Flip processes on graphs and dynamical systems they induce on graphons

Frederik Garbe
Masaryk University

Jan Hladký, Matas Šileikis
Czech Academy of Sciences

Fiona Skerman
Uppsala University

Erdos-Renyi random graph process

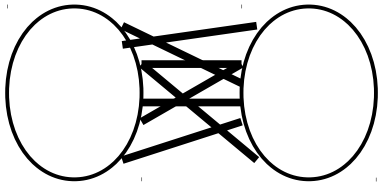
- G(n,p) *binomial Erdos-Renyi random graph*
  - n vertices, insert each potential edge with probability p
  - For this talk, $p \in (0,1)$ fixed
- G(n,m) uniform *Erdos-Renyi random graph*
  - Uniformly random graph with m edges.
  - For $m=pn^2/2$; $G(n,p) \approx G(n,m)$
- *Erdos-Renyi random graph process* (n vertices) $G_0, G_1, \ldots, G_{\binom{n}{2}}$
  - G_0 is edgeless, G_{r+1} is obtained from by turning a randomly selected nonedge into an edge
- With high probability, everything on this slide is *quasirandom*

Quasirandomness

- 1980's (Chung-Graham-Wilson, Szemeredi, ...)
- *Density* of a graph d=e(G)/$\binom{n}{2}$
- A graph is *ε-quasirandom* if for each set of vertices *U*

$$\left| e(G[U]) - d\binom{|U|}{2} \right| < \varepsilon n^2$$

- A nonquasirandom graph



Triangle removal process

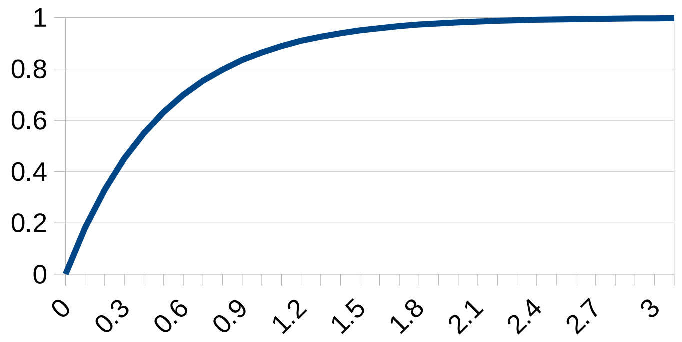
- Introduced by Bollobas-Erdos'90
- Start with G_0=clique
- In step r, pick a random triangle of G_r and delete it
- Bohman-Frieze-Lubetzky'15: *Triangle removal process typically terminates when there are $n^{3/2+o(1)}$ edges left.*
  - Key in the proof: quasirandomness during the evolution

Erdos-Renyi *flip* process

- Start with a graph G_0 (for now the edgeless graph)
- In each step, "replace" a uniformly chosen **pair** with an edge
- Density computation for G_r, $r=\alpha n^2$:

$$P[uv\ \text{is an edge}] = 1 - P[uv\ \text{is not an edge}]$$

$$\ldots = 1 - \left(1 - \frac{1}{\binom{n}{2}}\right)^r \approx 1 - \exp(-2r/n^2) = 1 - \exp(-2\alpha)$$



Erdos-Renyi 50:50 *flip* process

- Start with a graph G_0 on n vertices
- In each step, “replace” a uniformly chosen pair with an edge or an non-edge (50:50)
- “Converges to quasirandom graph of density 0.5”, after Cn^2 steps, $C \to \infty$

Triangle removal *flip* process

- Start with a graph G_0 (for now the complete graph)
- In each step r pick three random vertices u_1, u_2, u_3,
- If $G_r[u_1, u_2, u_3]$ induces a triangle then remove it…

