DISCONJUGACY AND DISFOCALITY CRITERIA FOR SECOND ORDER SINGULAR HALF-LINEAR DIFFERENTIAL EQUATIONS

Ondřej Došlý, Alexander Lomtatidze

Abstract. We establish Vallée Poussin type disconjugacy and disfocality criteria for half-linear second order differential equation

$$u'' = p(t)|u|^{\alpha}|u'|^{1-\alpha}\operatorname{sgn} u + g(t)u',$$

where $\alpha \in (0, 1]$ and the functions $p, g \in L_{\mathrm{loc}}(a, b)$ are allowed to have singularities at the end points $t = a$, $t = b$ of the interval under consideration.

1. Introduction.

In this paper we investigate oscillatory properties and solvability of two-points BVP's associated with the half-linear second order differential equation

$$u'' = p(t)|u|^{\alpha}|u'|^{1-\alpha}\operatorname{sgn} u + g(t)u', \tag{1.1}$$

where $\alpha \in (0, 1]$, $-\infty < a < b < \infty$, and the functions $p, g \in L_{\mathrm{loc}}(a, b)$ are allowed to have singularities at the end points $t = a$, $t = b$. Multiplying (1.1) by $\alpha|u'|^{1-\alpha}$ we get the half-linear equation in the form more frequently considered in the literature

$$(\Phi(u'))' = \alpha p(t)\Phi(u) + \alpha g(t)\Phi(u'), \quad \Phi(s) := |s|^{\alpha}\operatorname{sgn} s. \tag{1.2}$$

However, if $\alpha \neq 1$, equation (1.1) is of a slightly different kind than (1.2) and solution sets of (1.1) and (1.2) are not identical. Indeed, suppose that $a \le t_1 < t_2 < t_3 < t_4 \le b$ and that there exist solutions u, v of (1.1) such that $u(t_1) = 0 = u'(t_2)$, $v'(t_3) = 0 = v(t_4)$, moreover, since the solution space of (1.1) is homogeneous, we can suppose that $u(t_2) = v(t_3)$. Now, the function

$$\tilde u(t) = \begin{cases} u(t), & t_1 \le t \le t_2, \\ u(t_2) & t_2 \le t \le t_3, \\ v(t) & t_3 \le t \le t_4 \end{cases}$$

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is a solution of (1.1) but not of (1.2). This means, in particular, that the Sturmian separation theorem (which holds for (1.2)) does not extend to (1.1), i. e., (1.1) in general admits coexistence of a nonzero solution with a solution having two or more zeros in an interval under consideration.

Oscillation theory of (1.1) and (1.2) attracted a considerable attention in the recent years, see e. g. [2,5-7,12-16] and the references given therein. It was shown that oscillatory properties of (1.1), (1.2) are similar to those of the linear equation

$$u'' = p(t)u + g(t)u' \tag{1.3}$$

and methods like variational and Riccati technique, which are typical in linear oscillation theory, extend with minor modifications also to half-linear equations.

In this paper we prefer to write half-linear equations in the form (1.1) since then this equation can be regarded as a particular case of the nonlinear second order equation $u'' = f(t, u, u')$. The assumption $\alpha \in (0, 1]$ is enforced by the fact that for $\alpha > 1$ the right-hand-side of (1.1) becomes singular in the phase variable u'.

In 1929 de la Vallée Poussin [21] proved that the second order linear equation (1.3) (with g, p continuous on $[a, b]$) is disconjugate in the interval $[a, b]$, provided

$$b - a < 2\int_0^\infty \frac{dt}{1 + At + Bt^2}, \quad A = \max_{t\in[a,b]} |g(t)|, \ B = \max_{t\in[a,b]} |p(t)| \tag{1.4}$$

and as a consequence of (1.4) he showed that two consecutive zeros a, b of a non-trivial solution of (1.3) has to satisfy

$$2A(b-a) + \frac{1}{2}B(b-a)^2 > 1.$$

The last condition (given in the monograph of Sansone [20] as the "real" Vallée Poussin criterion) has been extended in various directions, see [9,19,22] and also the survey paper [1] containing a comprehensive bibliography about the problem. Finally, because of "misinterpretation" of the original result of Vallée Poussin (1.4), this condition was rediscovered by Cohn in [3]. Note also that an interesting treatment of Vallée-Poussin's condition (1.4) can be found in the paper of Opial [19].

