Tilings in graphons

Jan Hladký
Mathematics Institute,
Czech Academy of Sciences

joint with Martin Doležal (Math Inst Czech Academy)
Ping Hu (Uni Warwick)
and Diana Piguet (Comp Sci Inst Czech Academy)

JH's research is supported by a Marie Curie Intra European Fellowship within the 7th European Community Framework Programme.

Razborov 2008 Optimal function $g_3 : [0,1] \to [0,1]$ such that if G has $\alpha\binom{n}{2}$ edges then it has $\geq (g_3(\alpha) \pm o(1))\binom{n}{3}$ triangles.

g_2 trivial: $g_2 = \textit{identity}$
g_3 Razborov: graph limits
$g_4, \ldots$ Nikiforov, Reiher: graph limits inspired

Razborov 2008 Optimal function $g_3 : [0,1] \to [0,1]$ such that if G has $\alpha\binom{n}{2}$ edges then it has $\geq (g_3(\alpha) \pm o(1))\binom{n}{3}$ triangles.

g_2 trivial: $g_2 = \textit{identity}$
g_3 Razborov: graph limits
$g_4, \ldots$ Nikiforov, Reiher: graph limits inspired

Dense graph limits (either flag algebras or "graphons") have been very useful in obtaining results of the type:

density $\geq \alpha$ of graph F in G implies density $\geq \beta$ of H in G

Razborov 2008 Optimal function $g_3 : [0,1] \to [0,1]$ such that if G has $\alpha\binom{n}{2}$ edges then it has $\geq (g_3(\alpha) \pm o(1))\binom{n}{3}$ triangles.

g_2 trivial: $g_2 = identity$
g_3 Razborov: graph limits
$g_4, \ldots$ Nikiforov, Reiher: graph limits inspired

Dense graph limits (either flag algebras or "graphons") have been very useful in obtaining results of the type:

density $\geq \alpha$ of graph F in G implies density $\geq \beta$ of H in G

Allen-Böttcher-H-Piguet 2014 Optimal function $f_3 : [0,1] \to [0,1]$ such that if G has $\alpha\binom{n}{2}$ edges then it has $\geq (f_3(\alpha) \pm o(1))\frac{n}{3}$ vertex-disjoint triangles.

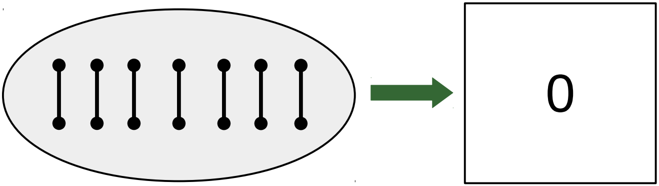
f_2 Erdős–Gallai 1959: "consider a maximum matching, ..."
f_3 Allen-Böttcher-H-Piguet 2014: modern tools but finite
$f_4, \ldots$??

Could graph limits help us in obtaining such **tiling** results?

In this talk, we focus on K_2-tilings=matchings. This is for notational convenience only. All the features of the basic theory hold for H-tilings as well. (Some advanced, like the half-integrality of the vertex cover polytope do not.)

Aim: notion of matchings of linear size in graphons.
$\Rightarrow$ matching number of a graphon

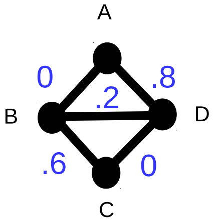
Bad news: normalized size of the maximum matching not continuous ...



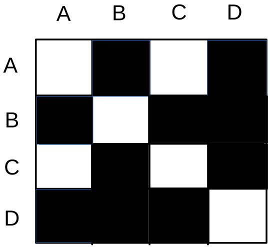
Good news: ... but lower semicontinuous, which is the more useful half of continuity

Aim: notion of **fractional** matchings in graphons.