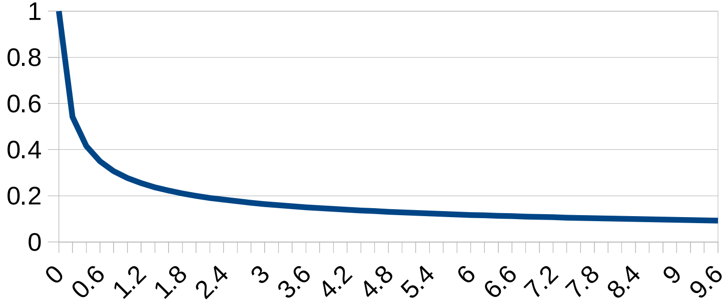
  ...otherwise $G_{r+1} := G_r$.
- Density computation: G_r, $r=\alpha n^2$, $e(\alpha):=e(G_r)$, $d(\alpha):=e(\alpha) / \binom{n}{2}$

$$P[u_1 u_2 u_3 \, is\, a\, triangle] \approx d(\alpha)^3$$

$$e(\alpha+\epsilon)-e(\alpha) \approx -3d(\alpha)^3 \cdot \epsilon n^2$$

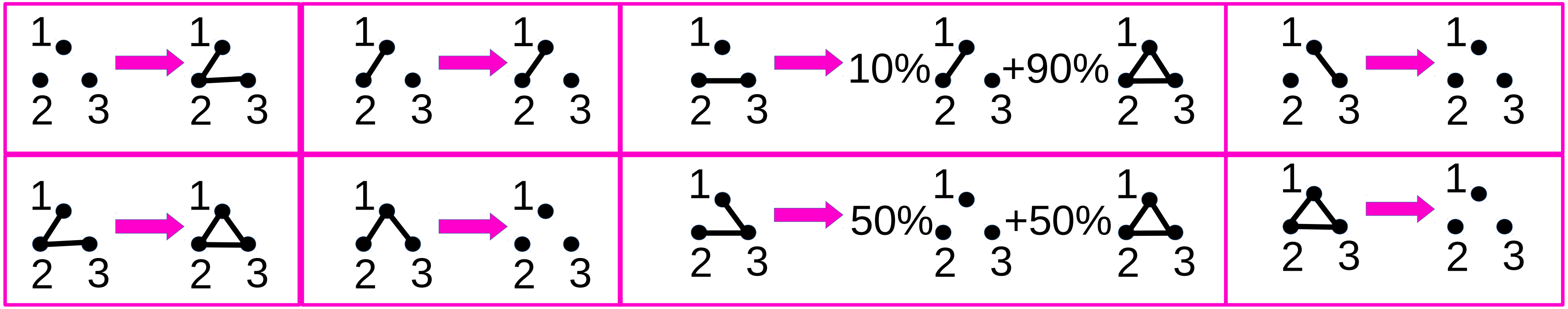
$$\frac{d(\alpha)}{d\alpha} = -6d(\alpha)^3 \quad \Rightarrow \quad d(\alpha) = \frac{1}{\sqrt{1+12\alpha}}$$

Separable first-order ODE



Flip process of order *k* (here, *k*=3)

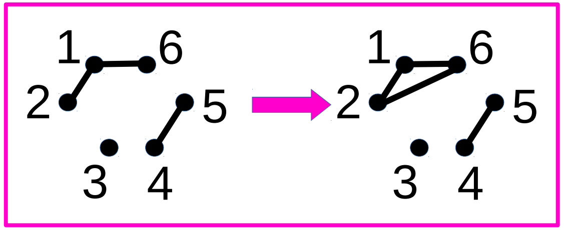
- Rule $\mathcal{R}$



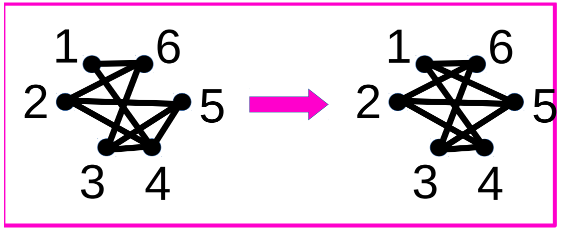
- Start with a (large) graph G_0
- Step $G_r \Rightarrow G_{r+1}$: Sample *k* vertices and replace the induced graph according to $\mathcal{R}$

More examples of flip processes

- Ignorant flip process
- Removal flip process
- Complement flip process
- Component completion flip process
- The stirring flip process
- The extremist flip process
- The polarizing flip process