Our main result, Theorem 3.1, concerns Vallée Poussin's type criterion for (1.1), where the coefficients g, p are allowed to be singular at the end points of the interval under consideration. Moreover, in contrast to the previous criteria of this kind, we use only one-side estimates of the functions g, p.

The paper is organized as follows. In the next section we formulate some auxiliary results concerning unique solvability of (1.1) with singular end points a, b and disconjugacy of this equation in terms of solvability of certain second order differential inequality. Section 3 is devoted to the main results of the paper – Vallée Poussin type disconjugacy and disfocality criteria for (1.1), whereas the last section contains some remarks concerning these statements.

2. Auxiliary results.

Throughout the paper we use the following notation: $\tilde{C}^1_{\mathrm{loc}}(a, b)$ denotes the set of functions $u : (a, b) \to \mathbf{R}$, such that u, u' are absolutely continuous on each compact subinterval of (a, b), $L(a, b)$ is the usual space of Lebesque integrable functions on (a, b).

Under the solution of equation (1.1) we understand a function $u \in \tilde{C}^1_{\text{loc}}(a, b)$ which satisfies (1.1) almost everywhere in (a, b). It is easy to see that if u is a solution of (1.1) then λu is a solution as well. The fact that the solution space of (1.1) has just one half of linearity (homogenity but not additivity) justifies the the terminology "half-linear equation" for (1.1).

Observe that if we denote

$$r(t) = \exp\left\{\int_{\frac{a+b}{2}}^{t} g(s)\, ds\right\}$$

then (1.1) can be written in the form

$$\left(\frac{u'}{r(t)}\right)' = \frac{p(t)}{r(t)}|u|^{\alpha}|u'|^{1-\alpha}\operatorname{sgn} u, \tag{2.1}$$

In this paper we investigate the problem of the existence of a nontrivial solution of (1.1) satisfying the boundary conditions

$$u(a_1+) = 0 = u(b_1-), \tag{2.2}$$

$$u(a_1+) = 0 = u'(b_1-) \tag{2.3}$$

for some $a_1, b_1 \in [a, b]$, $a_1 < b_1$. In our investigation we do not exclude the case when both functions p, q have singularities either at a and/or b. In particular, problem (1.1), (2.2) we will investigate under the assumptions

$$r(t) \in L(a, b), \quad p(t)\left(\frac{1}{r(t)}\int_a^t r(s)\, ds \cdot \int_t^b r(s)\, ds\right)^{\alpha} \in L(a, b) \tag{2.4}$$

and problem (1.1), (2.3) in case

$$r(t) \in L(a, b), \quad p(t)\left(\frac{\int_a^t r(s)\, ds}{r(t)}\right)^{\alpha} \in L(a, b). \tag{2.5}$$

Observe that (2.4) holds e.g. if

$$|p(t)| \le \frac{C}{[(t-a)(b-t)]^{1+\varepsilon}}, \quad |g(t)| \le C + \delta\left[\frac{1}{t-a} + \frac{1}{b-t}\right]$$

and condition (2.5) holds if

$$|p(t)| \le \frac{C}{(t-a)^{1+\varepsilon}}, \quad |g(t)| \le C + \frac{\delta}{t-a},$$

where $C > 0$, $\varepsilon \in (0, \alpha)$, $\delta \in [0, 1)$. Note also that condition (2.4) guarantees that each solution u of (1.1) has finte (one-side) limits at the end points a, b and that (2.5) implies that $\lim_{t \to b-} u'(t)$ exists and it is finite, see [2], Lemma 2.2.

Now we will show that if the integrability conditions (2.4) hold then the zero initial conditions determine trivial solution of (1.1).

