty$ at $r = r_s$, but also to the existence of the event horizon, where $g_{tt} = 0$ at $r = r_s$, in the Schwarzschild metric. This pathological effect is commonly addressed using Kruskal–Szekeres coordinates [7, 23], but it could also be elegantly resolved using Brillouin's coordinates [16], see Eq. (3).
- Second, it is often overlooked that the coordinate distance r in the Schwarzschild metric does not have a direct physical meaning, as it is not a standard distance R, measured in metres. Similarly, the Schwarzschild angular coordinates ϑ, φ, and ω do not correspond to the standard spherical angles θ, ϕ, and Ω, measured in degrees. For instance, Landau and Lifshitz [6], Misner et al. [7], Hartle [25], and Lambourne [26] emphasize the need for caution in interpreting Schwarzschild coordinates. The radial coordinate r in the Schwarzschild metric is not the actual physical distance R from the observer to the point mass. Instead, it is assumed to be a quantity related to the area of a sphere in Schwarzschild coordinates, its element da being defined as

  $$da = r^2 d\omega , \tag{5}$$

  where r and t are held constant in the Schwarzschild metric. The physical distance R must be calculated via an integral over r ([6, their eq. 97.16]; [26, his eq. 5.20]). Similarly, $d\omega$ in Eq. (5) must be rescaled to describe a physically meaningful solid angle measured in degrees, as shown in Appendix 1.
  Despite these distinctions, the difference between r and R is often neglected in the physical interpretation of the Schwarzschild metric. For example, the Schwarzschild radius r_s is frequently calculated for objects such as Earth, the Sun, or other stars, and is often misinterpreted as reflecting the physical size or volume of a hypothetical black hole [7, 12, 26, 27]. Similarly, orbits in a Schwarzschild gravitational field are commonly described as functions of the Schwarzschild coordinate r and misinterpreted as the geometry of actual physical orbits [7, 26, 28]. This misinterpretation also arises in studies of black hole collapse, where particle behaviour near the event horizon is analysed under the implicit assumption that the black hole physical size is directly described by the Schwarzschild radius [7, 12, 27, 29, 30]. Such studies often fail to acknowledge that the properties of the solution may differ significantly when expressed in physical coordinates.

In this paper, we address the challenges of interpreting the Schwarzschild spacetime when using Schwarzschild and other coordinate systems. Using the radial geodesics of photons, we demonstrate how to work properly with different coordinate systems. We examine the properties of several metrics commonly used to solve this problem and discuss their relationships. We point out that confusions arise from conflating invariant quantities with coordinate-dependent ones, as well as from neglecting the distinction between free-falling frames and non-inertial static frames. We emphasize the importance of properly eliminating coordinate dependence by transforming coordinate-dependent quantities into physical quantities. Through this process, we propose a unique and straightforward interpretation of the physical solution, even within non-inertial frames. Finally, we argue that the event horizon in the Schwarzschild metric may be an apparent feature, as it shrinks to a single point in physical space. This finding underscores the complexities of the Schwarzschild metric and its implications for the nature of event horizons.

2 Theory

2.1 Riemannian manifold and curvilinear coordinate systems

In general relativity, physical problems involving gravity are studied using a Riemannian spacetime (manifold). Unlike Euclidean space, which remains unaffected by the presence of mass and gravitational fields, a Riemannian manifold is curved, reflecting the influence of gravity. This curvature is physically manifested through the bending of geodesics, which are the paths traced by free-falling particles or light rays within the spacetime. The curvature of geodesics can result in intricate paths for light rays, leading to a complex topology for the manifold. The geometry of manifolds is parametrized using curvilinear coordinate systems. These systems map the flat four-dimensional Minkowski space onto the curved Riemannian manifold. Notably, a single manifold can be represented by various coordinate systems, each providing a different perspective on its geometry.

Let us assume that (x^0, x^1, x^2, x^3) is a specific choice of a coordinate system that covers the Riemannian manifold. These coordinates will be unique and differentiable functions of the Cartesian coordinates (y^0, y^1, y^2, y^3), covering the flat Minkowski space $\mathbf{R}^4$. The geometry of the Riemannian manifold is defined by the covariant and contravariant base vectors $\mathbf{g}_\mu$ and $\mathbf{g}^\mu$ [25, his eq. 20.43]

$$\mathbf{g}_\mu = \frac{\partial y^\beta}{\partial x^\mu}\,\mathbf{i}_\beta\,,\;\; \mathbf{g}^\mu = \frac{\partial x^\beta}{\partial y^\mu}\,\mathbf{i}^\beta\,, \tag{6}$$

and by the covariant and contravariant metric tensors $g_{\mu\nu}$ and $g^{\mu\nu}$ ([8, his eq. 4.2.6]; [25, his eq. 20.44]; [28, his eq. 2.48])

$$g_{\mu\nu} = \mathbf{g}_\mu \cdot \mathbf{g}_\nu = \frac{\partial y^\alpha}{\partial x^\mu}\frac{\partial y^\beta}{\partial x^\nu}\,\eta_{\alpha\beta}\,,\;\; g^{\mu\nu} = \mathbf{g}^\mu \cdot \mathbf{g}^\nu = \frac{\partial x^\alpha}{\partial y^\mu}\frac{\partial x^\beta}{\partial y^\nu}\,\eta^{\alpha\beta}\,, \tag{7}$$

where $\mathbf{i}_\beta = \mathbf{i}^\beta$ are the unit Cartesian base vectors in Minkowski space, and $\eta_{\alpha\beta} = \eta^{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$ is the Minkowski metric. In contrast to the base vectors $\mathbf{i}_\beta$, which are unit in length, the base vectors $\mathbf{g}_\mu$ and $\mathbf{g}^\mu$ are generally non-unit. Vector $\mathbf{v}$ and tensor $\mathbf{T}$ in curvilinear coordinates x^α are expressed as

$$\mathbf{v} = v_\alpha \mathbf{g}^\alpha = v^\alpha \mathbf{g}_\alpha\,, \tag{8}$$

and

$$\mathbf{T} = T_{\alpha\beta}\,\mathbf{g}^\alpha \mathbf{g}^\beta = T^{\alpha\beta}\,\mathbf{g}_\alpha \mathbf{g}_\beta\,. \tag{9}$$

The covariant and contravariant components of a vector $\mathbf{v}$ and tensor $\mathbf{T}$ are related as

$$v^\alpha = g^{\alpha\mu} v_\mu\,,\;\; T^{\alpha\beta} = g^{\alpha\mu} g^{\beta\nu}\,T_{\mu\nu}\,. \tag{10}$$

2.2 Physical (proper) quantities in curvilinear coordinates

Since the base vectors $\mathbf{g}_\mu$ and $\mathbf{g}^\mu$ are generally non-unit, the vector components v_α or v^α in Eq. (10) are not coordinate-invariant in curvilinear coordinates x^α. Hence, they do not represent physical quantities. A physically meaningful vector quantity, which is invariant to the choice of coordinates, is the vector magnitude v, expressed as

$$v = \sqrt{v_\mu v^\mu}\,, \tag{11}$$

being measured in physical experiments in true physical units.

To express physically meaningful vector components, the base vectors $\mathbf{g}_\mu$ and $\mathbf{g}^\mu$ must be replaced with normalized unit base vectors $\mathbf{e}_\mu$ and $\mathbf{e}^\mu$ [25, 31]

$$\mathbf{e}_\mu = \mathbf{g}_\mu/\sqrt{g_{\mu\mu}}\,,\;\; \mathbf{e}^\mu = \mathbf{g}^\mu/\sqrt{g^{\mu\mu}}\;\; (\text{no summation over}\,\mu)\,. \tag{12}$$

Consequently, the vector v is expressed as

$$\mathbf{v} = v^{(\mu)}\mathbf{e}_\mu = v_{(\mu)}\mathbf{e}^\mu\,, \tag{13}$$

where

$$v^{(\mu)} = v^{\mu} \sqrt{g_{\mu\mu}}\,, \;\; v_{(\mu)} = v_{\mu} \sqrt{g^{\mu\mu}} \;\; \text{(no summation over } \mu\text{)}\,, \tag{14}$$

are the physical (proper) components of the vector **v**. For orthogonal curvilinear coordinates, the physical components simplify to

$$v^{(\mu)} = v_{(\mu)}\,. \tag{15}$$

Another physically meaningful quantity is the infinitesimal distance in the Riemannian manifold defined as [7, their eq. 13.3]

$$\mathrm{d}s^2 = g_{\mu\nu}\, \mathrm{d}x^{\mu} \mathrm{d}x^{\nu}\,, \tag{16}$$

being independent of the choice of the coordinate system x^{α}. For static problems, where the Riemannian manifold is described by orthogonal coordinates (i.e., time is independent of spatial coordinates), the physical distance in the three-dimensional (3D) Riemannian manifold reduces to the distance in the standard 3D Euclidean space.