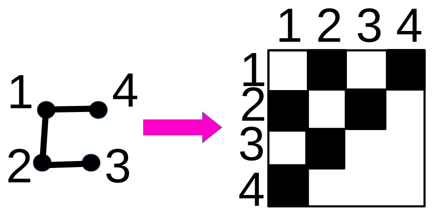
Component completion

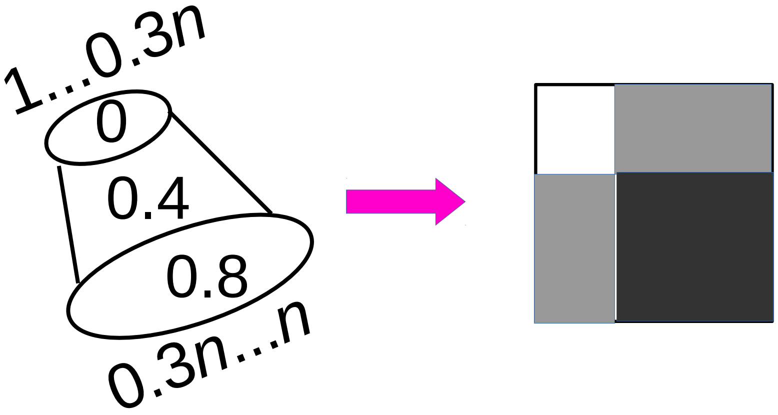


Polarizing

Graphons (limits of dense graphs)

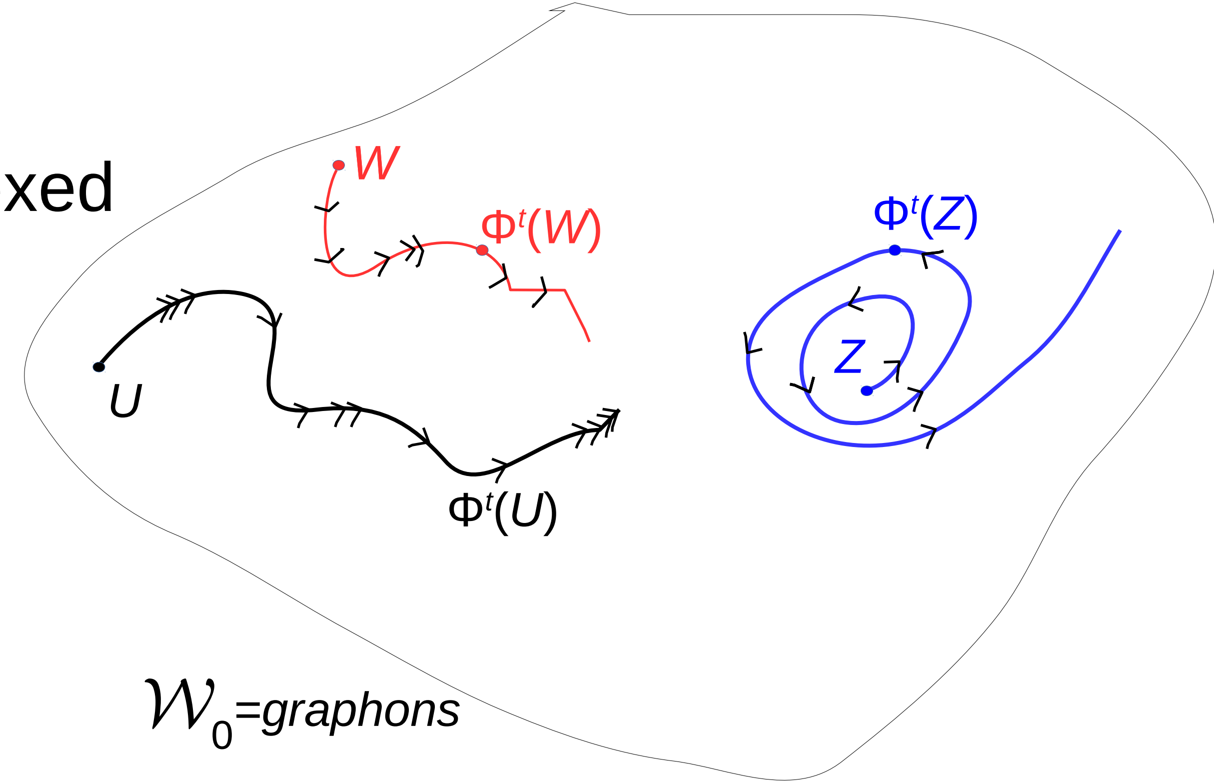
- Borgs-Chayes-Lovasz-Sos-Szegedy-Vesztergombi 2004
- Useful framework for extremal and probabilistic questions
- Graphon is a symmetric function W:$[0,1]^2 \to [0,1]$
- Cut norm measures how similar two graphons are