Lemma 2.1. *Suppose that (2.4) holds and $a_1 \in [a, b)$ ($b_1 \in (a, b]$). Then the trivial solution $u \equiv 0$ is the only solution of (1.1) satisfying the condition*

$$u(a_1+) = 0, \quad \liminf_{t\to a_1+} \left|\frac{u'(t)}{r^\alpha(t)}\right| = 0, \quad \left(u(b_1-) = 0, \quad \liminf_{t\to b_1-} \left|\frac{u'(t)}{r^\alpha(t)}\right| = 0\right).$$

Proof. See [2, Lemma 2.8].

Lemma 2.2. *Suppose that (2.4) holds and u is a nontrivial solution of (1.1) satisfying*

$$u(a+) = 0 \quad \big(u(b-) = 0\big).$$

Then for any continuously differentiable bounded function $v : (a, b) \to \mathbb{R}$ we have

$$\liminf_{t\to a+} \left|\frac{v'(t)u(t)}{u'(t)}\right| = 0, \quad \left(\liminf_{t\to b-} \left|\frac{v'(t)u(t)}{u'(t)}\right| = 0\right). \tag{2.6}$$

Proof. Suppose, by contradiction, that the statement fails to hold. Then by Lemma 2.1 and (2.6) there exists $\varepsilon > 0$ and $a_0 \in (a, b)$ such that

$$|u(t)| \neq 0 \quad \text{and} \quad |v'(t)| > \varepsilon \left|\frac{u'(t)}{u(t)}\right| \quad \text{for } a < t < a_0.$$

Hence

$$|v(t)| > \int_t^{a_0} |v'(s)|\, ds - |v(a_0)| > \varepsilon \lg \frac{u(a_0)}{u(t)} - |v(a_0)|$$

for $a < t < a_0$ which contradicts to the boundedness of v. The proof is complete. □

Now we turn our attention to disconjugacy and disfocality criteria for (1.1).

Definition. Equation (1.1) is said to be *disconjugate* in the interval $[a, b]$ if does not possess a nontrivial solution satisfying $u(a_1+) = 0 = u(b_1-)$ for any $a_1, b_1 \in [a, b]$, $a_1 < b_1$, and it is said to be *left (right) disfocal* in $[a, b]$ if (1.1) does not possess a nontrivial solution satisfying $u(a_1+) = 0 = u'(b_1+)$ ($u'(a_1+) = 0 = u(b_1-)$) for any $a_1, b_1 \in [a, b]$, $a_1 < b_1$.

Lemma 2.3. *Let (2.4) hold and suppose that there exist $c \in [a, b]$ and a bounded positive function $v \in \tilde{C}^1_{\text{loc}}(a, b)$ with finite limits $v(a+) > 0$, $v(b-) > 0$ such that $v'(c) = 0$ and*

$$v'' \le p(t)|v(t)|^\alpha |v'(t)|^{1-\alpha} + g(t)v'(t) \quad \textit{for } a < t < b,$$
$$v'(t)(c - t) > 0 \quad \textit{for } a < t < b,\ t \neq c.$$

Then (1.1) is disconjugate in [a,b].

Proof. We will prove the statement in case $c \in (a, b)$. When $c = a$ or $c = b$ we would proceed in a similar way. Suppose, by contradiction, that u is a nontrivial solution of (1.1) having at least two zeros in $[a, b]$, i.e., there exist $a_1 \in [a, b)$ and $b_1 \in (a, b]$ such that $u(a_1+) = 0 = u(b_1-)$. Then there exists $t_1, t_1 \in (a_1, b_1)$, $t_1 \le t_2$, such that $u'(t_1) = 0$, $u'(t_2) = 0$ and

$$u'(t) > 0 \text{ for } a_1 < t < t_1, \quad u'(t) < 0 \text{ for } t_2 < t < b_1.$$

Define the functions ρ_1 and ρ_2 by

$$\rho_1(t) = \left|\frac{u'(t)}{u(t)r(t)}\right|^\alpha, \quad \rho_2(t) = \left|\frac{v'(t)}{v(t)r(t)}\right|^\alpha.$$