4-vertex graph and its representation $W : \Omega^2 \to [0,1]$ (measure λ)



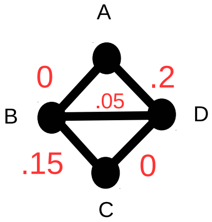
a fractional matching



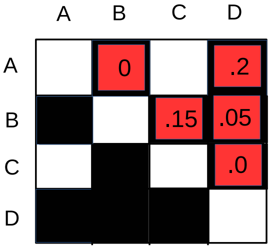
finite fractional matching		
weight incident with D $.8+.2=1$		

Aim: notion of **fractional** matchings in graphons.

4-vertex graph and its representation $W : \Omega^2 \to [0,1]$ (measure λ)



a normalized frac matching

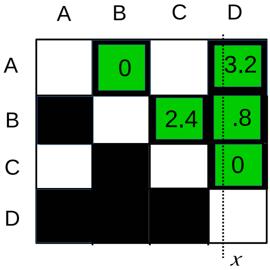


finite fractional matching	"brick measure" μ	
weight incident with D $.8+.2=1$	$\mu(D\times\Omega)$ $.2+.05=.25$	

Aim: notion of **fractional** matchings in graphons.

4-vertex graph and its representation $W : \Omega^2 \to [0,1]$ (measure λ)

area of an elementary rectangle=(1/4*1/4)=1/16



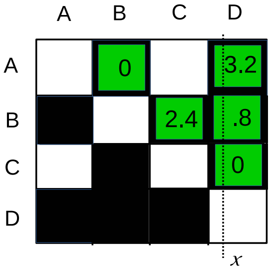
Radon-Nikodym derivative f

finite fractional matching	"brick measure" μ	Rad-Nyk der f
weight incident with D $.8+.2=1$	$\mu(D \times \Omega)$ $.2+.05=.25$	$\int_y f(x,y)d\lambda$ 1

Aim: notion of **fractional** matchings in graphons.

4-vertex graph and its representation $W : \Omega^2 \to [0,1]$ (measure λ)

area of an elementary rectangle=(1/4*1/4)=1/16



Radon-Nikodym derivative f

finite fractional matching	“brick measure” μ	Rad-Nyk der f
weight incident with D	$\mu(D \times \Omega)$	$\int_y f(x,y)d\lambda$
$.8+.2=1$	$.2+.05=.25$	1
General properties		
supported on edges		$\mathrm{supp} f \subset \mathrm{supp} W$
total weight at vertex≤ 1		$\int_y f(y,x)d\lambda \leq 1$
weights$\in [0,1]$		$f \geq 0$

$f \in L^1(\Omega^2)$ is a matching in a graphon W if:

- $supp(f) \subset supp(W)$ (?)
- for each $x \in \Omega$: $\int_y f(x,y)d\lambda \leq 1$, $\int_y f(y,x)d\lambda \leq 1$
- f non-negative

The size of f is $\frac{1}{2}\int_x \int_y f(x,y)$
The matching number of W is $match(W) = \sup_f size(f)$

Recall: no distinction between integral and fractional matchings in graphons.

$f \in L^1(\Omega^2)$ is a matching in a graphon W if:

- $supp(f) \subset supp(W)$ (?)
- for each $x \in \Omega$: $\int_y f(x,y)d\lambda \leq 1$, $\int_y f(y,x)d\lambda \leq 1$
- f non-negative

The size of f is $\frac{1}{2}\int_x \int_y f(x,y)$
The matching number of W is $match(W) = \sup_f size(f)$

Recall: no distinction between integral and fractional matchings in graphons.

A function $c : \Omega \to [0,1]$ is a fractional vertex cover of W if $W(x,y) = 0$ for almost every $(x,y) : c(x) + c(y) < 1$.
The size of c is $\int_x c(x)$.
The fractional cover number of W is $fcov(W) = \inf_c size(c)$

Results

Thm1 (finite versus limit)
If $G_n \to W$ then $\liminf_n \frac{match(G_n)}{n} \geq match(W)$.

Thm2 (semicontinuity of Matching Number for graphons)
If $W_n \to W$ then $\liminf_n match(W_n) \geq match(W)$.

Thm3 (semicontinuity of Cover Number for graphons)
If $W_n \to W$ (cut-norm) and c_n a vertex cover of W_n.
Then any weak* limit of c_n's is a vertex cover of W.