Trajectories

- Fixed rule $\mathcal{R}$ of order k
- We construct time-indexed trajectories $\Phi:\mathcal{W}_0\times[0,\infty)\rightarrow\mathcal{W}_0$
- Construction later



Transference theorem

Given $\mathcal{R}$ and corresponding

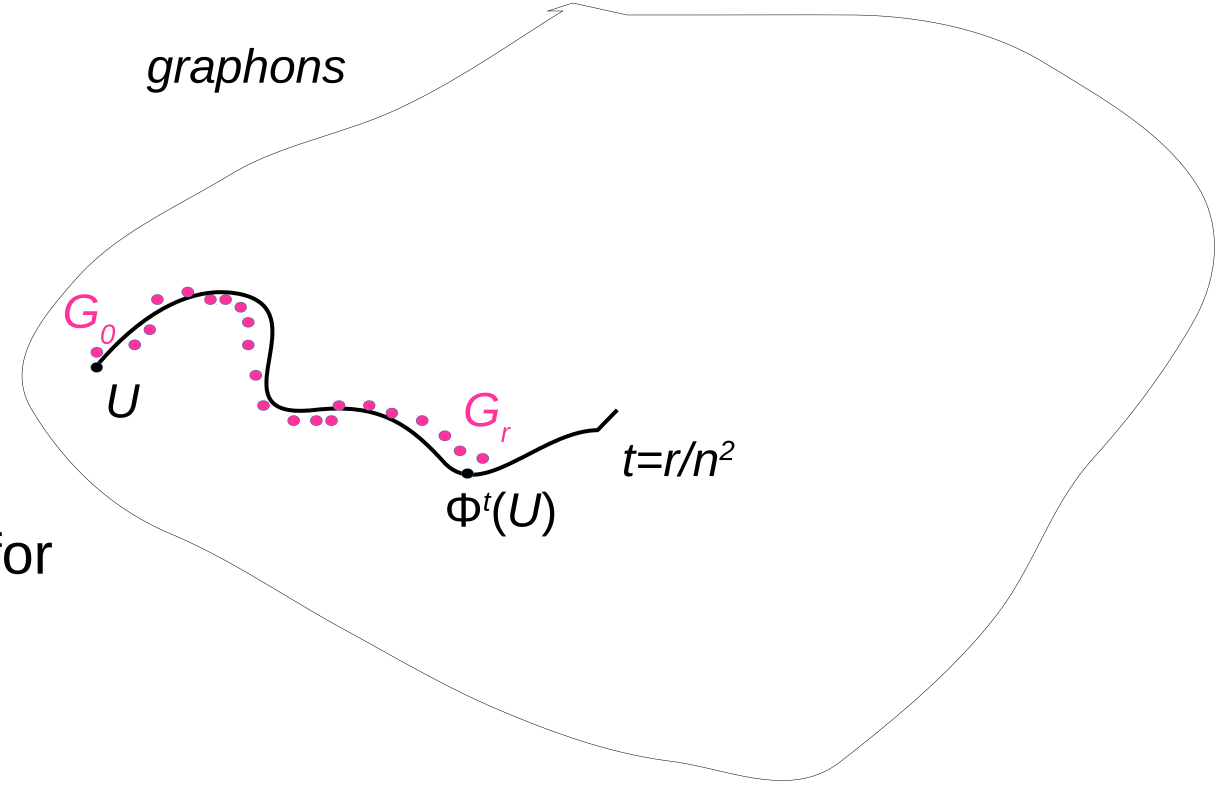
trajectories $\Phi:\mathcal{W}_0\times[0,\infty)\rightarrow\mathcal{W}_0$,