By a direct computation one can verify that

$$\begin{aligned} \rho_1'(t) &= \alpha\left(\frac{p(t)}{r^\alpha(t)} - r(t)\rho_1^{\frac{\alpha+1}{\alpha}}(t)\right)\operatorname{sgn}\frac{u'(t)}{u(t)}, \quad t\in(a_1,t_1)\cup(t_2,b_1), \\ \rho_2'(t) &\le \alpha\left(\frac{p(t)}{r^\alpha(t)} - r(t)\rho_2^{\frac{\alpha+1}{\alpha}}(t)\right)\operatorname{sgn} v'(t), \quad a_1<t<b_1. \end{aligned} \tag{2.7}$$

In addition, by Lemma 2.1 and Lemma 2.2 we have

$$\rho_1(a_1+)=\infty,\ \rho_1(b_1-)=\infty, \quad \liminf_{t\to a_1+}\frac{\rho_2(t)}{\rho_1(t)} = 0 = \liminf_{t\to b_1-}\frac{\rho_2(t)}{\rho_1(t)}. \tag{2.8}$$

Suppose first that $t_1<c$. Then by (2.8) there exist $\tau_1\in(a_1,t_1)$ and $\varepsilon\in(0,t_1-\tau_1)$ such that

$$\rho_1(t)<\rho_2(t) \text{ for } \tau_1<t<\tau_1+\varepsilon, \quad \rho_1(\tau_1)=\rho_2(\tau_1). \tag{2.9}$$

Taking into account (2.9), (2.7) implies

$$\begin{aligned} \rho_1(t) &= \rho_1(\tau_1) + \int_{\tau_1}^t \frac{p(s)}{r^\alpha(s)}\,ds - \int_{\tau_1}^t r(s)\rho_1^{\frac{\alpha+1}{\alpha}}(s)\,ds \\ &> \rho_2(\tau_1) + \int_{\tau_1}^t \frac{p(s)}{r^\alpha(s)}\,ds - \int_{\tau_1}^t r(s)\rho_2^{\frac{\alpha+1}{\alpha}}(s)\,ds \ge \rho_2(t) \end{aligned}$$

for $\tau_1<t<\tau_1+\varepsilon$ which contradicts to (2.9), hence $t_1\ge c$.

Now suppose that $t_1=c$. By the same argument as above there exist $c_1\in(a,c]$ and $\varepsilon\in(0,c_1-a_1)$ such that

$$\rho_2(t)<\rho_1(t) \quad \text{for } c_1-\varepsilon<t<c_1, \quad \rho_1(c_1)=\rho_2(c_1). \tag{2.10}$$

Denote $w(t)=\rho_1(t)-\rho_2(t)$ and

$$\varphi(x,y) = \begin{cases} \frac{1}{|x|-|y|}\left(|x|^{\frac{\alpha+1}{\alpha}} - |y|^{\frac{\alpha+1}{\alpha}}\right) & \text{for } |x|\ne|y| \\ 0 & \text{for } |x|=|y|. \end{cases}$$

Then

$$w'(t)\ge -\alpha r(t)w(t)\varphi(\rho_1(t),\rho_2(t)), \quad \text{for } c_1-\varepsilon<t<c_1. \tag{2.11}$$

Inequality (2.11) implies that for $t\in(c_1-\varepsilon,c_1)$

$$w(t)\le w(c_1)\exp\left[\alpha\int_t^{c_1} r(s)\varphi(\rho_1(s),\rho_2(s))\,ds\right].$$

Since $w(c_1)=0$, the last inequality contradicts to (2.10). Consequently, we have proved that $t_1>c$. In the same way one can prove that $t_2<c$, but this impossible, since $t_1\le t_2$. The proof is now complete. □

Remark 2.2. The statement of Lemma 2.3 remains valid if $c\in(a,b)$, $v\in\tilde{C}^1_{\rm loc}(a,c)\cup\tilde{C}^1_{\rm loc}(c,b)$ and $v'(c-)\ge 0\ge v'(c+)$.

In the same way as in Lemma 2.3 one can prove the following disfocality criterion for (1.1).