Note that the term ‘proper’ is more commonly used than ‘physical’ to denote true physical quantities in relativistic literature. However, the meaning of ‘proper’ is often restricted to physical quantities measured only in inertial or free-falling frames. In this work, we adopt the following convention: the term ‘physical’ refers to quantities measured in true physical units, regardless of the reference frame (whether free-falling or non-inertial), while the term ‘proper’ is mostly reserved for physical quantities measured in free-falling frames.

2.3 Physical time and physical radial distance in a spherically symmetric manifold

Let us assume that $(x^0, x^1, x^2, x^3) = (ct, r, \vartheta, \varphi)$ is an orthogonal coordinate system, which covers a spherically symmetric Riemannian manifold with the metric tensor expressed in Eq. (1). Although this coordinate system appears straightforward, we must be careful in interpreting quantities in this system because its base vectors are not normalized. Therefore, the coordinates $(ct, r, \vartheta, \varphi)$ are, in general, artificial parameters rather than true time, distance and spherical angles (cT, R, θ, ϕ). Consequently, the quantities evaluated in these coordinates may not represent true physical quantities (measured in seconds, metres and degrees, respectively).

To calculate physical quantities (cT, R, θ, ϕ) in this curvilinear coordinate system, a set of orthonormal base vectors must be applied [19, 25, 31]. Using Eqs. (1) and (14), we derive the following expressions for true physical coordinates [6, 19, 25, 26]

$$\mathrm{d}T = \sqrt{g_{tt}}\, \mathrm{d}t\,, \tag{17}$$

$$\mathrm{d}R = \sqrt{g_{rr}}\, \mathrm{d}r\,, \tag{18}$$

where $\mathrm{d}T$ is the physical time element, and $\mathrm{d}R$ is the physical radial distance element. For a time-independent (static) coordinate system, the physical time T and physical radial distance R are

$$T = \sqrt{g_{tt}}\, t\,, \tag{19}$$

$$R = \int \sqrt{g_{rr}}\, \mathrm{d}r\,. \tag{20}$$

Similarly, the angular coordinates (ϑ, φ) must be rescaled to yield the physical spherical angles (θ, ϕ)

$$\sqrt{g_{\vartheta\vartheta}}\, \mathrm{d}\vartheta = \sqrt{g_{\theta\theta}}\, \mathrm{d}\theta\,, \;\; \sqrt{g_{\varphi\varphi}}\, \mathrm{d}\varphi = \sqrt{g_{\phi\phi}}\, \mathrm{d}\phi\,. \tag{21}$$

Hence, we get for Schwarzschild coordinates

$$\mathrm{d}\theta = \frac{r}{R} \mathrm{d}\vartheta\,, \;\; \mathrm{d}\phi = \frac{r \sin\vartheta}{R \sin\theta} \mathrm{d}\varphi\,, \tag{22}$$

implying that angular Schwarzschild coordinates (ϑ, φ) are not standard spherical angles measured in degrees (for details, see Appendix 1). This finding might seem confusing and is often overlooked. However, the reason for it is clear: since $g_{rr} \neq 1$, the radial coordinate distance r is different from the radial physical distance R and this space deformation projects also into the distortion of angles (ϑ, φ).

2.4 Physical speed of light in a spherically symmetric manifold

The concept of the physical speed of light in different frames can be confusing. Even when using physical coordinates, the value of the physical speed of light is not always straightforward. This ambiguity arises because the speed of light depends on the reference frame, in which it is measured. For example, the speed of light measured in a laboratory at rest relative to a gravitational field differs from that measured in a laboratory situated in a free-falling frame. In non-inertial frames (e.g., at rest with respect to the gravitational field), the effects of gravity cannot be neglected as they are in free-falling frames. This is emphasized by Einstein [32] when he says: *‘the law of constancy of the velocity of light in vacuo, which constitutes one of the fundamental assumptions in the special theory* *of relativity and to which we have already frequently referred, cannot claim an unlimited validity... its results hold only so long as we are able to disregards the influences of gravitational fields on the phenomena (e.g. of light)'.*

Hence, the proper speed of light in free-falling frames in a gravitational field is exactly equal to c. However, special care must be taken when calculating the physical speed of light in non-inertial frames that are at rest relative to the gravitational field. The following discussion focuses on this non-trivial case.

Let us assume that light propagates radially ($\mathrm{d}\vartheta = 0$, $\mathrm{d}\varphi = 0$), so the line element of the metric is

$$\mathrm{d}s^2 = -g_{tt}c^2\mathrm{d}t^2 + g_{rr}\mathrm{d}r^2 \,. \tag{23}$$

For null geodesics, $\mathrm{d}s^2 = 0$, this simplifies to

$$g_{tt}c^2\mathrm{d}t^2 = g_{rr}\mathrm{d}r^2 \,. \tag{24}$$

The contravariant (coordinate-dependent) speed of light c_g^r along the radial direction is then

$$c_g^r = \frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{g_{tt}}{g_{rr}}}\, c \,, \tag{25}$$

where subscript g indicates that the speed is affected by gravity in a static non-inertial frame, and the coordinate time t is used as the affine parameter.

To calculate the proper speed of light, which is coordinate-invariant and measured in a static frame, we must express the speed of light in an orthonormal coordinate basis [25, 31]. Hence, the radial component of the physical speed of light is

$$c_{g(r)} = \frac{\mathrm{d}R}{\mathrm{d}t} = \sqrt{g_{rr}}\,\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{g_{rr}}\,c_g^r = \sqrt{g_{tt}}\,c \,. \tag{26}$$

Calculating the tangential components of the speed of light in an analogous way, the physical speed of light has the same magnitude in all directions. Therefore, we can write

$$c_g = \sqrt{g_{tt}}\,c \,. \tag{27}$$

Emphasize that the physical speed of light c_g is the value measured in a frame at rest relative to the static gravitational field, calculated with respect to the coordinate time t. This system is non-inertial and not free-falling, meaning it is influenced by gravity. As a result, the speed of light is not constant but varies depending on the observer's distance from the source of the gravitational field.

In contrast, the proper speed of light (i.e., the physical speed of light in a free-falling frame) is calculated with respect to the proper time T, rather than the coordinate time t:

$$c = \frac{\mathrm{d}R}{\mathrm{d}T} \,. \tag{28}$$

In such a frame, the speed of light remains constant and equal to c, just as in inertial frames. Hence, this analysis highlights that the time T is always associated with the physical time in the free-falling frame. In contrast, the coordinate time t is, in general, not physical, except when it is measured in a static frame relative to the gravitational field. In this specific non-inertial frame, the coordinate time t becomes physical and measures time in the presence of the gravitational field. The physical nature of the coordinate time t in the static non-inertial frame, and its dependence on the gravitational potential, is supported by the Pound and Rebka [33] and Pound and Snider [34] experiments, which studied the gravitational redshift. These experiments demonstrated that the gravitational field affects the passage of physical time in a static non-inertial frame. Therefore, we consider both times T and t as physical: time T refers to the proper time in the free-falling frame, and time t refers to the physical time in the static frame.

2.5 Gravitational field of point mass in various coordinate systems

In this section, we study the Schwarzschild spacetime, i.e., the Riemannian manifold produced by the gravitational field around a point mass situated in a vacuum, using various metrics. Since the space around the mass is empty, the Einstein field equations imply that the Ricci curvature tensor $R_{\mu\nu}$ [7, their eq. 8.47] and the scalar curvature $\mathcal{R}$ [7, their eq. 8.48] of the manifold are zero [28, his Eq. 4.46]. Although this metric is Ricci-flat, it remains curved in the sense that the Riemann tensor does not vanish.