whenever a large *n*-vertex G_0

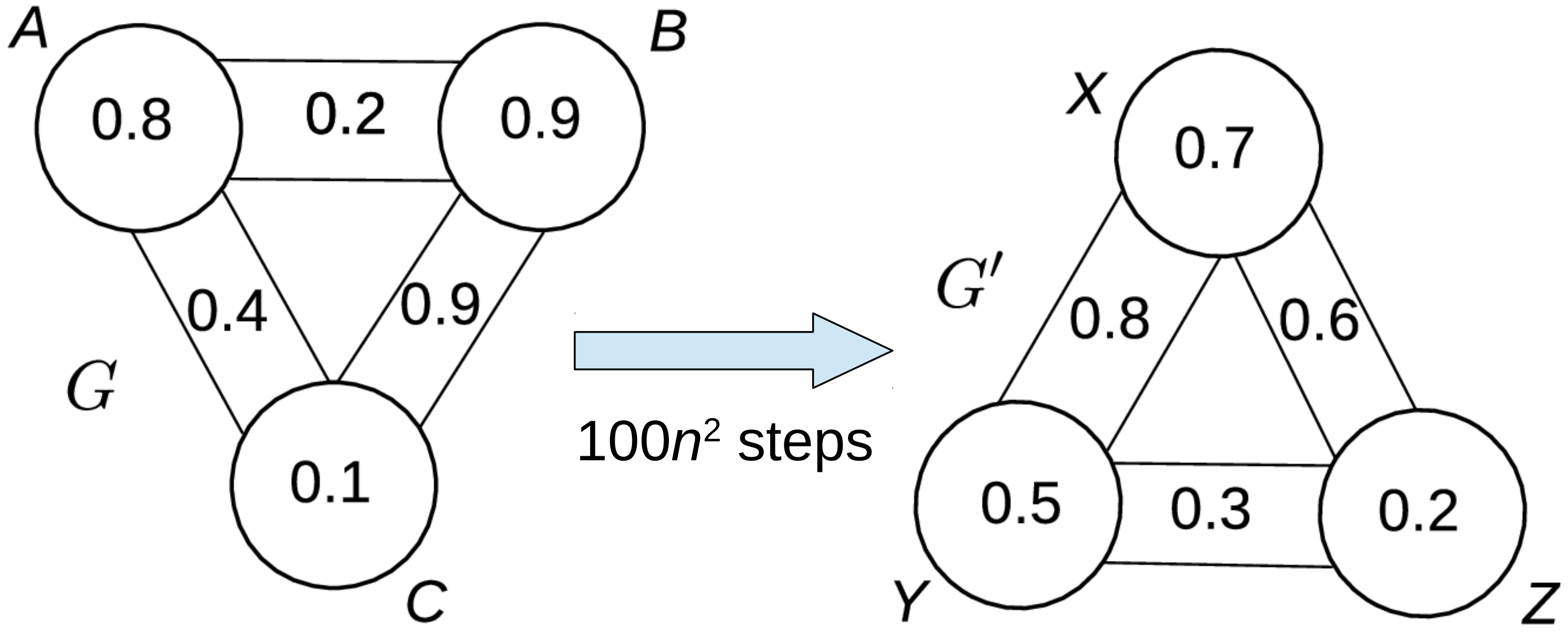
is close to *U* (in cut norm)

then w.h.p. G_r is close to $\Phi^t(U)$ for

$t:=r/n^2$



Cut norm, not cut distance



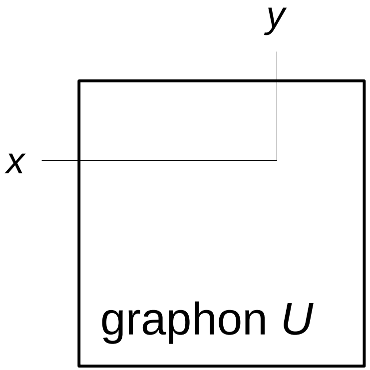
Constructing trajectories I

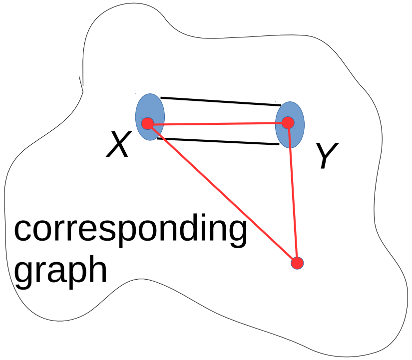
- In this example, consider the Triangle removal flip process
- $(\Phi^{\varepsilon}(U)-U)\,(x,y)$

  correspondence with a graph

  $|X|=|Y|=\gamma n$ and εn^2 steps

  $U(x,y)=e(X,Y)/(\gamma n)^2$





- Number of removed edges between X and Y in εn^2 steps:

