Limits of sparse graph sequences

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European
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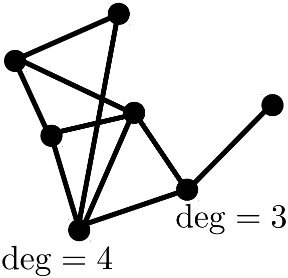
Graph limit theories



Graph limit theories

Limits of dense graph sequences$\sim$2004
Borgs, Chayes, Lovász, Razborov, Sós, Szegedy, Vesztergombi
works for all graph sequences but trivial when $e(G_n)/v(G_n)^2 \to 0$.

note $e(G_n) \leq \binom{v(G_n)^2}{2} \approx \frac{v(G_n)^2}{2}$.

$\Rightarrow$ breakthroughs in graph theory (extremal GrTh, random graphs)
$\Rightarrow$ stimulated developements in Higher Order Fourier Analysis
(Szegedy, Green–Tao, ...)

Limits of sparse graphs$\sim$2001, Benjamini–Schramm
one needs to fix $D \in \mathbb{N}$ and work in the category of graphs of maximum degree $\leq D$.

note $e(G_n) \leq \frac{D}{2} v(G_n)$

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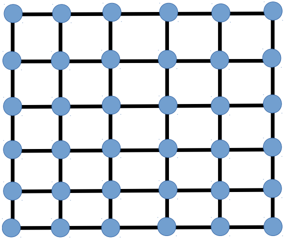

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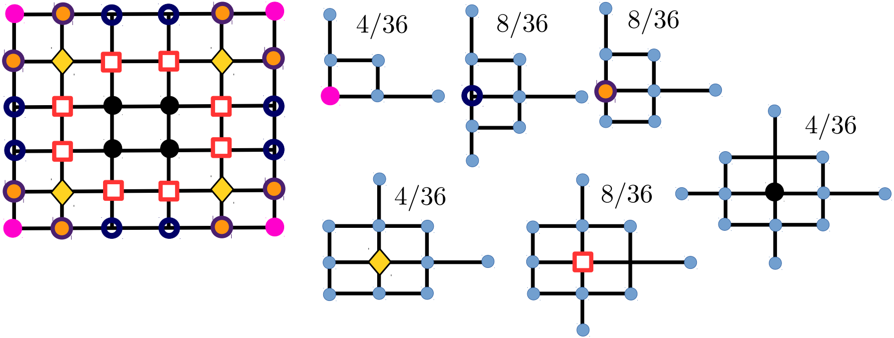

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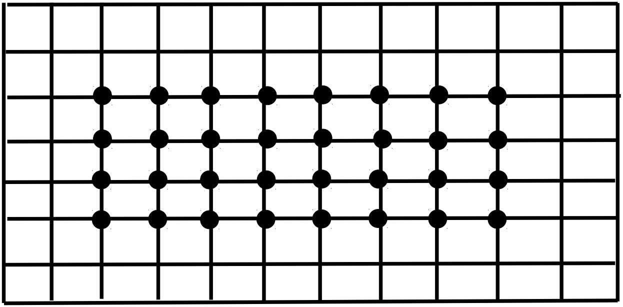

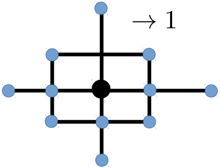

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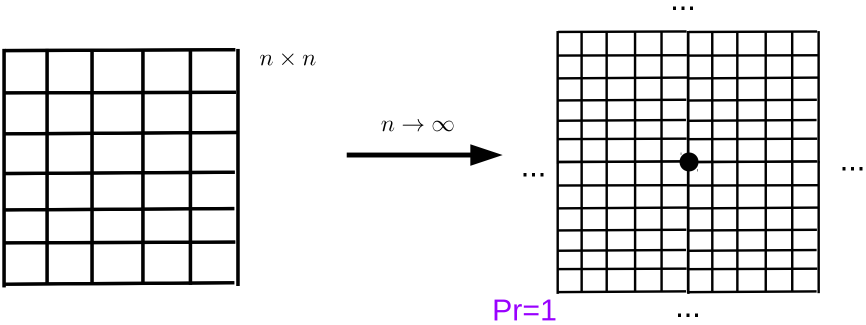
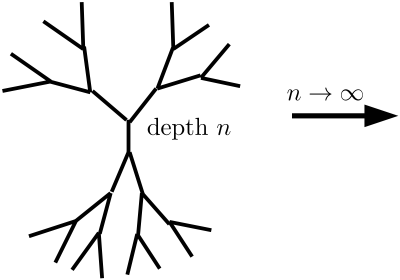
Observation Every sequence of uniformly degree-bounded graphs contains a convergent subsequence.