Thm4 (LP-duality)

$$fcov(W) = match(W)$$

attained   not necessarily attained

A new ???? form of the LP duality

Primal

maximize	$c^T x$	$\sum_j c_j x_j$
subject to	$Ax \leq b$:	$\forall i : \sum_j A_{ij} x_j \leq b_i$
and	$x \geq 0$	

Dual

minimize	$b^T y$
subject to	$A^T y \geq c$
and	$y \geq 0$

Applications in random graphs/extremal graph theory

F is an arbitrary "smallish" graph. The theory introduced above for for matchings generalizes to F-tilings.
$TIL(F, G)$, $TIL(F, W)$: size of the maximum tiling in G or in W

F-tilings in random graphs $\mathbb{G}(n, W)$

Thm For an fixed graph F, a.a.s.,

$$\lim \frac{TIL(F, \mathbb{G}(n, W))}{n} = TIL(F, W) \ .$$

Komlós's Theorem

Thm Suppose G is on n vertices and that $\delta(G) \geq \alpha n$. Then

$$TIL(F, G) \geq h_F(\alpha)n \pm o(n) \ ,$$

where the function $h_F : [0, 1] \to [0, 1]$ is best possible.

Property testing in dense graphs

$\mathcal{G}$...all finite graphs
A function $f : \mathcal{G} \to \mathbb{R}$ is testable if for each $\epsilon > 0$ there exists a number $r \in \mathbb{N}$ and a function $g : \mathcal{G} \to \mathbb{R}$ (tester) such that

$$\mathbb{P}\big[\, |f(G) - g(G[X])| > \epsilon \,\big] < \epsilon \,,$$

where X is a uniformly random r-tuple of vertices in G.

..work of Alon, Shapira...
Observation: A function is testable if and only if it is continuous in the cut-distance.

In particular, the matching ratio is not testable.

Property testing in dense graphs

$\mathcal{G}$...all finite graphs

A function $f : \mathcal{G} \to \mathbb{R}$ is testable if for each $\epsilon > 0$ there exists a number $r \in \mathbb{N}$ and a function $g : \mathcal{G} \to \mathbb{R}$ (tester) such that

$$\mathbb{P}\big[\, |f(G) - g(G[X])| > \epsilon \,\big] < \epsilon \,,$$

where X is a uniformly random r-tuple of vertices in G.

..work of Alon, Shapira...

Observation: A function is testable if and only if it is continuous in the cut-distance.

In particular, the matching ratio is not testable.

Define

$$match_\epsilon(G) = \min\left\{match(G') : G' \subset G, e(G') > e(G) - \epsilon n^2\right\}$$

Theorem: For each $\epsilon > 0$, $\frac{match_\epsilon}{n}$ is testable.

Combinatorial optimization of graphons

Recall: if $G = (V, E)$ is a finite graph, then we write

- $FMATCH(G) \subset [0,1]^E$ for set of all fractional matchings (fractional matching polytope)
- $FCOV(G) \subset [0,1]^V$ for set of all fractional vertex covers (fractional vertex cover polytope)

A basic fact: The following are equivalent:

- G bipartite
- all vertices of $FMATCH(G)$ integral
- all vertices of $FCOV(G)$ integral

Combinatorial optimization of graphons

Recall: if $G = (V, E)$ is a finite graph, then we write

- $FMATCH(G) \subset [0,1]^E$ for set of all fractional matchings (fractional matching polytope)
- $FCOV(G) \subset [0,1]^V$ for set of all fractional vertex covers (fractional vertex cover polytope)

A basic fact: The following are equivalent:

- G bipartite
- all vertices of $FMATCH(G)$ integral
- all vertices of $FCOV(G)$ integral

- $MATCH(W) \subset [0,\infty)^{\Omega^2}$: matching polyton
- $FCOV(W) \subset [0,1]^{\Omega}$: fractional vertex cover polyton

Theorem: The following are equivalent:

- W bipartite
- ??
- all extreme points of $FCOV(W)$ integral