$$\epsilon n^2 \cdot \gamma^2 \cdot t_{xy}^{\bullet\bullet}(K_{3,}U)$$

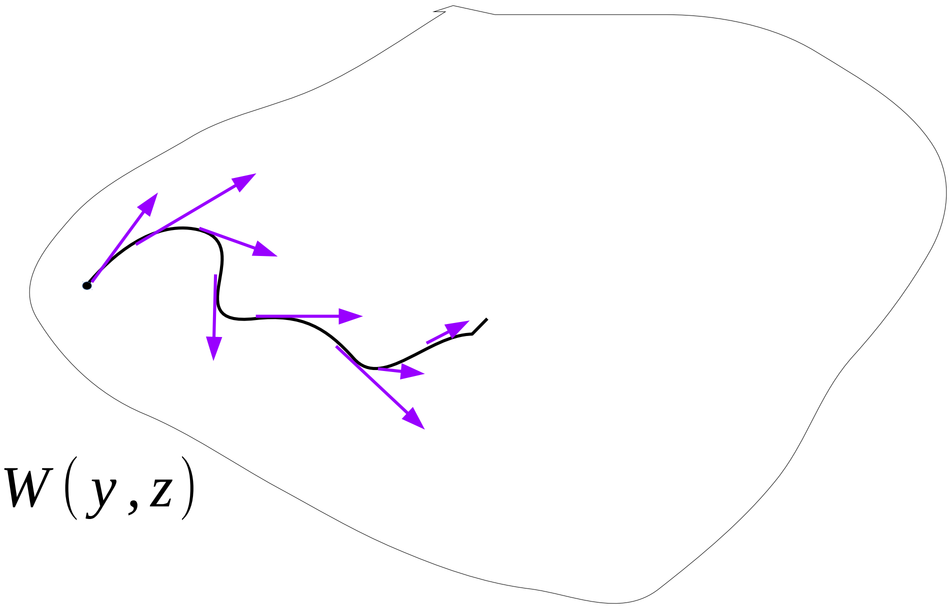
Density change at (x,y): $-\epsilon \cdot t_{xy}^{\bullet\bullet}(K_{3,}U)$

$$t_{xy}^{\bullet\bullet}(K_{3,}W)=\int_z W(x,y)W(x,z)W(y,z)$$

Constructing trajectories II

- Construct a velocity field $V: \mathcal{W}_0 \rightarrow \mathcal{W}$ (signed graphons)