As mentioned earlier, the manifold can be covered by various types of coordinates. Here, we examine the Schwarzschild and Brillouin metrics as described by Eqs. (2) and (3). Additionally, we will consider metrics with isotropic coordinates defined as [7, 8, 20]

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{4r}\right)^2 \Big/ \left(1 + \frac{r_s}{4r}\right)^2 c^2\mathrm{d}t^2 + \left(1 + \frac{r_s}{4r}\right)^4 \left(\mathrm{d}r^2 + r^2\mathrm{d}\omega^2\right) , \tag{29}$$

and with harmonic coordinates defined as [8]

$$\mathrm{d}s^2 = -\left(1 - \frac{r_s}{2r}\right) \Big/ \left(1 + \frac{r_s}{2r}\right) c^2\mathrm{d}t^2 + \left(1 + \frac{r_s}{2r}\right) \Big/ \left(1 - \frac{r_s}{2r}\right) \mathrm{d}r^2 + r^2\left(1 + \frac{r_s}{2r}\right)^2 \mathrm{d}\omega^2 \,. \tag{30}$$

Using Eqs. (20) and (27), the physical distance R and the physical speed of light c_g for the four coordinate systems are as follows:

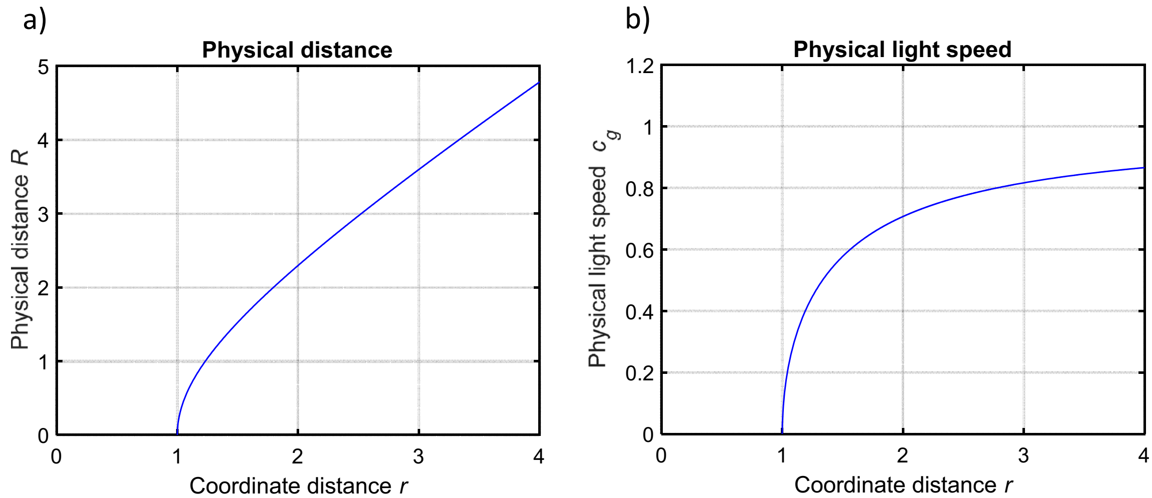

Fig. 1 **a** Physical distance R (normalized to r_s), and **b** physical speed of light c_g (normalized to c) as a function of the Schwarzschild coordinate distance r

- Schwarzschild coordinates (Fig. 1)

$$R = \int_{r_s}^{r} \frac{1}{\sqrt{1-\frac{r_s}{r}}}\,\mathrm{d}r\,,\ c_g = c\sqrt{1-\frac{r_s}{r}}\,,\ r \ge r_s\,, \tag{31}$$

- Brillouin coordinates (Fig. 2)

$$R = \int_{0}^{r} \sqrt{1+\frac{r_s}{r}}\,\mathrm{d}r\,,\ c_g = c/\sqrt{1+\frac{r_s}{r}}\,,\ r \ge 0\,, \tag{32}$$

- isotropic coordinates

$$R = \int_{r_s/4}^{r} \left(1+\frac{r_s}{4r}\right)^2 \mathrm{d}r\,,\ c_g = c\left(1-\frac{r_s}{4r}\right)/\left(1+\frac{r_s}{4r}\right),\ r \ge \frac{r_s}{4}\,, \tag{33}$$

- harmonic coordinates

$$R = \int_{r_s/2}^{r} \sqrt{\left(1+\frac{r_s}{2r}\right)/\left(1-\frac{r_s}{2r}\right)}\,\mathrm{d}r\,,\ c_g = c\sqrt{\left(1-\frac{r_s}{2r}\right)/\left(1+\frac{r_s}{2r}\right)}\,,\ r \ge \frac{r_s}{2}\,. \tag{34}$$

The integrals in Eqs. (31)–(34) can be solved easily and expressed in closed form (see Appendix 2). The range of the coordinate distance r is chosen so that the physical distance R and the physical speed of light c_g to be positive or zero. The limiting values of r are: r_s, 0, $\frac{1}{4}r_s$ and $\frac{1}{2}r_s$ for the Schwarzschild, Brillouin, isotropic and harmonic coordinates, respectively. These limits correspond to the singularity at the origin of 3D Euclidean coordinates. Values of r below these limits are unphysical, as they result in negative or complex values for R or c_g. Consequently, it is important to note that a point in the Schwarzschild metric with $r = 0$ does not correspond to the position of the point mass in physical space, as commonly assumed, but rather has no counterpart in physical space—for a detailed discussion, see the next section.

2.6 Position of the point mass singularity in different metrics

The key question in solving the Schwarzschild problem is to determine the coordinate distance r corresponding to the position of the point mass in various metrics. Since the point mass represents a physical singularity, all metrics describing the problem must exhibit singular behaviour at this point. The situation is particularly simple in Brillouin's coordinates, which are singular only at $r = 0$. In this case, the origin of Brillouin's curvilinear coordinates coincides with the origin of Euclidean space. However, two singularities exist in the Schwarzschild, isotropic and harmonic coordinates: (1) at $r = 0$, and (2) at $r = r_s$, $\frac{1}{4}r_s$, and $\frac{1}{2}r_s$, respectively. Thus, we must select which of the two singularities corresponds to the position of the point mass.

In the case of the Schwarzschild metric, the physical radial distance R from a point mass to an observer is calculated using an integral in Eq. (31), where the lower limit of the integral defines the position of the point mass in this metric. If we choose $r = 0$, the integral becomes complex-valued for any upper limit $r > 0$. Similarly, the physical speed of light c_g, calculated from Eq. (31) becomes complex-valued for $0 \le r < r_s$, and equals zero at $r = r_s$. Therefore, the singularity at $r = 0$ in the Schwarzschild metric is purely a mathematical artefact with no physical meaning. Consequently, the position of the point mass must correspond to the other singularity of the metric tensor, located at $r = r_s$, as indicated by the lower limit of the integral for R in Eq. (31). The same

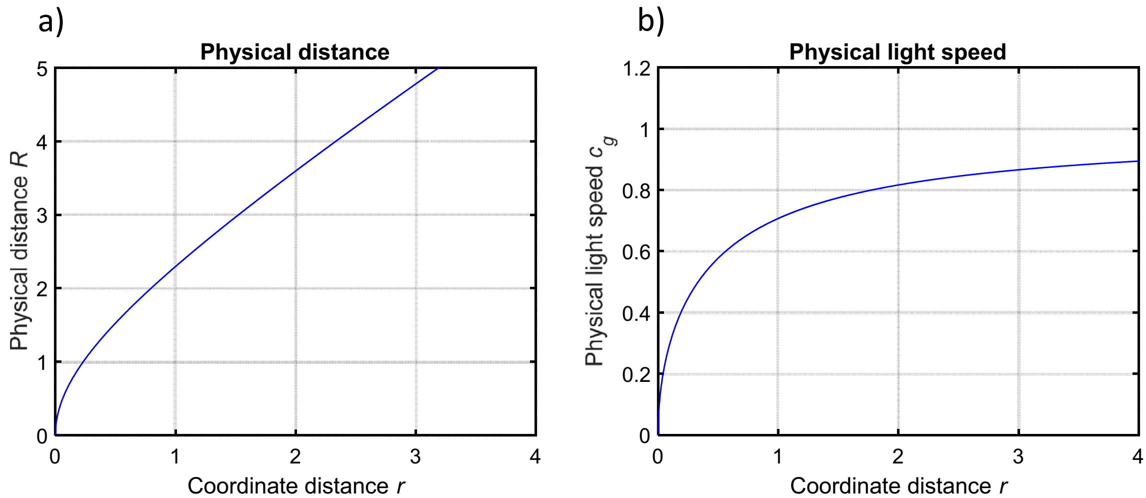

Fig. 2 **a** Physical distance R (normalized to r_s), and **b** physical speed of light c_g (normalized to c) as a function of the Brillouin coordinate distance r

reasoning applies to the isotropic and harmonic coordinates, where the point mass is located at $r = \frac{1}{4}r_s$ and $r = \frac{1}{2}r_s$, respectively, as shown in the integrals for R in Eqs. (33) and (34).