Lemma 2.4. *Suppose that there exists a bounded function $v \in \tilde{C}^1_{\rm loc}(a,b)$ with finite limits $v(a+) > 0$, $v'(b-) > 0$, $v'(t) > 0$ on (a,b), and satisfying the inequality*

$$v''(t) \le p(t)v^{\alpha}(t)\left(v'(t)\right)^{1-\alpha} + g(t)v(t), \quad a < t < b. \tag{2.12}$$

Then the boundary value problem (1.1), (2.3) has only trivial solution for every $a_1, b_1 \in [a,b]$.

Remark 2.3. In the same way as in Lemma 2.4 we can prove that the boundary value problem (1.1), $u'(a+) = 0$, $u(b-) = 0$ has no nontrivial solution provided there exists a bounded function $v \in \tilde{C}^1_{\rm loc}(a,b)$ such that $v'(a+) < 0$, $v(b-) > 0$, $v'(t) < 0$ and (2.12) holds.

3. Main results.

The main results of our paper, the Vallée Poussin type disconjugacy and disfocality criteria for (1.1), read as follows.

Theorem 3.1. *Suppose that (2.4) holds, there exist $c \in [a,b]$, $l_{ij} \ge 0$, $i,j = 1,2$, and differentiable functions β, γ such that $\beta(t) > 0$ for $t \in (a,c)$, $\gamma(t) > 0$ for $t \in (c,b)$,*

$$\beta^{-\frac{1}{\alpha}} \in L(a,c), \quad \gamma^{-\frac{1}{\alpha}} \in L(c,b)$$

and

$$\int_0^\infty \frac{ds}{l_{11} + l_{12}s + s^{\frac{\alpha+1}{\alpha}}} > \alpha \int_a^c \beta^{-\frac{1}{\alpha}}(s)\,ds, \tag{3.1$_1$}$$

$$\int_0^\infty \frac{ds}{l_{21} + l_{22}s + s^{\frac{\alpha+1}{\alpha}}} > \alpha \int_c^b \gamma^{-\frac{1}{\alpha}}(s)\,ds, \tag{3.1$_2$}$$

where

$$\beta^{\frac{\alpha+1}{\alpha}}(t)p(t) \ge -l_{11}, \quad \beta^{\frac{1}{\alpha}}(t)\left[g(t) + \frac{\beta'(t)}{\alpha\beta(t)}\right] \ge -l_{12}, \quad a < t < c, \tag{3.2$_1$}$$

$$\gamma^{\frac{\alpha+1}{\alpha}}(t)p(t) \ge -l_{21}, \quad \gamma^{\frac{1}{\alpha}}(t)\left[g(t) + \frac{\gamma'(t)}{\alpha\gamma(t))}\right] \le l_{22}, \quad c < t < b. \tag{3.2$_2$}$$

Then BVP (1.1), (2.2) has only the trivial solution for any $a \le a_1 < b_1 \le b$, i.e., (1.1) is disconjugate in $[a,b]$.

Proof. We prove the statement in case $c \in (a,b)$, the cases $c = a$ or $c = b$ may treated in the same way. Without loss of generality we can suppose that both $l_{11} \ne 0$, $l_{21} \ne 0$. Let $\gamma_1 > 0$ and $\gamma_2 > 0$ be such that

$$\int_0^{\gamma_1} \frac{ds}{l_{11} + l_{12}s + s^{\frac{\alpha+1}{\alpha}}} = \alpha \int_a^c \beta^{-\frac{1}{\alpha}}(s)\,ds,$$
$$\int_0^{\gamma_2} \frac{ds}{l_{21} + l_{22}s + s^{\frac{\alpha+1}{\alpha}}} = \alpha \int_c^b \gamma^{-\frac{1}{\alpha}}(s)\,ds.$$