$$V(W) = \lim_{\epsilon \to 0} \frac{\Phi^{\epsilon}(W) - W}{\epsilon}$$



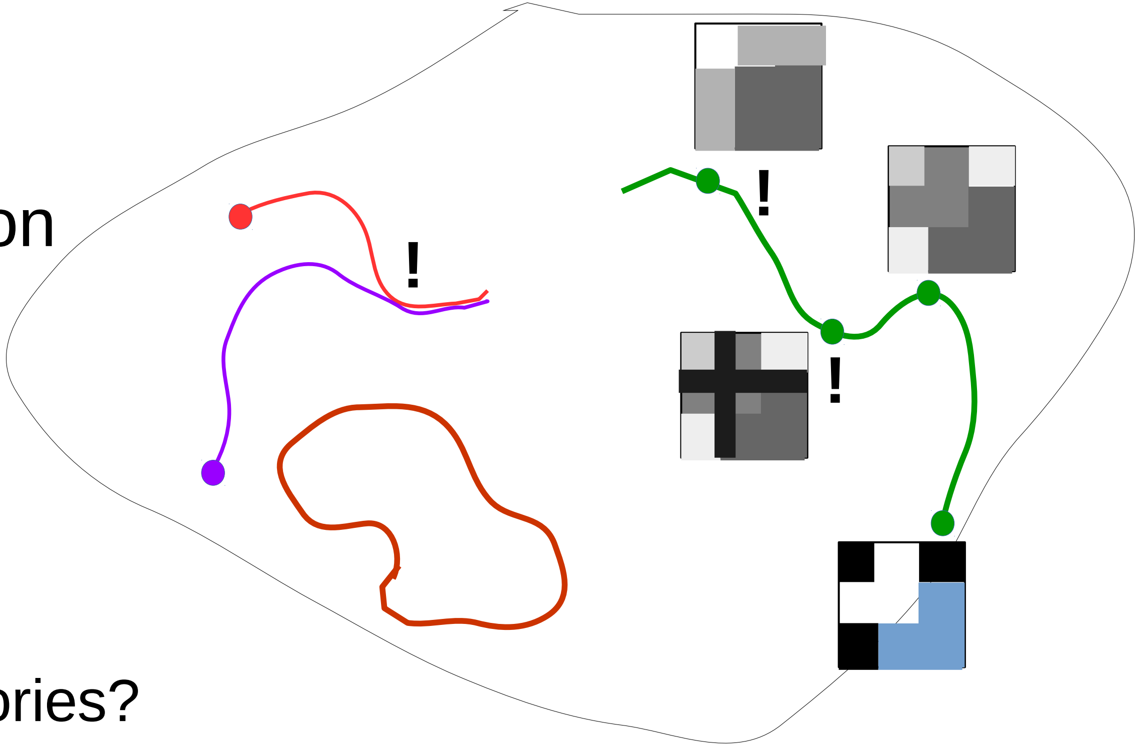
- Triangle removal flip process

$$V(W)(x,y) = -W(x,y) \int_z W(x,z) W(y,z)$$

- Velocity is continuous in L^{∞} and cut norm

What is all this good for ?

- No confluences
- Going back in time
- Block structure preservation
- Limits $t \to \infty$:
  - Stable and unstable fixed points (often constants)
  - Periodic trajectory
  - Really complicated trajectories?
- Speed of convergence

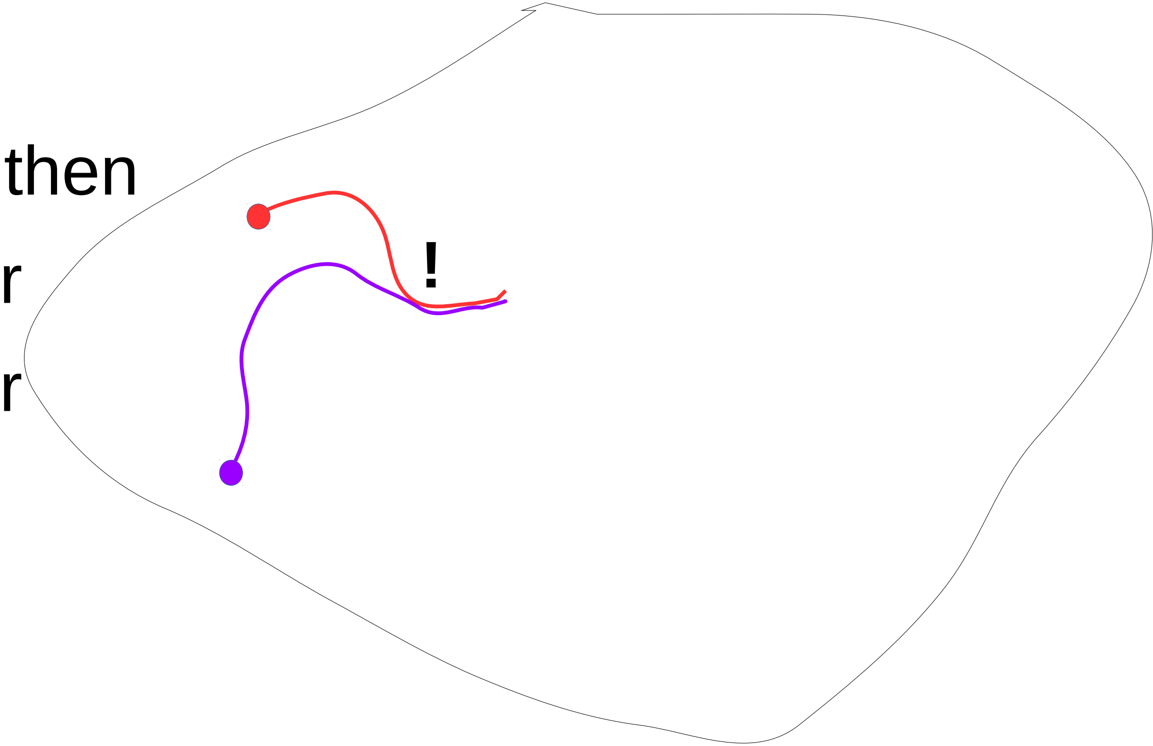


No confluences

Theorem: Fix a rule $\mathcal{R}$.