For completeness, it should be noted that R does not necessarily represent the physical distance from the singularity. If we use other integral limits in Eqs. (31)–(34), such as r_A and r_B in Eq. (31) for the Schwarzschild coordinates:

$$R = \int_{r_A}^{r_B} \frac{1}{\sqrt{1 - \frac{r_s}{r}}} \mathrm{d}r \ , \ r_B \geq r_A \geq r_s \ , \tag{35}$$

then R represents the physical distance between two points with Schwarzschild coordinates r_A and r_B.

2.7 Asymptotic properties of analysed metrics

Far from the point mass ($r \to \infty$), the formulas for the physical distance coincide in all four coordinate systems to the first-order approximation of $1/r$:

$$\mathrm{d}T = \left(1 - \frac{r_s}{2r}\right)\mathrm{d}t + O\left(r^{-2}\right), \tag{36}$$

$$\mathrm{d}R = \left(1 + \frac{r_s}{2r}\right)\mathrm{d}r + O\left(r^{-2}\right). \tag{37}$$

The physical speed of light c_g also behaves similarly in all four coordinate systems:

$$c_g = \left(1 - \frac{r_s}{2r}\right)c + O\left(r^{-2}\right), \tag{38}$$

with c_g converging to c for $r \to \infty$.

Equations (36) and (37) imply that the differences in scales for proper and coordinate time, as well as for physical and coordinate distance, vanish as $r \to \infty$:

$$\lim_{r\to\infty} \mathrm{d}T = \mathrm{d}t \ , \ \lim_{r\to\infty} \mathrm{d}R = \mathrm{d}r \ . \tag{39}$$

Similarly, from Eqs. (22), (36) and (37), the differences in physical and coordinate angular scales also disappear:

$$\lim_{r\to\infty} \mathrm{d}\theta = \mathrm{d}\vartheta \ , \ \lim_{r\to\infty} \mathrm{d}\phi = \mathrm{d}\varphi \ . \tag{40}$$

Consequently, the metric tensors for all four analysed coordinates converge to the metric of Minkowski space for $r \to \infty$:

$$\lim_{r\to\infty} g_{tt} = 1 \ , \ \lim_{r\to\infty} g_{rr} = 1 \ , \ \lim_{r\to\infty} g_{\vartheta\vartheta} = R^2 \ , \ \lim_{r\to\infty} g_{\varphi\varphi} = R^2 \sin^2\theta \ . \tag{41}$$

Considering Eq. (41) and the fact that the Ricci tensor equals zero ($R_{\mu\nu} = 0$), for all four analysed metrics, and that their Riemann tensor asymptotically vanishes (see Appendix 3), all the metrics are asymptotically flat. Furthermore, they can be uniquely transformed into a system defined by proper coordinates ($cT, \ R, \ \theta, \ \phi$), which is globally flat and corresponds to Minkowski space:

$$\mathrm{d}s^2 = -c^2\mathrm{d}T^2 + \mathrm{d}R^2 + R^2\mathrm{d}\theta^2 + R^2 \sin^2\theta \, \mathrm{d}\phi^2 \ . \tag{42}$$

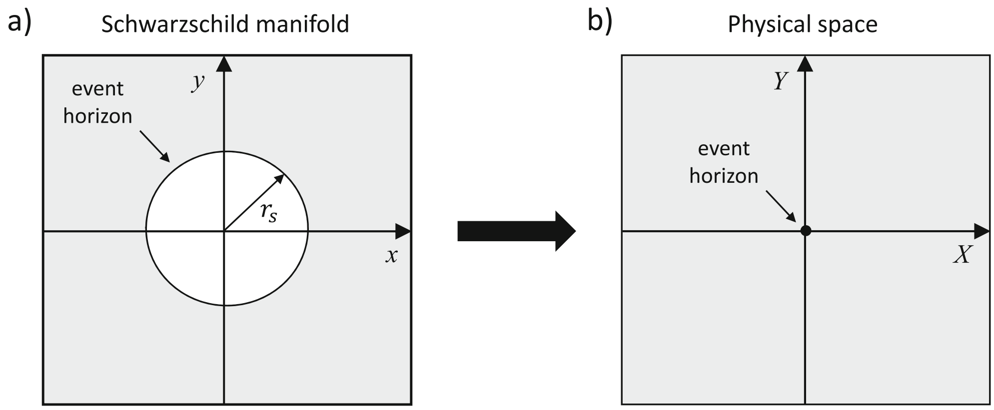

Fig. 3 Projection of the manifold covered by Schwarzschild coordinates (**a**) onto physical Euclidean space (**b**). The sphere with the Schwarzschild radius r_s in the Schwarzschild manifold is projected to the origin of the Cartesian coordinates in Euclidean space. The axes x and y represent the Schwarzschild coordinates, while X and Y represent the physical Cartesian coordinates. The white area in (**a**) is not projected onto the physical Euclidean space (**b**). Consequently, there is no need to cover this region using other coordinate systems with some prescribed mathematical properties

2.8 Equivalence of different metrics and Birkhoff's theorem

We showed in the previous section that all four analysed metrics equivalently describe the static gravitational field of a point mass:

- They are spherically symmetric, meaning that their metric tensor components g_{tt}, g_{rr}, and $g_{\omega\omega}$ depend only on the radial coordinate distance r, where $r = r(R)$.
- They satisfy Einstein's field equations (vacuum solution).
- They are static and asymptotically flat.
- They can equivalently be used to calculate the physical speed of light as a function of the physical distance R in a static frame.

This demonstrates that Schwarzschild metric is not exceptional or unique for describing the Schwarzschild solution. Hence, there are no mathematical or physical reasons to prefer this metric over others.

This statement appears to contradict Birkhoff's theorem [35], which asserts: *Let the geometry of a given region of spacetime (1) be spherically symmetric, and (2) be a solution to the Einstein field equations in vacuum. Then that geometry is necessarily a piece of the Schwarzschild geometry* [7, §32.2].

The key difficulty in deriving Birkhoff's theorem lies in the assumption that the metric tensor component $g_{\omega\omega}$ of a spherically symmetric metric can be chosen, without loss of generality, in the form $g_{\omega\omega} = r^2$. Once this assumption is made, the metric is uniquely defined and takes the form of the Schwarzschild metric. However, this assumption lacks justification, and there is no reason to restrict the metric in this way a priori. Therefore, it is incorrect to argue that the Schwarzschild metric is the unique description of the Schwarzschild solution. Other metrics, such as Brillouin's, isotropic or harmonic coordinates, are also suitable for solving this problem.

2.9 Uniqueness of the physical solution

Uniqueness is an essential property of any physical solution, including the gravitational field of a point source. A physically meaningful theory must yield a unique solution to a given problem; otherwise, it lacks utility. The fact that the Schwarzschild metric does not uniquely describe the Schwarzschild solution suggests that none of the derived metrics, including the Schwarzschild metric, represent the final physical solution to the problem. Instead, these metrics are merely tools for finding a unique physical solution. The final, unique physical solution can be obtained by evaluating physical quantities using the orthonormal tetrad of basis vectors defined within each metric. As we demonstrate numerically in the next section, this approach yields consistent and unique results.

Therefore, the event horizon at $r = r_s$, which characterizes the Schwarzschild solution, cannot represent a physical phenomenon. This conclusion applies equally to isotropic and harmonic coordinates, which exhibit analogous singularities at $r = \frac{1}{4}r_s$ and $r = \frac{1}{2}r_s$, respectively. In contrast, Brillouin's coordinates do not exhibit an event horizon at all. Evidently, the radius of the event horizon varies depending on the chosen coordinate system. Since this feature is not coordinate-independent, it cannot be a property of a true, unique physical solution.

