Jan Hladký, TU Dresden

Graphons as weak* limits

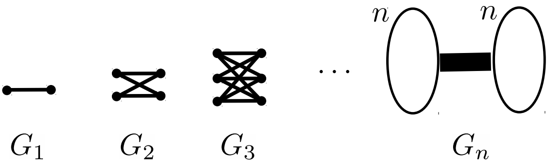
(1) "entropy minimization" with Doležal (arXiv: 1705.09160)
(2) "Vietoris topology" with Doležal, Grebík, Rocha, Rozhoň
(3) hypergraphons with Noel, Piguet, Rocha, Saumell

Limits of dense graph sequences

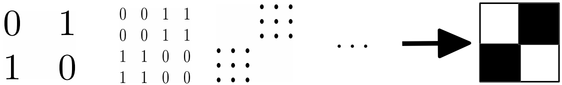
Borgs, Chayes, Lovász, Sós, Szegedy, Vesztergombi 2006

idea: convergence notion for sequences of finite graphs
compactification of the space of finite graphs $\Rightarrow$
$\ldots$ *graphons* symmetric Lebesgue-m. functions $\Omega^2 \to [0,1]$
Ω=separable atomless probability space $\cong [0,1]$

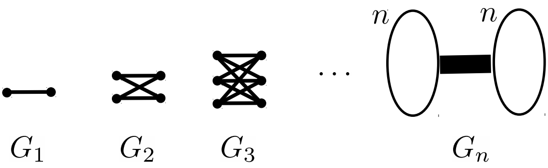
Graphons



Represent these graphs by their adjacency matrices:



Graphons



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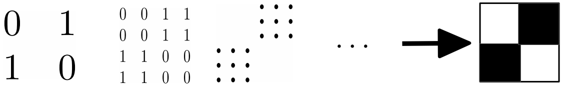
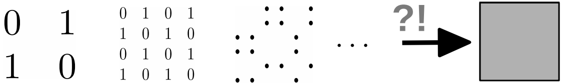


. . . works if you do things the right way. But, . . .



The cut-distance topology

Step 1: "Comparing the number of edges inside any vertex set"

$$d_{\square}(U, W) = \sup_{S \subset \Omega} \left| \int_S \int_S U(x,y) - W(x,y) \right| .$$

Step 2: "Permuting the adjacency matrix"

$$\delta_{\square}(U, W) = \inf_{\pi} d_{\square}(U, W^{\pi}) ,$$

where $\pi : \Omega \to \Omega$ runs through all measure-preserving bijections and $W^{\pi}(x,y) := W(\pi(x), \pi(y))$ version of W

Many important graph parameters still continuous

Lovász&Szegedy'06 $\delta_{\square}$ is a compact topology (on $\Omega^2 \to [0,1]$)

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$$\mathbf{ACC}_\square(\Gamma_1, \Gamma_2, \ldots) := \{\delta_\square\text{-acc pts of } \Gamma_1, \Gamma_2, \ldots\}$$
$$= \bigcup_{\pi_1, \pi_2, \ldots} \{d_\square\text{-acc pts of } \Gamma_1^{\pi_1}, \Gamma_2^{\pi_2}, \ldots\}$$
$$\mathbf{LIM}_\square(\Gamma_1, \Gamma_2, \ldots) := \bigcup_{\pi_1, \pi_2, \ldots} \{d_\square\text{-limit of } \Gamma_1^{\pi_1}, \Gamma_2^{\pi_2}, \ldots\}$$

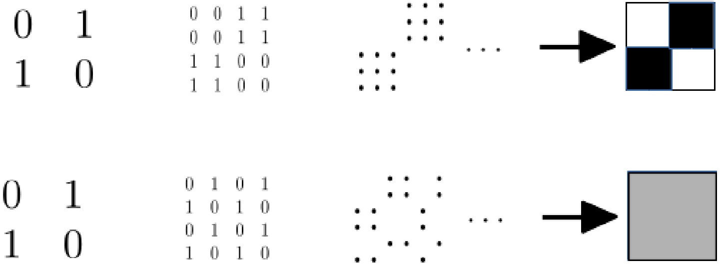
Lovász&Szegedy'06 For any sequence $\Gamma_1, \Gamma_2, \ldots$ we have that $\mathbf{ACC}_\square(\Gamma_1, \Gamma_2, \ldots) \neq \emptyset$.

Proofs of the Lovász–Szegedy Theorem

1. Lovász–Szegedy: Using Szemerédi's Regularity lemma
2. Elek–Szegedy (2012): Ultraproducts
3. Aldous–Hoover theorem on exchangeable arrays (1981)
   Persi Diaconis&Svante Janson and Tim Austin, 2008
4. **our proof(s) based on weak* convergence**

Comparing the weak* and cut-distance topology

Weak* converg.: $\Gamma_1, \Gamma_2, \ldots \xrightarrow{w^*} \Gamma$ iff $\forall X \subset \Omega^2$: $\lim_n \int_X \Gamma_n = \int_X \Gamma$



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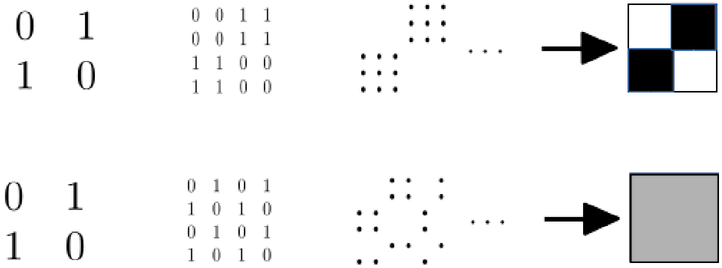
$$\mathbf{ACC}_{w*}(\Gamma_1, \Gamma_2, \ldots) := \bigcup_{\pi_1, \pi_2, \ldots} \{\text{w*-acc pts of } \Gamma_1^{\pi_1}, \Gamma_2^{\pi_2}, \ldots\}$$

