# Evolution systems — an ordinal rank measuring universality

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## Abstract

We introduce a natural ordinal rank on the objects of an evolution system. In the model-theoretic setup, the rank measures how far the structure in question is from being universal in the class of all countable structures of a given class. (Research supported by GA ČR grant EXPRO 20-31529X) (August 19, 2024)

## Main definition

Let $\mathcal{E} = \langle \mathfrak{V}, \mathcal{T}, \Theta \rangle$ be a fixed regular evolution system and let $M$ be a $\mathfrak{V}$-object. We denote by $\operatorname{Fin} M$ the class of all $\mathfrak{V}$-arrows from $\mathcal{E}$-finite objects to $M$.

Given $f \colon A \to M$ in $\operatorname{Fin} M$, we define its ***rank*** $\operatorname{rk}(f)$ by the folowing condition.

(†) $\operatorname{rk}(f) \geq \alpha + 1$ if and only if for every nontrivial transition $t \colon A \to A'$ there exists a $\mathfrak{V}$-arrow $f' \colon A' \to M$ satisfying $f = f' \circ t$ and $\operatorname{rk}(f') \geq \alpha$.

We define $\operatorname{rk}(f) = \alpha$ iff $\operatorname{rk}(f) \geq \alpha$ and $\operatorname{rk}(f) \not\geq \alpha + 1$. We agree that $\infty$ is above all ordinals; hence $\operatorname{rk}(f) = \infty$ if and only if $\operatorname{rk}(f) \geq \alpha + 1$ for every ordinal $\alpha$.

Finally, we define the ***rank*** of $M$ as

$$\operatorname{rk}(M) := \sup\{\operatorname{rk}(f) \colon f \in \operatorname{Fin} M\}.$$

## Remarks

- Note that $\operatorname{rk}(f) = 0$ if and only if some nontrivial transition $t$ from $\operatorname{dom}(f)$ cannot be "realized" in $M$, namely, no $f'$ satisfies $f = f' \circ t$.
- The rank depends heavily on the universe $\mathfrak{V}$. For example, $\mathfrak{V}$ may be a category of all embeddings of some first-order structures, while also $\mathfrak{V}$ may consist of all homomorphisms.
- By regularity, $\operatorname{Fin} M$ is closed under isomorphisms, therefore typically it is a proper class. On the other hand, we shall always assume that $\mathcal{E}$ is ***locally small***, that is, for every finite object $A$ there is a set $\mathcal{S}(A)$ of transitions from $A$ such that every $t \in \mathcal{T}(A)$ is left-isomorphic to some $s \in \mathcal{S}(A)$.
- Note that if $f \colon A \to M$ is a $\mathfrak{V}$-arrow and $h \colon B \to A$ is an isomorphism then $\operatorname{rk}(f) = \operatorname{rk}(f \circ h)$.

An object $M$ is ***countable*** if there exists an evolution with colimit $M$.

## Proposition 1

*Assume $\mathcal{E}$ is locally countable and $M$ is a countable object. Then $\operatorname{rk}(M)$ is either a countable ordinal or $\infty$.*

## Example 1

Let $\mathfrak{V}$ be the category of sets with one-to-one mappings and let $\Theta = \emptyset$. Then $\operatorname{rk}(M) = \infty$ if and only if $M$ is an infinite set, otherwise $\operatorname{rk}(M) = |M|$.

Recall that an object $A$ is ***initial*** in $\mathfrak{V}$ if for every $\mathfrak{V}$-object $X$ there exists a unique $\mathfrak{V}$-arrow from $A$ to $X$.

## Proposition 2

*Assume the origin $\Theta$ is initial in $\mathfrak{V}$ and let $e_M$ denote the unique $\mathfrak{V}$-arrow from $\Theta$ to $M \in \operatorname{Obj}(\mathfrak{V})$. Then $\operatorname{rk}(M) = \operatorname{rk}(e_M)$.*

## Theorem 1 (Universality)

*Let $\mathcal{E}$ be an evolution system and let $M$ be a $\mathfrak{V}$-object. If $\operatorname{rk}(M) = \infty$ then for every countable object $X$ there exists a $\mathfrak{V}$-arrow $g \colon X \to M$.*