2.10 Maximal extension of the Schwarzschild solution for $r < r_s$

Equation (31) indicates that the spatial Schwarzschild coordinates (r, ϑ, φ) with $r > r_s$ uniquely cover all points in the Euclidean space $\mathbf{R}^3$ except for the origin of coordinates. This Euclidean space can be parametrized, for example, by spherical coordinates (R, θ, ϕ), where the physical distance R is positive, $R > 0$. Therefore, the mapping between these two spaces is diffeomorphic, and the

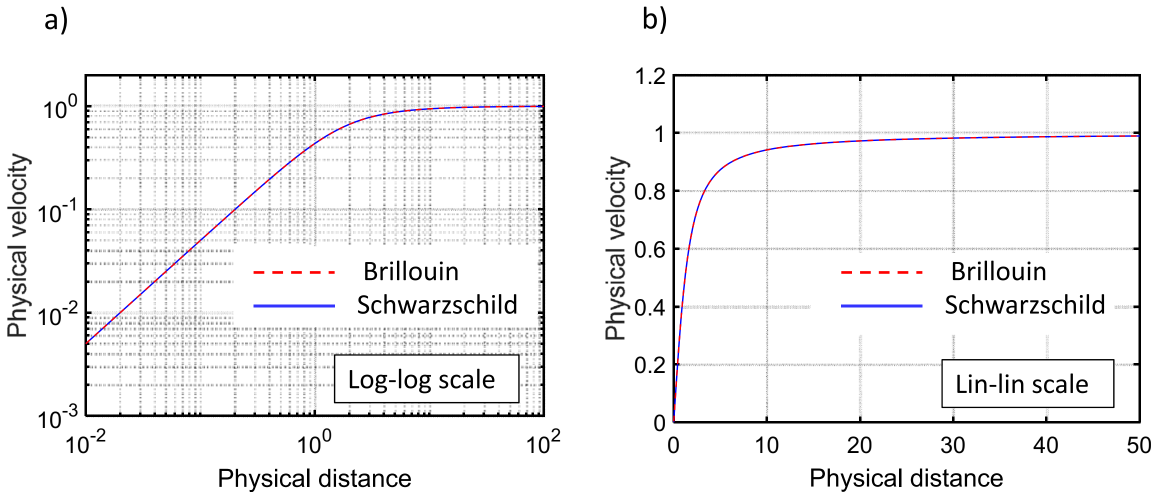

Fig. 4 The physical speed of light c_g as a function of the physical distance R from the observation point to the point mass. **a** Logarithmic scale, **b** linear scale. The Schwarzschild solution is represented by the solid blue line, and the Brillouin solution by the dashed red line. Note that the physical solutions in isotropic and harmonic coordinates are identical to the Schwarzschild and Brillouin solutions. The axes are in relative units: the speed of light c_g is normalized to c and the physical distance R is normalized to the Schwarzschild radius r_s

Schwarzschild manifold conforms to Euclidean geometry, except at the singularity at the origin of the coordinates ($R = 0$). Since the range of the Schwarzschild coordinate distance r ($r \geq r_s$) is sufficient to cover all points in the physical Euclidean space, points within the Schwarzschild sphere with $r < r_s$ do not correspond to any physical solution.

This finding contrasts with the standard concept of black holes, where the Schwarzschild coordinate distance r is assumed to range from zero to infinity, $0 \leq r < \infty$, for a complete physical description of the Schwarzschild black hole solution. This misconception leads to confusion regarding the physical interpretation of the Schwarzschild solution for r in the interval $0 \leq r \leq r_s$. It also prompts speculation about the existence and visibility of the trapped surfaces at $r \leq r_s$, as considered by many authors [7, 26, 29, 30, 36, 37].

Figure 3 schematically illustrates the correspondence between the Schwarzschild manifold, covered by Schwarzschild coordinates, and the Euclidean space, covered by Cartesian coordinates. The white area in Fig. 3a ($r < r_s$) does not map onto the physical Euclidean space shown in Fig. 3b. Therefore, there is no need to extend the Schwarzschild solution using alternative coordinate systems, such as Eddington–Finkelstein or Kruskal–Szekeres coordinates, which are designed to remove pathological properties of the Schwarzschild metric in the region defined by $r \leq r_s$ [7, 11, 12, 20, 23].

Although Kruskal–Szekeres coordinates provide a complete, global, and maximal extension of Schwarzschild spacetime, their application is unnecessary because they include regions of Schwarzschild spacetime that have no physical meaning. This conclusion also applies to other exotic spacetime geometries constructed in the unphysical region with $r < r_s$, such as the Einstein–Rosen bridge [38–40], Schwarzschild wormholes [26, 41], and trapped surfaces [7, 26, 29, 30, 36, 37]. All rays in the physical space are fully described using the standard Schwarzschild coordinates with $r \geq r_s$. This supports the conclusion that the Schwarzschild distance r is not a physical distance [25], and the Schwarzschild radius r_s does not represent a real physical quantity.

The same principle applies to isotropic and harmonic coordinates, which are defined for $r \geq \frac{1}{4}r_s$ and $r \geq \frac{1}{2}r_s$, respectively (see Eqs. (33) and (34)). Again, regions where $r < \frac{1}{4}r_s$ in isotropic coordinates or $r < \frac{1}{2}r_s$ in harmonic coordinates have no physical counterpart. Consequently, there is no need to extend these coordinates with other coordinate systems to cover these unmapped areas. Thus, Schwarzschild coordinates alone are fully sufficient for solving geodesics of massless or massive particles in the gravitational field of a central body. This conclusion also applies to the other mentioned alternative coordinate systems, such as Brillouin, isotropic, or harmonic coordinates.

3 Numerical example

3.1 Radial geodesics of photons

In this section, we numerically examine the properties of the radial geodesics of photons in Schwarzshild spacetime. Specifically, we investigate the physical speed of light measured by a static observer and expressed as a function of the physical distance from the point mass at rest. Since the Einstein field equations should yield a unique solution to this problem, we expect that the physical solutions derived using different coordinate systems will coincide.

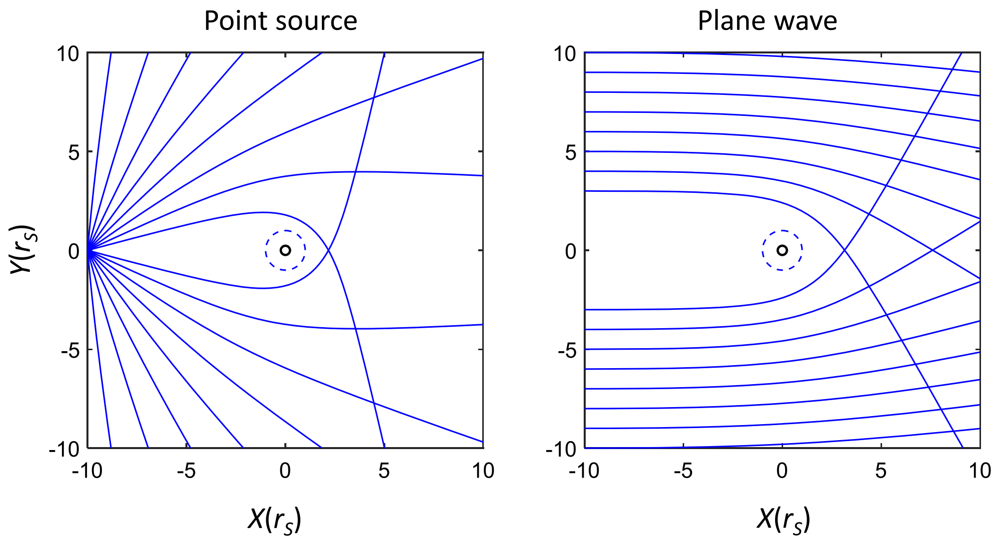

Fig. 5 Deflection of light rays in the vicinity of a black hole at distances larger than the Schwarzschild radius r_s. **a** Light is emitted by a point source, **b** the incident light wavefront is a plane wave. The ray fields are shown in Euclidean space, where the coordinates $X(r_s)$ and $Y(r_s)$ are normalized to the Schwarzschild radius r_s. The position of the black hole is indicated by a black open circle. In plot (**a**), the light source is located at a distance of 10 r_s from the black hole. The blue dashed circle around the black hole represents the Schwarzschild radius r_s

Figure 4 shows the physical radial speed of light c_g as a function of the physical radial distance R from the observation point to the central point mass. The figure demonstrates that all four coordinate systems (Schwarzschild, Brillouin, isotropic, and harmonic) yield the same dependence, $c_g = c_g(R)$. This confirms our expectation that the physical solution is unique and independent of the choice of coordinates. In this sense, all four mentioned metrics are equivalent.

Interestingly, the speed of light in the non-inertial static frame varies depending on the position of the observation point. The speed of light c_g is zero directly at the point mass but increases rapidly with distance. This increase of c_g slows down with greater distance, and ultimately c_g converges to c at a very large distance, which characterizes the speed of light in a region free of the gravitational field.

Since c_g is zero directly at the point mass (Fig. 4), no photon can be emitted from this point. This result aligns with the widely accepted notion that outgoing photons cannot cross the event horizon and escape the black hole. However, it is important to note that the event horizon is not a sphere in physical space; instead, it shrinks to a single point. This point coincides with the location of the point mass, where the mass density is infinite. In a more realistic scenario, where a black hole is characterized by a finite mass density, the speed of light c_g is nonzero even at the surface of the black hole. As a result, a true physical event horizon does not exists, and outgoing photons cannot be permanently trapped by the black hole. Photons outgoing from the vicinity of the black hole can travel to any point in outer space within a finite time.