Note: $\mathbf{ACC}_{w^*}(\Gamma_1, \Gamma_2, \ldots)$ is nonempty by Banach–Alaoglu Thm



Comparing the weak* and cut-distance topology

$$W_n \xrightarrow{d_\square} W \iff \limsup_n \left\{ \sup_{S\subset\Omega} \left| \int_{x\in S}\int_{y\in S} W_n(x,y) - W(x,y) \right| \right\} = 0$$

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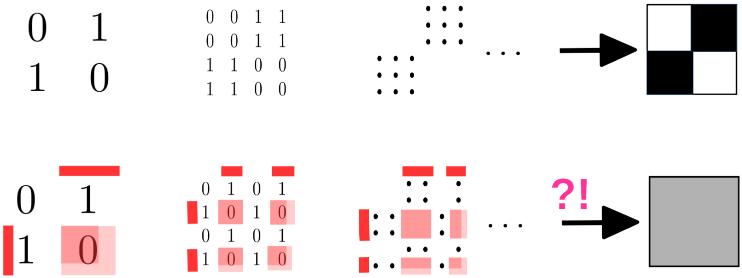

Lovász&Szegedy'06 $\delta_\square$ is a compact topology.
Proof (Doležal-H) Suppose that $W_1, W_2, \ldots : \Omega^2 \to [0,1]$.

- We need to find an accumulation point w.r.t. cut-distance.
- Lets search only in $\mathbf{ACC}_{w^*}(W_1, W_2, \ldots)$
- From $\mathbf{ACC}_{w^*}(W_1, W_2, \ldots)$ take a most structured graphon a prove that it is also a cut-distance accumulation point:

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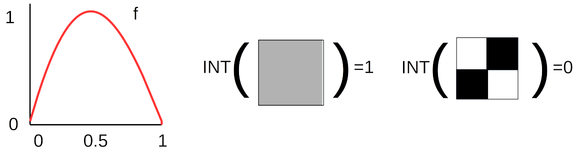
Fix concave function $f : [0,1] \to \mathbb{R}$. Define $INT(W) := \int_{x,y} f(W(x,y))$



Take $\Gamma \in \mathbf{ACC}_{w^*}(W_1, W_2, \ldots)$ that minimizes $INT(\Gamma)$
Lemma If $U_1, U_2, U_3, \ldots$ converges weak* but not in $d_\square$ to K. Then there exists a subsequence of versions $U_{n_1}^{\pi_{n_1}}, U_{n_2}^{\pi_{n_2}}, U_{n_3}^{\pi_{n_3}}, \ldots$ that weak* converges to some L, $INT(L) < INT(K)$

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??? $\mathbf{ACC}_{w^*}(W_1, W_2, \ldots)$ or $\mathbf{LIM}_{w^*}(W_1, W_2, \ldots)$???

Needs for using **ACC**: • nonempty
•in the Lemma, we pass to subsequence
Need for using **LIM**: • infimum of $INT(\cdot)$ is attained

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Graphons and the Vietoris topology (with Doležal-Grebík-Rocha-Rozhoň)

Theorem A For every sequence $W_1, W_2, \ldots$ there exists a subsequence so that

$$\mathbf{ACC}_{w^*}(W_{n_1}, W_{n_2}, \ldots) = \mathbf{LIM}_{w^*}(W_{n_1}, W_{n_2}, \ldots) \; .$$

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Envelopes and the structurdness order

$\langle W \rangle := \mathbf{LIM}_{w^*}(W, W, \ldots)$

$U \preceq W \quad \text{iff} \quad \langle U \rangle \subseteq \langle W \rangle$

Minimal elements ... constant graphons

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Key tool Vietoris topology (hyperspace)
Abstractly (X, d):
(1) points of $K(X)$: closed sets of X w.r.t. d.
(2) distance on $K(X)$: how far two closed sets are
Fact: if X is metric compact then $K(X)$ is compact.

Proof Apply this with $X = \mathcal{W}$, d ≈weak* topology
$W \mapsto \langle W \rangle$ is a homeomorphism of $\mathcal{W}/_{\delta_\square = 0}$ to a closed subset of $K(\mathcal{W})$.

Cut-distance identifying graphon parameters (with Doležal-Grebík-Rocha-Rozhoň)

Motivation: The Chung-Graham-Wilson Theorem:

Among all graphons with edge density p, the constant-p graphon is the only graphon U satisfying any of the following:

- $t(C_4, U) \leq p^4$, Sidorenko's Conj: $t(B, U) \leq p^{e(B)}$
- $|\lambda_1(U)| \leq p$ and $|\lambda_2(U)| \leq 0$.
- $INT_f(U) = \int_x \int_y f(U(x, y)) \leq f(p)$ for a fixed convex function f.

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Definition $F : \mathcal{W} \to \mathbb{R}$ is a cut-distance identifying graphon parameter (CDIGP) if for each $U \prec W$ we have $F(U) < F(W)$.

Results:

- $t(C_4, \cdot)$, $t(C_6, \cdot)$, $t(C_8, \cdot), \ldots$ are CDIGPs
- generalized Sidorenko conjecture not true, i.e., $t(P_3, \cdot)$ is not CDIGP
- each kth eigenvalue is CDIGP (not precise)

Hypergraphons (with Noel-Piguet-Rocha-Saumell)

We can (??) construct limits of k-uniform hypergraph(on)s in a similar manner.

Martingale approach by Yufei Zhao $\Rightarrow$ weak* limits