If X and Y are graphons, then

- $\Phi^t(X)=Y$ for some $t \geq 0$, or
- $\Phi^t(Y)=X$ for some $t \geq 0$, or
- The trajectories of X and Y are disjoint



No confluences, proof

We want to prove that if $X \neq Y$ then

for each $t>0$, $\Phi^t(Y) \neq \Phi^t(X)$.

- Introduce

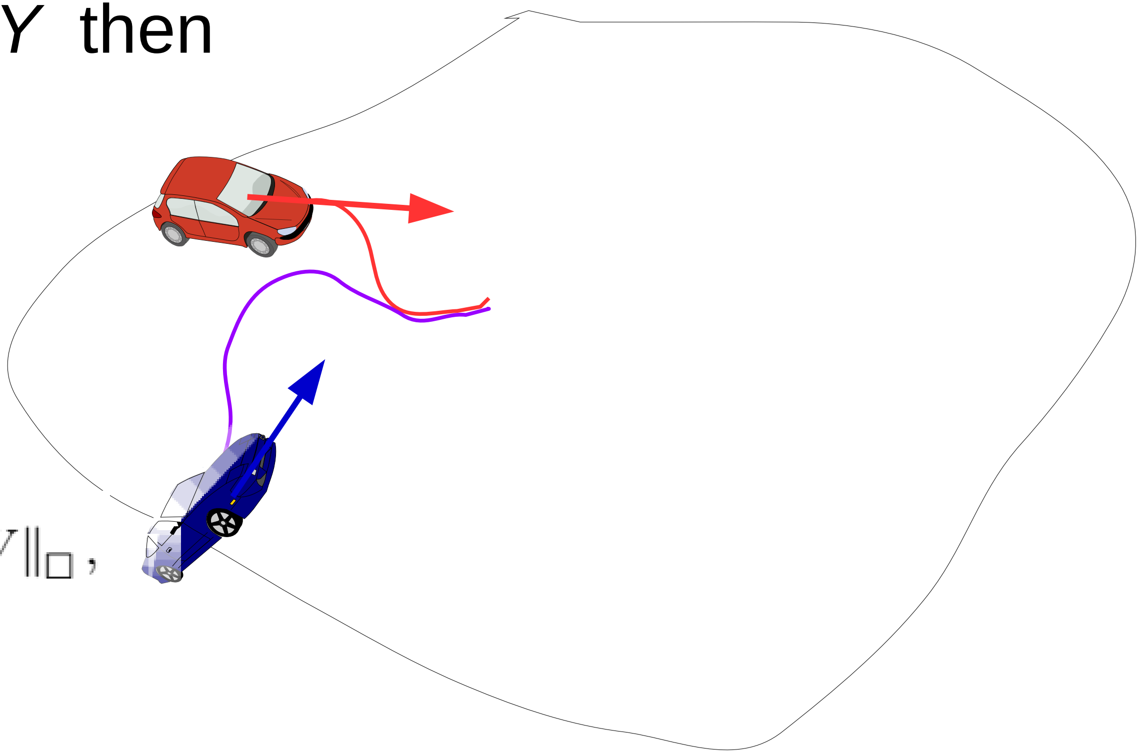
$$h(t) := \|\Phi^t(Y) - \Phi^t(X)\|_{\square}$$

- Prove

$$\|V_{\mathcal{R}}(U) - V_{\mathcal{R}}(W)\|_{\square} \leq C_k \|U - W\|_{\square},$$

- Leading to

$$\frac{d}{dt} h(t) \geq -C_k h(t)$$ and hence $$h(t) \geq \exp(-C_k t) h(0)$$



Behaviour of indivdual trajectories???

Fix a rule $\mathcal{R}$. X is a graphon.

- “Typically”, trajectory ($\Phi^t(X)$: t) converges to a graphon
- We have an example of a periodic nonconstant trajectory,

  $$\Phi^t(X)=\Phi^{t+7}(X)$$

- What else? Does there exist a really complicated trajectory?

  $$T:=\{\Phi^t(X): t\in[0,+\infty)\}$$

  Can the set T be totally unbounded (non-compact, after closeru)? (with respect to cut-norm)