Note that the concept of a varying speed of light does not contradict the basic principles of GR because our analysis focuses on non-inertial frames rather than free-falling frames. As mentioned earlier, these frames are not equivalent to inertial frames because the effects of gravitational acceleration are not cancelled in non-inertial frames. This influences the measured speed of light in such frames [32].

3.2 Deflection of light rays (ray bending)

Here, we examine the behaviour of non-radial rays near black holes. As in the previous section, we solve the geodesic equation for photons:

$$c_g^2 \mathrm{d}t^2 = g_{ii} \mathrm{d}x^i \mathrm{d}x^i = \mathrm{d}l^2 \,, \tag{43}$$

where dl is the physical distance element in 3D Euclidean space, and the Einstein summation convention over the index $i = 1, 2, 3$ is applied. Consequently,

$$c_g^2 \frac{\mathrm{d}t^2}{\mathrm{d}l^2} = c_g^2 p_{(i)} p_{(i)} = 1 \,, \tag{44}$$

where $p_{(i)}$ is the i-th physical component of the slowness vector $\mathbf{p} = \mathrm{d}t/\mathrm{d}\mathbf{x}$. Defining the Hamiltonian in the form

$$H = \frac{1}{2}\left(c_g^2 p_{(i)} p_{(i)} - 1\right) = 0 \,, \tag{45}$$

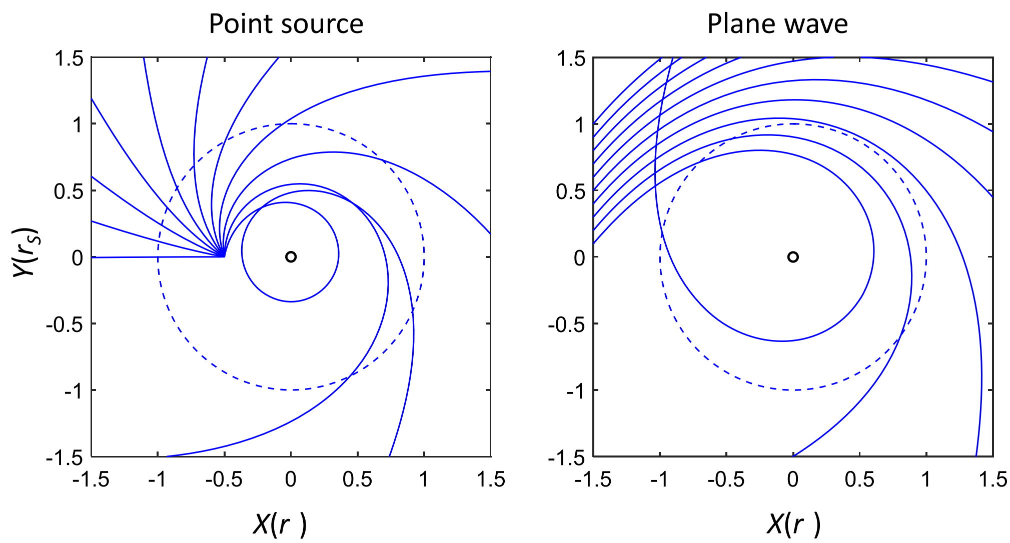

Fig. 6 Deflection of light rays in the vicinity of a black hole at distances smaller than the Schwarzschild radius r_s. **a** Light is emitted from a point source, **b** the incident light wavefront is locally a plane wave. In plot (**a**), the light source is located at a distance of 0.5 r_s from the black hole, and only rays emitted within a limited range of take-off angles are shown: $-90° \leq \theta \leq 10°$ with a step of 10°. For other details, see the caption of Fig. 5. The figure illustrates that no barrier exists to prevent light rays from escaping the vicinity of the black hole

and using Hamilton's equations with the travel time τ as the affine parameter, see [31, his Appendix 7] or [22, his chapter 11]

$$\frac{\mathrm{d}x^{\mu}}{d\tau} = \frac{\partial H}{\partial p_{\mu}}, \quad \frac{\mathrm{d}p_{\mu}}{\mathrm{d}\tau} = -\frac{\partial H}{\partial x^{\mu}}, \tag{46}$$

we obtain the ray-tracing equations:

$$\frac{\mathrm{d}x_{(i)}}{\mathrm{d}\tau} = c_g^2 p_{(i)}, \quad \frac{\mathrm{d}p_{(i)}}{\mathrm{d}\tau} = -\frac{1}{c_g}\frac{\partial c_g}{\partial x_{(i)}}, \quad \text{where } i = 1, 2, 3. \tag{47}$$

These ray-tracing equations are used to model the geometry of light rays in the gravitational field near a black hole. The equations are expressed using the physical components of the position and slowness vectors **x** and **p**, allowing us to calculate and visualize the ray geometry in Euclidean space. The physical speed of light c_g is calculated using Eq. (31) derived for the Schwarzschild metric. However, identical results are obtained when using Eqs. (32–34) for other metrics.

Figure 5 shows the deflection of light rays at distances larger than the Schwarzschild radius r_s. The initial wavefront is either spherical, emitted by a point source (Fig. 5a), or locally planar (Fig. 5b). The figure illustrates how the ray deflection from straight-line trajectories depends on the distance from the black hole. The strongest ray bending occurs for rays close to the black hole, where the speed of light c_g changes more rapidly.

The anomalous behaviour of light rays distorted by the gravitational field becomes even more pronounced in Fig. 6, which shows rays at distances comparable to or smaller than the Schwarzschild radius r_s. The figure clearly demonstrates that no barriers exist to prevent light rays from escaping the black hole. The closer the rays are to the black hole, the more complex their geometry becomes. Photons can even orbit the black hole once or several times before eventually leaving its vicinity.

4 Discussion

In general relativity, confusion often arises in distinguishing between formal parameters that describe the geometry of Riemannian manifolds representing gravitational fields and quantities with physical meaning. Riemannian manifolds can be described by an infinite number of coordinate systems, with metric tensors satisfying the Einstein field equations. However, selecting a suitable coordinate system and metric is not sufficient. The goal is to solve the geodesic equation and express its solution in physical, coordinate-independent quantities. Unlike the ambiguity associated with coordinate choices, the physical solution must be unique and unambiguous.

This concept is illustrated by solving the radial geodesics for photons using four different metrics that describe the Schwarzschild spacetime. The spacetime is Ricci-flat and topologically equivalent to Euclidean space, except at the origin of the coordinates. The physical speed of light c_g is expressed as a function of the physical distance R from the point mass, $c_g = c_g(R)$. It is shown that this dependence is unique, and all the coordinate systems under study yield the same solution. This conclusion also holds when tracing non-radial rays near a black hole. Numerical modelling demonstrates that no barriers exist to prevent light rays from escaping the vicinity of the black hole.

The physical speed of light c_g in the Schwarzschild black hole solution, as measured in a non-inertial static frame, varies depending on the position of the observation point. The speed c_g is zero directly at the point mass but increases rapidly with distance, and eventually converges to c, which represents the physical speed of light in a gravity-free region. Since c_g is zero at the point mass (Fig. 4), photons cannot be emitted from this point. This behaviour is a consequence of the infinite mass density associated with the point mass approximation. In contrast, for real black holes with a finite mass density, the speed of light c_g remains nonzero even at their surface, allowing photons to be freely emitted from their surface into outer space.

Importantly, the constant proper speed of light c in free-falling frames and the varying physical speed of light c_g in non-inertial frames are not contradictory. As emphasized by Einstein [32], gravitational effects influence c_g in non-inertial frames, unlike in free-falling frames. Both speeds are perfectly physical, measurable by clocks and rigid rods, and independent of the choice of coordinates within their respective frames.

It is essential to recognize that specific properties of metrics associated with individual coordinate systems cannot be directly interpreted in physical terms. For example, the pathological behaviour of the Schwarzschild metric at the Schwarzschild radius ($r = r_s$), where g_{tt} becomes zero, must be understood as a coordinate artefact. This artefact is eliminated when the Schwarzschild solution is expressed using the physical distance R instead of the Schwarzschild coordinate r. Importantly, physical space corresponds only to a subregion of Schwarzschild spacetime, defined by r in the range $r_s \le r < \infty$. The event horizon at the Schwarzschild radius $r = r_s$ does not correspond to a surface in physical space but rather to the point in physical space, where the central mass (black hole) is located (Fig. 3). The origin of Schwarzschild spacetime ($r = 0$) is often assumed to be associated with the position of the black hole; however, such physical interpretation lacks a clear foundation.