Define the functions ρ_1, ρ_2 (implicitly) by the equalities

$$\int_{\rho_1(t)}^{\gamma_1} \frac{ds}{l_{11} + l_{12}s + s^{\frac{\alpha+1}{\alpha}}} = \alpha \int_a^t \beta^{-\frac{1}{\alpha}}(s)\, ds, \quad a < t \le c,$$

$$\int_{\rho_2(t)}^{\gamma_2} \frac{ds}{l_{21} + l_{22} + s^{\frac{\alpha+1}{\alpha}}} = \alpha \int_t^b \gamma^{-\frac{1}{\alpha}}(s)\, ds, \quad c \le t < b.$$

Obviously, $\rho_1(t) > 0$ for $a < t < c$, $\rho_1(a+) = \gamma_1$, $\rho_1(c) = 0$, $\rho_2(t) > 0$ for $c < t < b$, $\rho_2(b-) = \gamma_2$, $\rho_2(c) = 0$. Let

$$v(t) = \begin{cases} \exp\left[-\int_t^c \beta^{-\frac{1}{\alpha}}(s)\rho_1^{\frac{1}{\alpha}}(s)\, ds\right] & \text{for } a < t \le c \\ \exp\left[-\int_c^t \gamma^{-\frac{1}{\alpha}}(s)\rho_2^{\frac{1}{\alpha}}(s)\, ds\right] & \text{for } c < t < b. \end{cases} \tag{3.3}$$

Then

$$v(t) > 0 \ \text{ for } \ a < t < b, \ v'(t)(c-t) > 0 \ \text{ for } \ t \in (a,c) \cup (c,b), \ v'(c) = 0,$$

there exist finite limits $v(a+) > 0$, $v(b-) > 0$ and we have for $t \in (a,c)$

$$v'(t) = \left(\frac{\rho_1(t)}{\beta(t)}\right)^{\frac{1}{\alpha}} v(t), \quad v''(t) = v(t)\left(\frac{\rho_1(t)}{\beta(t)}\right)^{\frac{2}{\alpha}} + v(t)\frac{1}{\alpha}\left(\frac{\rho_1(t)}{\beta(t)}\right)^{\frac{1}{\alpha}-1}\left(\frac{\rho_1(t)}{\beta(t)}\right)'.$$

Consequently,

$$\begin{aligned} &v''(t) - p(t)v^{\alpha}|v'(t)|^{1-\alpha} - g(t)v'(t) \\ &= \frac{1}{\alpha}\left(\frac{\rho_1(t)}{\beta(t)}\right)^{\frac{1}{\alpha}-1} v(t)\left[\left(\frac{\rho_1(t)}{\beta(t)}\right)' + \alpha\left(\frac{\rho_1(t)}{\beta(t)}\right)^{\frac{\alpha+1}{\alpha}} - \alpha p(t) - \alpha g(t)\frac{\rho_1(t)}{\beta(t)}\right]. \end{aligned}$$

The estimates (3.2) now implies that

$$v''(t) \le p(t)|v(t)|^{\alpha}|v'(t)|^{1-\alpha} + g(t)v'(t), \quad a < t < c,$$

the same inequality holds also for $c < t < b$ as can be verified by a direct computation. Consequently, by Lemma 2.3 every solution of (1.1) has at most one zero in $[a,b]$. The proof is complete. $\square$

Remark 3.1. A closer examination of the proof of the previous theorem reveals that this statement is actually a "double disfocality criterion". Indeed, (3.1_1), (3.2_1) imply that (1.1) is left disfocal in $[a,c]$ and (3.1_2), (3.2_2) imply right disfocality in $[c,b]$. This, in turn, implies disconjugacy of (1.1) in $[a,b]$.

This observation immediately suggests the following (left) disfocality criterion.

Theorem 3.2. *Suppose that (2.5) holds and there exists a differentiable function β satisfying $\beta(t) > 0$ for $t \in (a,b)$, $\beta^{-\frac{1}{\alpha}} \in L(a,b)$, such that*

$$\int_0^{\infty} \frac{dt}{l_{11} + l_{12}s + s^{\frac{\alpha+1}{\alpha}}} > \alpha \int_a^b \beta^{-\frac{1}{\alpha}}(s)\, ds,$$

where

$$\beta^{\frac{\alpha+1}{\alpha}}(t)p(t) \ge -l_{11}, \quad \beta^{\frac{1}{\alpha}}(t)\left[g(t) + \frac{\beta'(t)}{\beta(t)}\right] \ge -l_{12}, \quad a < t < b.$$

Then the BVP (1.1), (2.3) has only the trivial solution, i.e., (1.1) is left disfocal in $[a,b]$.

