Tilings in graphons

Jan Hladký
Institute for geometry,
TU Dresden

with P. Hu and D. Piguet (arXiv:1606.03113, 1607.08415)
and M. Doležal (arXiv: 1606.06958)

Razborov 2008 Optimal function $g_3 : [0,1] \to [0,1]$ such that if G has $\alpha\binom{n}{2}$ edges then it has $\geq (g_3(\alpha) \pm o(1))\binom{n}{3}$ triangles.

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g_3 Razborov: graph limits
$g_4, \ldots$ Nikiforov, Reiher: graph limits inspired

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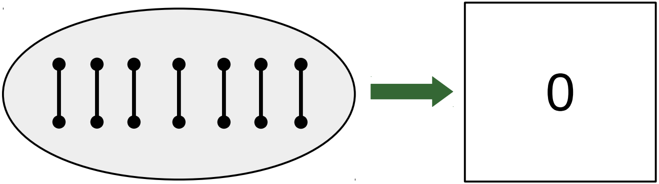
Allen-Böttcher-H-Piguet 2014 Optimal function $f_3 : [0,1] \to [0,1]$ such that if G has $\alpha\binom{n}{2}$ edges then it has $\geq (f_3(\alpha) \pm o(1))\frac{n}{3}$ vertex-disjoint triangles.

f_2 Erdős–Gallai 1959: "consider a maximum matching, ..."
f_3 Allen-Böttcher-H-Piguet 2014: modern tools but finite
$f_4, \ldots$??

Could graph limits help us in obtaining such **tiling** results?

In this talk, we focus on K_2-tilings=matchings. This is for notational convenience only. All the features of the basic theory hold for H-tilings as well. (Some advanced, like the half-integrality of the vertex cover polytope do not.)

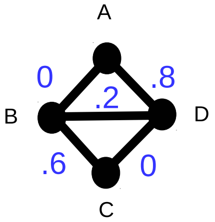
Aim: notion of matchings of linear size in graphons.
Bad news: normalized size of the maximum matching not continuous ...



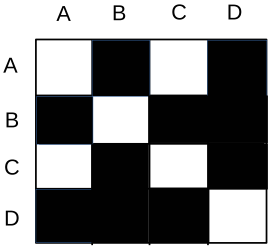
Good news: ... but lower semicontinuous, which is the more useful half of continuity

Aim: notion of **fractional** matchings in graphons.

4-vertex graph and its representation $W : \Omega^2 \to [0,1]$ (measure λ)



a fractional matching



finite fractional matching		
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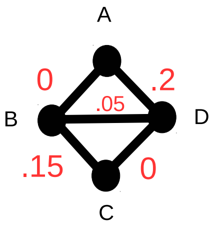
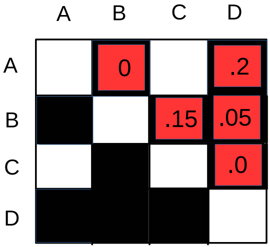


a normalized frac matching



finite fractional matching	"brick measure" μ	
weight incident with D $.8+.2=1$	$\mu(D \times \Omega)$ $.2+.05=.25$	

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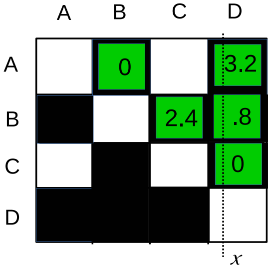

Radon-Nikodym derivative f

finite fractional matching	"brick measure" μ	Rad-Nyk der f
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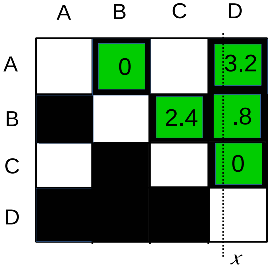

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General properties		
supported on edges		$\text{supp} f \subset \text{supp} W$
total weight at vertex ≤ 1		$\int_y f(y,x)d\lambda \leq 1$
weights $\in [0,1]$		$f \geq 0$

