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ON PERIODIC SOLUTIONS OF SECOND ORDER FUNCTIONAL DIFFERENTIAL EQUATIONS

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Let $C([a,b])$ be the set of continuous functions $u : [a,b] \to R$ and let $L([a,b])$ be the set of functions $p :]a,b[\to R$ which are Lebesque integrable on $[a,b]$. Moreover, let $F : C([a,b]) \to L([a,b])$ be an operator. We say that F is monotone if for any $u, v \in C([a,b])$ with $u(t) \le v(t)$ $(u(t) \ge v(t))$ for $a \le t \le b$, we have $F(u)(t) \le F(v)(t)$ $(F(u)(t) \ge F(v)(t))$ for $a \le t \le b$ and

$$F(1)\min\{w(t) : a \le t \le b\} \le F(w)(t) \le F(1)\max\{w(t) : a < t < b\}$$
$$\Big(F(1)\max\{w(t) : a \le t \le b\} \le F(w)(t) \le F(1)\min\{w(t) : a < t < b\} \Big)$$

for any $w \in C([a,b])$ taking both positive and negative values.

Under the solution of the equation

$$u''(t) = F(u)(t) \tag{1}$$

we mean a function $u : [a,b] \to R$ which along with its first derivative is absolutely continuous almost everywhere on $[a,b]$ and satisfies (1).

In this paper, we are especially interested in the question whether there exists a solution of (1) satisfying

$$u(a) = u(b), \quad u'(a) = u'(b). \tag{2}$$

Theorem 1. *Let $F : C([a,b]) \to L([a,b])$ be a monotone operator, $F(0)(t) \equiv 0$ and*

$$\int_a^b |F(1)(s)|\,ds < \frac{16}{b-a}.$$

Then the problem (1), (2) *has only the zero solution.*

In the case where (1) is a linear ordinary differential equation, an analogous result (slightly general) can be found in [1].

Theorem 1 covers both linear and nonlinear equations and also those with discontinuous right-hand side. In particular, for the linear equation with deviating argument

$$u''(t) = g(t)u(\tau(t)), \tag{3}$$

where $g \in L([a,b])$, $g \not\equiv 0$, is a function of constant signs and $\tau : [a,b] \to [a,b]$ is measurable, we have

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Corollary 1. *Let*

$$\int_a^b |g(s)|\,ds < \frac{16}{b-a}. \tag{4}$$

Then the problem (3), (2) *has only the zero solution.*

Corollary 1 remains valid for nonlinear equations of the type

$$u''(t) = g(t)\max\{u(s) : \alpha(t) \le s \le \beta(t)\}$$

and

$$u''(t) = g(t)\min\{u(s) : \alpha(t) \le s \le \beta(t)\},$$

where $\alpha, \beta : [a,b] \to [a,b]$ are measurable and $\alpha(t) \le \beta(t)$ for $a \le t \le b$.

Note that the inequality (4) for an equation of the type (3) is exact and it cannot be weakened. Indeed, let $a = -4$, $b = 4$,

$$v(t) = \begin{cases} t^2(19-12t) & \text{for } 0 \le t \le 1 \\ (t-1)(3-t)+7 & \text{for } 1 < t < 3 \\ (4-t)^2(12t-29) & \text{for } 3 \le t \le 4 \end{cases},$$

$$u(t) = \begin{cases} -v(t+4) & \text{for } -4 \le t \le 0 \\ v(t) & \text{for } 0 < t \le 4 \end{cases},$$

$$\tau(t) = \begin{cases} 2 & \text{for } t \in [-3,-1] \cup [0,1] \cup [3,4] \\ -2 & \text{for } t \in [-4,-3[\cup]-1,0[\cup]1,3[\end{cases},$$

$$g(t) = \begin{cases} \frac{1}{8}|v''(t+4)| & \text{for } -4 \le t \le 0 \\ \frac{1}{8}|v''(t)| & \text{for } 0 < t \le 4 \end{cases}.$$

We can easily see that $\int_{-4}^4 g(s)ds = 2$ and u is a periodic solution of (1).

Let us finally consider

$$u''(t) = g(t)T(u)(t), \tag{5}$$

where $g \in L([a,b])$ is the function of constant signs, and $T : C([a,b]) \to L([a,b])$ is the operator defined by the equality

$$T(u)(t) = \begin{cases} u(t) & \text{if } u(t) < 1 \text{ for } a \le t \le b \\ 1 & \text{if for some } t_0 \in [a,b],\ u(t_0) > 1 \end{cases}.$$

According to Theorem 1, if the inequality (4) is fulfilled, then the problem (5),(2) has only the zero solution. It is not difficult to construct an example from which one could easily see that in this case inequality (4) cannot be also weakened.

References

1. A. Lasota and Z. Opial, Sur les solutions périodiques des équations differentielles ordinaires. *Ann. Pol. Math.* **16**(1964), No. 1, 69–94.

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