Example: grids



Example: 3-regular trees



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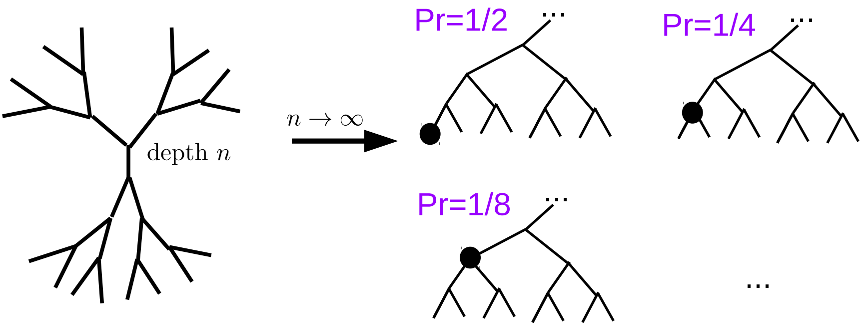
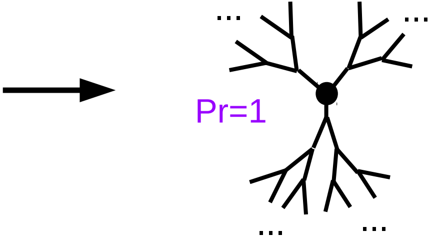
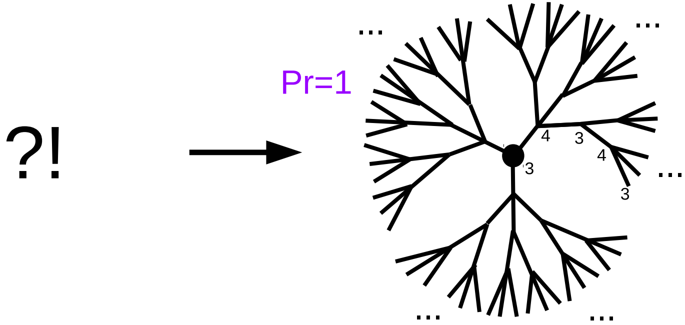


Obtaining the 3-regular tree in the limit

3-regular graphs
with no short cycles



The Aldous–Lyons conjecture



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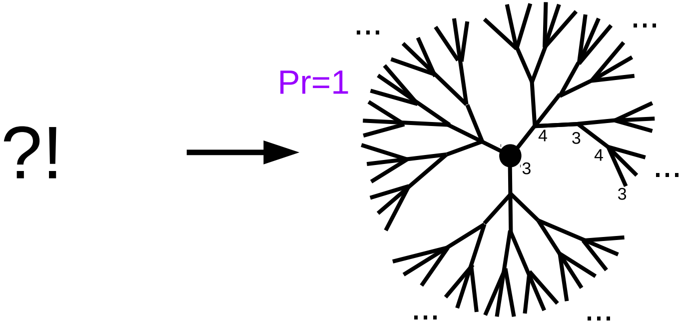
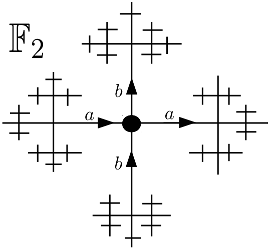


If μ is a limit distribution then sampling according to μ and moving to a random neighbor must give the original law μ (weighted by degrees) $\Rightarrow$ **unimodular distributions**

Conjecture (Aldous–Lyons'07) Every unimodular distribution can be obtained as a limit.

Sofic groups (Gromov 1990)

A (finitely generated) group $\Gamma = \langle S \rangle$ is **sofic** if the Dirac measure on the Cayley graph (Γ, S) can be approximated by finite graphs.
(in the actual definition, one has to move to the category of edge-labelled directed graphs)



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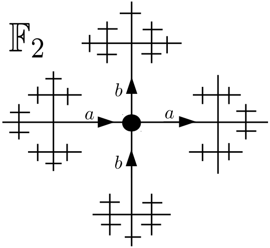

This definition does not depend on the choice of S.
Alternative definition: subgroup of a metric ultraproduct of $\mathbb{S}_n$'s
Gromov 1990: It could perhaps be the case (?!) that every group is sofic???
Elek–Szabó 2005: Every sofic group is hyperlinear.