$f \in L^1(\Omega^2)$ is a matching in a graphon W if:

- $supp(f) \subset supp(W)$ (?)
- for each $x \in \Omega$: $\int_y f(x,y) d\lambda \leq 1$, $\int_y f(y,x) d\lambda \leq 1$
- f non-negative

The size of f is $\frac{1}{2} \int_x \int_y f(x,y)$

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A function $c : \Omega \to [0,1]$ is a fractional vertex cover of W if $W(x,y) = 0$ for almost every $(x,y) : c(x) + c(y) < 1$.
The size of c is $\int_x c(x)$ The cover number of W is $CN(W) = \inf_c size(f)$

Results

Thm1 (finite versus limit)
If $G_n \to W$ then $\liminf_n \frac{MN(G_n)}{n} \geq MN(W)$.

Thm2 (semicontinuity of Matching Number for graphons)
If $W_n \to W$ then $\liminf_n MN(W_n) \geq MN(W)$.

Thm3 (semicontinuity of Cover Number for graphons)
If $W_n \to W$ (optimally overlaid) and c_n a vertex cover of W_n.
Then any weak* limit of c_n's is a vertex cover of W.

Thm4 (LP-duality)

$$CN(W) = MN(W)$$

attained   not necessarily attained

Applications

F is an arbitrary "smallish" graph. The theory introduced above for for matchings generalizes to F-tilings.
$TIL(F,G)$, $TIL(F,W)$: size of the maximum tiling in G or in W

F-tilings in random graphs $\mathbb{G}(n,W)$

Thm For an fixed graph F, a.a.s.,

$$\lim \frac{TIL(F,\mathbb{G}(n,W))}{n} = TIL(F,W) \; .$$

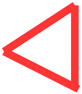
Komlós's Theorem

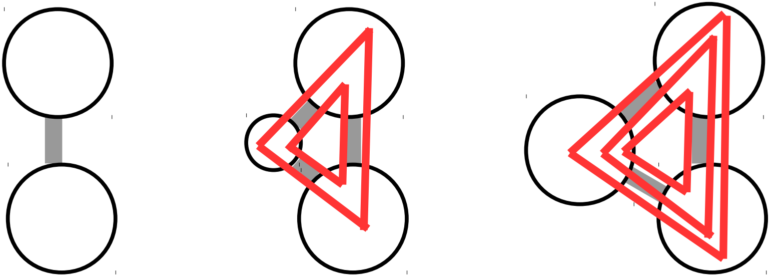
Thm Suppose G is on n vertices and that $\delta(G) \geq \alpha n$. Then

$$TIL(F,G) \geq h_F(\alpha)n \pm o(n) \; ,$$

where the function $h_F : [0,1] \to [0,1]$ is best possible.

Komlós's Theorem

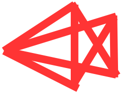
F= 

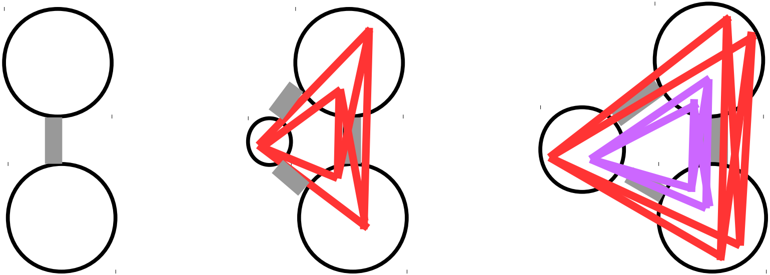


$\delta = \frac{n}{2}$

$\delta = \frac{2n}{3}$

Komlós's Theorem

F= 



$\delta = \frac{n}{2}$ $\delta = \frac{3}{5} \cdot n$

Other graph(ON) parameters and LP-duality relations?

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Graphons: Not true

Ongoing work with Israel Rocha.