Typical example of functions β, γ which meet assumptions of Theorem 3.1 are specified in the following corollary.

Corollary 3.1. *Let* $\lambda_1, \lambda_2 \in [0, \alpha)$, $c \in [a, b]$, $l_{ij} \geq 0$, $i, j = 1, 2$, *and*

$$\begin{aligned} &\int_0^\infty \frac{dt}{l_{11} + l_{12}s + s^{\frac{\alpha+1}{\alpha}}} > \frac{\alpha^2}{\alpha - \lambda_1}(c-a)^{\frac{\alpha-\lambda_1}{\alpha}}, \\ &\int_0^\infty \frac{dt}{l_{21} + l_{22}s + s^{\frac{\alpha+1}{\alpha}}} > \frac{\alpha^2}{\alpha - \lambda_2}(b-c)^{\frac{\alpha-\lambda_2}{\alpha}}, \end{aligned} \tag{3.4}$$

where

$$(t-a)^{\lambda_1 \frac{\alpha+1}{\alpha}} p(t) \geq -l_{11}, \quad (t-a)^{\frac{\lambda_1}{\alpha}} \left[g(t) + \frac{\lambda_1}{\alpha(t-a)} \right] \geq -l_{12}, \quad a < t < c,$$
$$(b-t)^{\lambda_2 \frac{\alpha+1}{\alpha}} p(t) \geq -l_{21}, \quad (b-t)^{\frac{\lambda_2}{\alpha}} \left[g(t) - \frac{\lambda_2}{\alpha(b-t)} \right] \leq l_{22}, \quad c < t < b.$$

Then (1.1) is disconjugate in $[a, b]$.

Proof. Substitute $\beta(t) = (t-a)^{\lambda_1}$, $\gamma(t) = (b-t)^{\lambda_2}$ in Theorem 3.1. □

4. Remarks and Comments.

In this final section we give some remarks concerning the results of the previous section.

(i) The Vallée Poussin criterion (1.4) is a special case of Theorem 3.1. Indeed, it suffices to substitute $c = \frac{b+a}{2}$, $\beta \equiv 1 \equiv \gamma$, $\alpha = 1$ and take into account that

$$\int_0^\infty \frac{dt}{1 + Mt + Nt^2} = \int_0^\infty \frac{dt}{N + Mt + t^2}.$$

Theorem 3.1 also extends the results of [10], Theorem 4.4 and [11], Lemma 1.7 to half-linear equations.

(ii) Harris [8] proved that (1.3) is disconjugate in $[a, b]$ if there exists a differentiable function μ such that (1.4) holds whereby A, B are given by

$$A = \max_{t \in [a,b]} |g(t) + 2\mu(t)|, \quad B = \max_{t \in [a,b]} |p(t) - g(t)\mu(t) - \mu^2(t) + \mu'(t)|.$$

Substituting $\mu \equiv 0$ we get (1.4). The criterion of Harris is essentially based on the Riccati substitution $\rho = \frac{y'}{y} + \mu$ whereas our criterion from Theorem 3.1 is based on the substitution

$$\rho(t) = \beta(t) \left(\frac{y'(t)}{y(t)} \right)^\alpha.$$

This suggests to investigate disconjugacy of (1.1) using the combined substitution

$$\rho(t) = \beta(t) \left(\frac{y'(t)}{y(t)} \right)^\alpha + \mu(t).$$

We have tried to follow this idea, but till now we have obtained no result analogous to those of Harris.