Applications of soficity: equations in groups

$\Gamma \ldots$ group, $k_1, k_2, \ldots, k_n \in \mathbb{Z}$, $\gamma_1, \gamma_2, \ldots, \gamma_n \in \Gamma$.
We want to find a solution $x \in \Gamma$,

$$\gamma_1 x^{k_1} \gamma_2 x^{k_2} \ldots \gamma_n x^{k_n} = 1 \, .$$

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$$\alpha x \beta x^{-1} = 1 \quad \text{when } ord(\alpha) \neq ord(\beta)$$

An equation is **regular** if $\sum k_i \neq 0$.
Conjecture Any regular equation (in a group Γ) has a solution over some extension $\Lambda \supseteq \Gamma$.
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Baby version For each regular equation in a finite group Γ has a solution over some $\Lambda \supseteq \Gamma$.
Proof $\Gamma \leq \mathbb{S}_n \leq O(n)$, and we have
Gerstenhaber–Rothaus'62: $O(n)$ is algebraically close.

Applications of soficity: group rings

Conjecture (Kaplansky 1969): For any group G and commutative field K, the group algebra $K(G)$ is directly finite. That is $ab = 1_K$ implies $ba = 1_K$.

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Theorem (Elek–Szabó'04): For any sofic group G and commutative field K, the group algebra $K(G)$ is directly finite. That is $ab = 1_K$ implies $ba = 1_K$.

An application in global analysis

Theorem (Lück 1994, Abért, Thom, Virág 201?):
Let X be a finite connected simplicial complex. Let $\pi_1(X) \geq \Gamma_1 \geq \Gamma_2 \geq \dots$ be a chain of normal subgroups of finite index in $\pi_1(X)$ with $\cap_n \Gamma_n = 1$, and let $X_n = \tilde{X}/\Gamma_n$. Then

$$\lim_n \frac{b_k(X_n)}{|\Gamma : \Gamma_n|} = \beta_k^{(2)}(X) \ .$$

($\beta_k^{(2)} \dots$ k-th L^2 Betti number)

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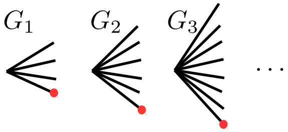

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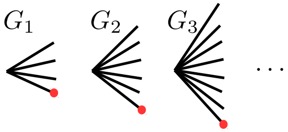

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A sequence of probability measures $\mu_1, \mu_2, \ldots$ on $\mathcal{X}$ is **tight** if for every $\epsilon > 0$ there exists a **finite** $K \subset \mathcal{X}$ such that $\mu_n(K) \geq 1 - \epsilon$ for all n.

Lyons'07: The concept of Benjamini–Schramm limit can be extended to sequences $G_1, G_2, \ldots$ where for each $r \in \mathbb{N}$, the sequence $\rho_r(G_1), \rho_r(G_2), \ldots$ is tight. AND NOT FURTHER

Ongoing work with Lukasz Grabowski & Oleg Pikhurko

Theorem (Elek'10)
The Aldous–Lyons conjecture holds for measures supported on bounded-degree trees.

Theorem (Elek–Lippner'10) (Borel Oracles Method)
The matching ratio is Benjamini–Schramm continuous for bounded-degree graphs.

Definition A **graphing** is a unimodular Borel graph whose each degree is finite and bounded by an absolute constant $D \in \mathbb{N}$.

Theorem (Hatami–Lovász–Szegedy'13)
For every Benjamini–Schramm convergent sequence of graphs of degree$\leq D$ there is a graphing that is its local-global limit.

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...and perhaps almost all of the theory can be extended

Sparse graphs with unbounded maximum degree

Erdős–Rényi random graph $\mathbb{G}(n, p)$ (Erdős–Rényi, Gilbert, 1959): Take $V(G) = \{1, \ldots, n\}$. To randomly generate the edges, we put $ij \in E(G)$ with probability p.

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Why important? Questions from dynamical systems. Previously, more complicated model of random D-regular graphs.