Consequently, calculating geodesics for massive or massless particles using Kruskal–Szekeres coordinates may lack physical relevance inside the event horizon ($r < r_s$). Similarly, widely discussed concepts, such as the Einstein–Rosen bridge, Schwarzschild wormholes, and trapped surfaces, could potentially lead to misinterpretations, as they are associated with the region within $r < r_s$ in Schwarzschild spacetime, which may not correspond to a physically realizable domain. In physical space, it appears that no trapped surfaces exist to prevent photons from escaping the vicinity of the black hole. Photons outgoing from the vicinity of the black hole can travel to any point in outer space within a finite time.

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Appendix 1: A common misconception in the interpretation of the physical radius of the event horizon in the Schwarzschild metric

The proper size of the event horizon is often derived using the geometrical concept of nested spheres in Schwarzschild coordinates. The coordinate radius r of these spheres is determined from their proper circumference L [26, his fig. 5.5]

$$r = \frac{L}{2\pi}\,, \tag{48}$$

or from their proper area A [7, their eq. 23.9']

$$r = \sqrt{\frac{A}{4\pi}}\,, \tag{49}$$

because the circumference and the area of the sphere is calculated as ([26, his exercise 5.4]; [7, their eq. 23.9])

$$l = \int r\mathrm{d}\vartheta = 2\pi r\,,\ a = \int r^2 \sin\vartheta\, \mathrm{d}\vartheta\, \mathrm{d}\varphi = 4\pi r^2\,, \tag{50}$$

where l and a are the circumference and area of a two-dimensional sphere, and L and A are rescaled to proper length units. Writing r_s instead of r in Eqs. (48) and (49), the proper circumference and the proper area of the event horizon sphere are defined as [26, his fig. 5.5]

$$L_s = 2\pi r_s\,,\ A_s = 4\pi r_s^2\,. \tag{51}$$

In this way, it is deduced that the event horizon has a finite size because its physical circumference and area, calculated by Eq. (51), are nonzero.

However, this derivation is misleading because it incorrectly mixes coordinate and physical quantities. If we wish to substitute coordinate quantities with physical quantities, it must be done rigorously, rather than by simple rescaling. The correct procedure is as follows.

Let us consider a Riemannian manifold covered by spatial Schwarzschild coordinates (r, ϑ, φ) with distances $r > r_s$, and Euclidean space $\mathbf{R}^3$ parametrized by spherical coordinates (R, θ, ϕ), where $R > 0$. Equation (31) ensures that the mapping between these two spaces is diffeomorphic, and the Riemannian manifold conforms to Euclidean geometry. Let us define a transformation between Cartesian coordinates (X, Y, Z) and spherical coordinates (R, θ, ϕ) as

$$X = R(r)\sin\theta\cos\phi\,, Y = R(r)\sin\theta\sin\phi\,, Z = R(r)\cos\theta\,, \tag{52}$$

where $R = R(r)$ is the proper radius (measured in metres) as a function of the Schwarzschild coordinate distance r, and angles θ and ϕ are standard spherical angles in Euclidean space (measured in degrees). Time transformations are omitted here since we focus only on spatial properties of the Schwarzschild manifold. The line element in spherical coordinates is given by

$$\mathrm{d}s^2 = \mathrm{d}R^2 + R^2\mathrm{d}\theta^2 + R^2\sin^2\theta\mathrm{d}\phi^2 = \left(\frac{\mathrm{d}R}{\mathrm{d}r}\right)^2\mathrm{d}r^2 + R^2\mathrm{d}\Omega^2\,. \tag{53}$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$.

The Schwarzschild metric is (see Eq. (2))

$$\mathrm{d}s^2 = g_{rr}\mathrm{d}r^2 + r^2\mathrm{d}\vartheta^2 + r^2\sin^2\vartheta\mathrm{d}\varphi^2 = g_{rr}\mathrm{d}r^2 + r^2\mathrm{d}\omega^2\,, \tag{54}$$

where $g_{rr} = \left(1 - \frac{r_s}{r}\right)^{-1}$ and $\mathrm{d}\omega^2 = \mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2$.

Comparing Eq. (53) with Eq. (54), it is evident that the Schwarzschild angles ϑ and φ differ from the spherical angles θ and ϕ, because $g_{\theta\theta} \neq g_{\vartheta\vartheta}$ and $g_{\phi\phi} \neq g_{\varphi\varphi}$. Specifically, angles ϑ and φ are not measured in degrees and do not span the usual ranges: $0 \leq \vartheta \leq \pi$ and $0 \leq \varphi \leq 2\pi$. Their relationship to the physical spherical angles is (see Eq. (22))

$$\mathrm{d}\vartheta = \frac{R}{r}\,\mathrm{d}\theta\,,\ \mathrm{d}\varphi = \frac{R\sin\theta}{r\sin\vartheta}\,\mathrm{d}\phi\,. \tag{55}$$

Consequently, the integrals in Eq. (50) must be corrected using Eq. (55)

$$l = \int_0^{2\pi R/r} r\mathrm{d}\varphi = R\int_0^{2\pi}\mathrm{d}\phi = 2\pi R\,, \tag{56}$$

$$a = \int_0^{\pi R/r}\int_0^{2\pi R/r} r^2\sin\vartheta\,\mathrm{d}\vartheta\mathrm{d}\varphi = 4\pi R^2\,. \tag{57}$$

Hence, Eq. (51) must be modified to

$$L_s = 2\pi R_s\,,\ A_s = 4\pi R_s^2\,. \tag{58}$$

Considering that the proper radius $R_s = R(r_s)$, calculated from Eq. (31), is zero, $R_s = 0$, the proper circumference and the proper area of the event horizon are also zero: $L_s = 0$ and $A_s = 0$. This confirms that the event horizon is indeed shrunk to a single point in proper coordinates.

In summary, the misinterpretation of the physical size of the event horizon sphere arises from the incorrect assumption that the Schwarzschild angles ϑ and φ are measured in degrees and behave like the standard spherical angles θ and ϕ, for which Eq. (50) applies. Similarly, the Schwarzschild solid angle ω in Eq. (5) is often assumed to be the standard solid angle Ω in spherical coordinates. However, ϑ and φ are formal parameters, not spherical angles. Consequently, their integrals $\int \mathrm{d}\vartheta$ or $\int \sin\vartheta\,\mathrm{d}\vartheta\mathrm{d}\varphi$ are not simply 2π or 4π as in standard spherical geometry but require rescaling.

Appendix 2: Closed-form solutions for the physical distance *R*

Integrals for proper distance R in Eqs. (31)–(34) can be expressed in closed form as follows:

- Schwarzschild coordinates (see Fig. 1a)

$$R = r\sqrt{1 - \frac{r_s}{r}} + \frac{1}{2}r_s\ln\left(2\frac{r}{r_s}\sqrt{1 - \frac{r_s}{r}} + 2\frac{r}{r_s} - 1\right),\ r \geq r_s\,, \tag{59}$$

- Brillouin coordinates (see Fig. 2a)

$$R = r\sqrt{1 + \frac{r_s}{r}} + \frac{1}{2}r_s\ln\left(2\frac{r}{r_s}\sqrt{1 + \frac{r_s}{r}} + 2\frac{r}{r_s} + 1\right),\ r \geq 0\,, \tag{60}$$

- isotropic coordinates

$$R = r - \frac{1}{16}\frac{r_s^2}{r} + \frac{1}{2} r_s \ln\left(4\frac{r}{r_s}\right), \; r \geq \frac{r_s}{4}, \tag{61}$$

- harmonic coordinates

$$R = r\sqrt{1 - \frac{r_s^2}{4r^2}} + \frac{1}{2} r_s \ln\left(2\frac{r}{r_s}\sqrt{1 - \frac{r_s^2}{4r^2}} + 2\frac{r}{r_s}\right), \; r \geq \frac{r_s}{2}. \tag{62}$$

Appendix 3: Asymptotic behaviour of nonzero components of the Riemann curvature tensor