## Theorem 2

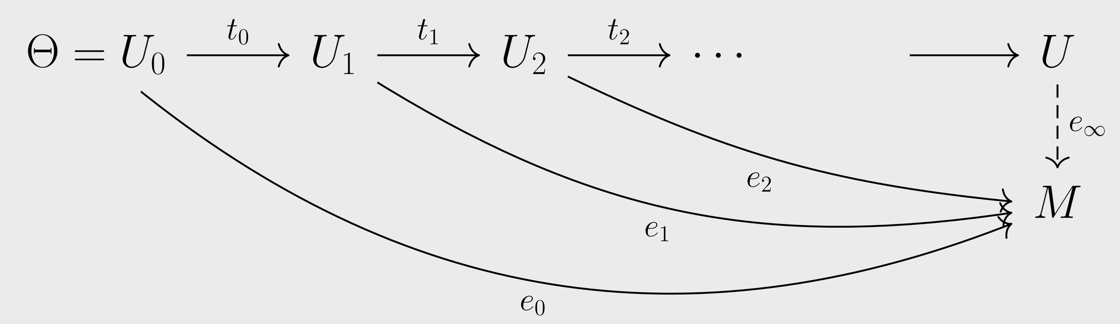
*Assume $\mathcal{E}$ is locally countable and has the transition amalgamation property. Let $U$ be the $\mathcal{E}$-generic object, namely, the colimit of an evolution with the absorption property. Then for every $\mathfrak{V}$-object $M$ the following properties are equivalent.*

(a) $\operatorname{rk}(M) = \infty$.

(b) *There exists a $\mathfrak{V}$-arrow from $U$ to $M$.*

## Proof.

Clearly $\operatorname{rk}(U) = \infty$. Note that $\operatorname{rk}(f) = \infty$ implies that for every transition $t$ with $\operatorname{dom}(t) = \operatorname{dom}(f)$ there exists $f'$ satisfying $f = f' \circ t$ and $\operatorname{rk}(f') = \infty$. The rest of the proof is visualized in the following diagram, where $\operatorname{rk}(e_n) = \infty$ for every $n \in \mathbb{N}$.




□

## General example

Let $\operatorname{Obj}(\mathfrak{V}) = \mathcal{F}$ be a fixed hereditary class of relational structures, closed under unions of chains. Let $\Theta$ be the trivial (empty) structure. We declare $\mathfrak{V}$-arrows to be all possible embeddings. Transitions are all isomorphisms and all one-point extensions. Specifically, an embedding $t \colon X \to Y$ is declared to be a transition if $|Y \setminus t[X]| \leq 1$. We shall call $\mathcal{E}$ the ***natural*** evolution system induced by $\mathcal{F}$.

Note that $\operatorname{Fin} M$ can be identified with all finite substructures of $M$.

Note also that the rank is hereditary, namely, $\operatorname{rk}(N) \leq \operatorname{rk}(M)$ whenever $N$ is a substructure of $M$. More generally, $\operatorname{rk}(N) \leq \operatorname{rk}(M)$ whenever there exists a $\mathfrak{V}$-arrow from $N$ to $M$.

## Example 2: Linearly ordered sets

Let $\mathcal{E}$ be the natural evolution system of embeddings between linearly ordered sets with $\Theta = \emptyset$. Then $\operatorname{rk}(X) = \infty$ if and only if $\mathbb{Q}$ embeds into $X$. In other words, $\operatorname{rk}(X) < \infty$ if and only if $X$ is a scattered linearly ordered set. Hausdorff theorem says that scattered linear orderings can be built recursively starting from ordinals and using two operations: reversing the order, and lexicographic sums along ordinals. This allows defining an ordinal rank, different from ours. For example, if $X$ is finite then $\operatorname{rk}(X) \approx \log |X|$, while on the other hand the rank coming from Hausdorff theorem would be zero for all ordinals. Note also that $\operatorname{rk}(\omega, <) = \omega$, while $\operatorname{rk}(\mathbb{Z}, <) = \omega + 1$.

## Proposition 2

*Let $\mathcal{E}$ be the natural system induced by a hereditary relational class in a finite language, such that finite structures have the amalgamation property. Then for every finite $n$ there is a finite list of finite structures $F_0, \dots, F_{k-1}$ such that $\operatorname{rk}(M) > n$ if and only if $F_i$ embeds into $M$ for some $i < k$.*

**Problem:** Find necessary and sufficient conditions for a Fraïssé class such that countable structures of rank $\leq \alpha < \omega_1$ has a universal element.