Graphons as weak* limits

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arXiv: 1705.09160

+ongoing work with M. Doležal, J. Grebík, I. Rocha,
V. Rozhoň, J. Venters

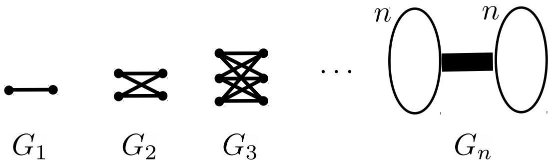
Limits of dense graph sequences

Lovász, Szegedy *JCTB'06* (Fulkerson Prize'12)
Borgs, Chayes, Lovász, Sós, Vesztergombi *Adv.Math.'06*
Borgs, Chayes, Lovász, Sós, Vesztergombi *Ann.Math.'12*

idea: convergence notion for sequences of finite graphs
compactification of the space of finite graphs $\Rightarrow$
…*graphons* symmetric Lebesgue-m. functions $\Omega^2 \to [0,1]$

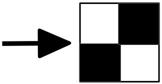
Why? same story as with $\mathbb{Q}$ vs $\mathbb{R}$: only the latter allows
reasonable e.g. variational and integral calculus
for example $argmin(x^3 - 2x)$

Graphons

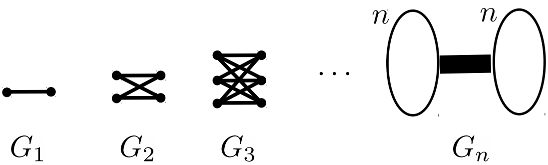


Represent these graphs by their adjacency matrices:

$$\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{matrix} \qquad \cdots$$



Graphons



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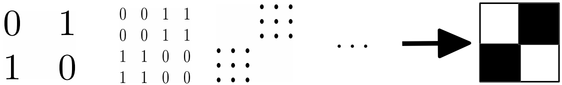
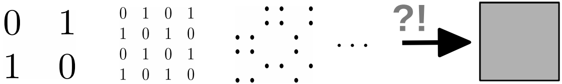


. . . works if you do things the right way. But, . . .



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ground space =symmetric Lebesgue measurable $U : \Omega^2 \to [0,1]$

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$$\delta_1(U, W) = \inf_\pi d_1(U, W^\pi)$$

where π ranges over all measure preserving bijections, and $W^\pi(x, y) = W(\pi(x), \pi(y))$. (pseudometric)

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extremely strong topology...
...continuity of many graph parameters
Bad news: not compact (e.g., chessboards)

The cut-distance topology

$$d_\square(U,W) = \sup_{S\subset\Omega} \left| \int_S \int_S U(x,y) - W(x,y) \right| .$$

$$\delta_\square(U,W) = \inf_\pi d_\square(U, W^\pi)$$

Implicitly used by graph theorists since the 1990's

Many important graph parameters still continuous

Lovász&Szegedy'06 $\delta_\square$ is a compact topology (on $\Omega^2 \to [0,1]$)

A sample application (A version of Turán's/Mantel's Theorem)
Theorem For any $\epsilon > 0$ there exists $\delta > 0$ such that if an n-vertex graph has more than $(\frac{1}{2}+\epsilon)\binom{n}{2}$ edges than it has more than δn^3 many triangles.
Proof By contradiction: There exists $\epsilon > 0$ and a sequence of graphs of edge densities $> (\frac{1}{2}+\epsilon)$ and vanishing triangle densities.
Accumulation point W. $\int_x \int_y W(x,y) \geq \frac{1}{2}+\epsilon$ and
$\int_x \int_y \int_z W(x,y)W(y,z)W(z,x) = 0$

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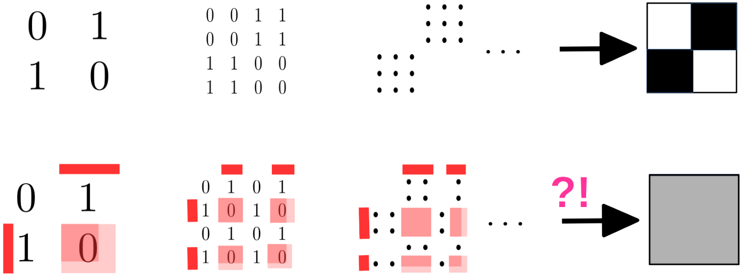
Proofs of the Lovász–Szegedy Theorem

1. Lovász–Szegedy: Using Szemerédi's Regularity lemma
2. Elek–Szegedy (2012): Ultraproducts
3. Via the Aldous–Hoover theorem on exchangeable arrays (1981, realized by Persi Diaconis& Svante Janson and Tim Austin, 2008)
4. our proof based on weak* convergence

Comparing the weak* and cut-distance topology

$$W_n \xrightarrow{d_\square} W \iff \limsup_{n} \left\{ \sup_{S\subset\Omega} \left| \int_{x\in S} \int_{y\in S} W_n(x,y) - W(x,y) \right| \right\} = 0$$

$$W_n \xrightarrow{\mathrm{w}^*} W \iff \sup_{S\subset\Omega} \left\{ \limsup_{n} \left| \int_{x\in S} \int_{y\in S} W_n(x,y) - W(x,y) \right| \right\} = 0$$



Lovász&Szegedy'06 $\delta_\square$ is a compact topology.
Our proof Suppose that $W_1, W_2, W_3 \ldots : \Omega^2 \to [0,1]$.

- We need to find an accumulation point w.r.t. cut-distance.

- Take all possible $W_1^{\pi_1}, W_2^{\pi_2}, W_3^{\pi_3}, \ldots$ and take all their weak* accumulation points (Banach–Alaoglu Theorem) $\to \mathcal{ACC}_{w^*}$

- From $\mathcal{ACC}_{w^*}$ take a most structured graphon a prove that it is also a cut-distance accumulation point:

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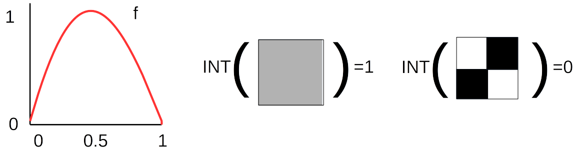
- From $\mathcal{ACC}_{w^*}$ take a most structured graphon a prove that it is also a cut-distance accumulation point:

Fix concave function $f : [0,1] \to \mathbb{R}$. Define $INT(W) := \int_{x,y} f(W(x,y))$



Take $\Gamma \in \mathcal{ACC}_{w^*}$ that minimizes $INT(\Gamma)$ (infimum is attained, nontrivial)

Lemma If $U_1, U_2, U_3, \ldots$ converges weak* but not in $d_\square$ to K. Then there exists a subsequence of versions $U_{n_1}^{\pi_{n_1}}, U_{n_2}^{\pi_{n_2}}, U_{n_3}^{\pi_{n_3}}, \ldots$ that weak* converges to some L, $INT(L) < INT(K)$