The leading terms of the nonzero components of the Riemann curvature tensor for the Schwarzschild, Brillouin, isotropic and harmonic coordinates are equally expressed as follows:

$$R^t_{rtr} = -R^t_{rrt} = \frac{r_s}{r^3}, \tag{63}$$

$$R^t_{\vartheta t \vartheta} = -R^t_{\vartheta\vartheta t} = -\frac{1}{2}\frac{r_s}{r}, \tag{64}$$

$$R^t_{\varphi t \varphi} = -R^t_{\varphi\varphi t} = -\frac{\sin^2\vartheta}{2}\frac{r_s}{r}, \tag{65}$$

$$R^r_{ttr} = -R^r_{trt} = \frac{r_s}{r^3}, \tag{66}$$

$$R^r_{\vartheta r \vartheta} = -R^r_{\vartheta\vartheta r} = -\frac{1}{2}\frac{r_s}{r}, \tag{67}$$

$$R^r_{\varphi r \varphi} = -R^r_{\varphi\varphi r} = -\frac{\sin^2\vartheta}{2}\frac{r_s}{r}, \tag{68}$$

$$R^\vartheta_{t\vartheta t} = -R^\vartheta_{tt\vartheta} = \frac{1}{2}\frac{r_s}{r^3}, \tag{69}$$

$$R^\vartheta_{r\vartheta r} = -R^\vartheta_{rr\vartheta} = -\frac{1}{2}\frac{r_s}{r^3}, \tag{70}$$

$$R^\vartheta_{\varphi\vartheta\varphi} = -R^\vartheta_{\varphi\varphi\vartheta} = \sin^2\vartheta\,\frac{r_s}{r}, \tag{71}$$

$$R^\varphi_{t\varphi t} = -R^\varphi_{tt\varphi} = \frac{1}{2}\frac{r_s}{r^3}, \tag{72}$$

$$R^\varphi_{r\varphi r} = -R^\varphi_{rr\varphi} = -\frac{1}{2}\frac{r_s}{r^3}, \tag{73}$$

$$R^\varphi_{\vartheta\varphi\vartheta} = -R^\varphi_{\vartheta\vartheta\varphi} = \frac{r_s}{r}. \tag{74}$$

All these components vanish for $r \to \infty$.

References

1. P. Fromholz, E. Poisson, C.M. Will, The Schwarzschild metric: it's the coordinates, stupid! Am. J. Phys. **82**(4), 295–300 (2014). https://doi.org/10.1119/1.4850396. arXiv:1308.0394 [gr-qc]
2. J. Droste, The field of a single centre in Einstein's theory of gravitation, and the motion of a particle in that field. Koninklijke Nederlandse Akademie van Wetenschappen Proc. Series B Phys. Sci. **19**, 197–215 (1917)
3. D. Hilbert, Die Grundlagen der Physik. Nachr. Ges. Wiss. Göttingen, Math. Phys. Kl., 53–76 (1917)
4. K. Schwarzschild, Über das Gravitationsfeld eines Massenpunktes nach der Einsteinschen Theorie. In: *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pp. 189–196 (1916)
5. K. Schwarzschild, Über das Gravitationsfeld einer Kugel aus inkompressibler Flüssigkeit nach der Einsteinschen Theorie. In: *Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften zu Berlin*, pp. 424–434 (1916)
6. L.D. Landau, E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon Press, New York, 1971)
7. C.W. Misner, K.S. Thorne, J.A. Wheeler, *Gravitation* (Princeton University Press, Princeton, 1973)
8. S. Weinberg, *Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity* (Wiley, New York, 1972)
9. R.M. Wald, *General Relativity* (The University of Chicago Press, Chicago, 1984)
10. W. Rindler, Visual horizons in world models. MNRAS **116**, 662 (1956). https://doi.org/10.1093/mnras/116.6.662
11. D. Finkelstein, Past-future asymmetry of the gravitational field of a point particle. Phys. Rev. **110**(4), 965–967 (1958). https://doi.org/10.1103/PhysRev.110.965
12. S.W. Hawking, G.F.R. Ellis, *The Large-Scale Structure of Space-Time* (Cambridge University Press, Cambridge, 1973)
13. P. Painlevé, La mécanique classique et la théorie de la relativité. Comptes Rendus Academie des Sciences (serie non specifiee) **173**, 677–680 (1921)
14. S.J. Crothers, On the general solution to Einstein's vacuum field and its implications for relativistic degeneracy. Prog. Phys. **1**(1), 68–73 (2005)

15. S.J. Crothers, On the generation of equivalent 'black hole' metrics: a review. Space Sci. Int. **3**(2), 28–44 (2015). https://doi.org/10.3844/ajssp.2015.28.44
16. M. Brilloin, The singular points of Einstein's Universe. Le Journal de Physique et Le Radium **23**, 43 (1923)
17. L.S. Abrams, Black holes. The legacy of Hilbert's error. Can. J. Phys. **67**(9), 919–926 (1989). https://doi.org/10.1139/p89-158
18. C. Corda, A clarification on the debate on "the original Schwarzschild solution". arXiv e-prints, 1010–6031 (2010) https://doi.org/10.48550/arXiv.1010.6031arXiv:1010.6031 [gr-qc]
19. R.J. Cook, Physical time and physical space in general relativity. Am. J. Phys. **72**(2), 214–219 (2004). https://doi.org/10.1119/1.1607338
20. A.S. Eddington, *The Mathematical Theory of Relativity* (Cambridge University Press, Cambridge, 1923)
21. G. Lemaître, L'Univers en expansion. Annales de la Societe Scientifique de Bruxelles **53**, 51 (1933)
22. J.L. Synge, The gravitational field of a particle. In: *Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences*, (1950)
23. M.D. Kruskal, Maximal extension of Schwarzschild metric. Phys. Rev. **119**(5), 1743–1745 (1960). https://doi.org/10.1103/PhysRev.119.1743
24. W. Lin, J. Li, B. Yang, Coordinates in general relativity: orbit, velocity, and time form perihelion to aphelion. Gravit. Cosmol. **29**(1), 95–102 (2023). https://doi.org/10.1134/S0202289323010127
25. J.B. Hartle, *Gravity: An Introduction to Einstein's General Relativity* (Addison Wesley, San Francisco, 2003)
26. R.J.A. Lambourne, *Relativity Gravitation and Cosmology* (Cambridge University Press, Cambridge, 2010)
27. M.P. Hobson, G. Efstathiou, A.N. Lasenby, *General Relativity: An Introduction for Physicists* (Cambridge University Press, Cambridge, 2005)
28. S.M. Carroll, Spacetime and geometry. In: *An Introduction to General Relativity* (Addison Wesley, San Francisco, 2004)
29. R. Penrose, Gravitational collapse and space-time singularities. Phys. Rev. Lett. **14**(3), 57–59 (1965). https://doi.org/10.1103/PhysRevLett.14.57
30. R. Penrose, Gravitational collapse: the role of general relativity. Nuovo Cimento Rivista Serie **1**, 252 (1969)
31. C. Møller, *The Theory of Relativity* (Clarendon Press, Oxford, 1972)
32. A. Einstein, *Relativity, the Special and the General Theory*. p. 168, Henry Holt and Company, New York (1920). Authorized translation by Robert W. Lawson
33. R.V. Pound, G.A. Rebka, Apparent weight of photons. Phys. Rev. Lett. **4**(7), 337–341 (1960). https://doi.org/10.1103/PhysRevLett.4.337
34. R.V. Pound, J.L. Snider, Effect of gravity on nuclear resonance. Phys. Rev. Lett. **13**(18), 539–540 (1964). https://doi.org/10.1103/PhysRevLett.13.539
35. G.D. Birkhoff, *Relativity and Modern Physics* (Harvard University Press, Cambridge, 1923)
36. S.W. Hawking, R. Penrose, The singularities of gravitational collapse and cosmology. Proc. R. Soc. Lond. Ser. A **314**(1519), 529–548 (1970). https://doi.org/10.1098/rspa.1970.0021
37. S.W. Hawking, Black holes in general relativity. Commun. Math. Phys. **25**(2), 152–166 (1972). https://doi.org/10.1007/BF01877517
38. A. Einstein, N. Rosen, The particle problem in the general theory of relativity. Phys. Rev. **48**(1), 73–77 (1935). https://doi.org/10.1103/PhysRev.48.73
39. M. Visser, *Lorentzian Wormholes: From Einstein to Hawking Computational and Mathematical Physics* (Springer, New York, 1996)
40. N.J. Popławski, Radial motion into an Einstein–Rosen bridge. Phys. Lett. B **687**(2–3), 110–113 (2010). https://doi.org/10.1016/j.physletb.2010.03.029. arXiv:0902.1994 [gr-qc]
41. M.S. Morris, K.S. Thorne, Wormholes in spacetime and their use for interstellar travel: a tool for teaching general relativity. Am. J. Phys. **56**(5), 395–412 (1988). https://doi.org/10.1119/1.15620