(iii) The strict inequalities in (3.1) cannot be replaced by equalities as shows the following example. Let $c \in (a, b)$, λ_1, λ_2 and l_{12}, $i, j = 1, 2$, be the same as in Corollary 3.1 and

$$\int_0^\infty \frac{ds}{l_{11} + l_{12}s + s^{\frac{\alpha+1}{\alpha}}} = \frac{\alpha^2}{\alpha - \lambda_1}(c-a)^{\frac{\alpha-\lambda_1}{\alpha}},$$
$$\int_0^\infty \frac{ds}{l_{21} + l_{22}s + s^{\frac{\alpha+1}{\alpha}}} = \frac{\alpha_2}{\alpha - \lambda_2}(b-c)^{\frac{\alpha-\lambda_2}{\alpha}}.$$

Define the functions ρ_1, ρ_2 by

$$\int_{\rho_1(t)}^\infty \frac{ds}{l_{11} + l_{12}s + s^{\frac{\alpha+1}{\alpha}}} = \frac{\alpha^2}{\alpha - \lambda_1}(c-a)^{\frac{\alpha-\lambda_1}{\alpha}}, \quad a < t \le c,$$
$$\int_{\rho_2(t)}^\infty \frac{ds}{l_{21} + l_{22} + s^{\frac{\alpha+1}{\alpha}}} = \frac{\alpha_2}{\alpha - \lambda_2}(b-c)^{\frac{\alpha-\lambda_2}{\alpha}}, \quad c \le t < b.$$

Then

$$\rho_1(a+) = \infty,\ \rho_2(b-) = \infty,\ \rho_1(c) = 0,\ \rho_2(c) = 0,$$
$$\rho_1(t) > 0 \text{ for } a < t < c, \quad \rho_2(t) > 0 \text{ for } c < t < b.$$

Now it is not difficult to verify that there exists $\epsilon > 0$ (sufficiently small) and $\delta \in (0,1)$ such that

$$\frac{\alpha^2}{\alpha - \lambda_1}(t-a)^{\frac{\alpha-\lambda_1}{\alpha}} > \delta \int_{\rho_1(t)}^\infty \frac{ds}{s^{\frac{\alpha+1}{\alpha}}} = \alpha\delta\rho_1^{-\frac{1}{\alpha}}(t) \quad \text{for } a < t < a + \epsilon,$$
$$\frac{\alpha^2}{\alpha - \lambda_2}(b-t)^{\frac{\alpha-\lambda_2}{\alpha}} > \alpha\delta\rho_2^{-\frac{1}{\alpha}}(t) \quad \text{for } b - \epsilon < t < b.$$

Consequently,

$$\lim_{t\to a+} \int_t^c (s-a)^{-\frac{\lambda_1}{\alpha}} \rho_1^{\frac{1}{\alpha}}(s)\, ds = \infty = \lim_{t\to b-} \int_c^t (b-s)^{-\frac{\lambda_2}{\alpha}} \rho_2^{\frac{1}{\alpha}}(s)\, ds. \tag{4.1}$$

Directly one can verify that the function v given by (3.3) (with $\beta(t) = (t-a)^{\lambda_1}$, $\gamma(t) = (b-t)^{\lambda_2}$) is a solution of (1.1) with

$$p(t) = -l_{11}(t-a)^{-\lambda_1\frac{1+\alpha}{\alpha}}, \quad g(t) = \left(-l_{12} - \frac{\lambda_1}{\alpha(t-a)}\right)^{-\frac{\lambda_1}{\alpha}} \quad \text{for } a < t < c,$$
$$p(t) = -l_{21}(b-t)^{-\lambda_2\frac{\alpha+1}{\alpha}}, \quad g(t) = \left(l_{22} + \frac{\lambda_2}{\alpha(b-t)}\right)^{-\frac{\lambda_2}{\alpha}} \quad \text{for } c < t < b.$$

On the other hand, by (4.1) $v(a+) = 0, v(b-) = 0$, hence (1.1) is not diconjugate in $[a, b]$.

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Address:
Ondřej Došlý,

Mathematical Institute,
Czech Academy of Sciences,
Žižkova 22,
CZ-616 62 Brno,
Czech Republic.

Alexander Lomtatidze,
Department of Mathematics,
Masaryk University,
Janáčkovo nám. 2a,
CZ-662 95 Brno,
Czech Republic.