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ON PERIODIC SOLUTIONS OF FIRST ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS OF NON–VOLTERRA'S TYPE

Abstract. Nonimprovable effective sufficient conditions for the existence and uniqueness of an ω-periodic solution of the equation

$$u'(t) = f(u)(t),$$

where $f : C_\omega(R) \to L_\omega(R)$ is a continuous operator satisfying the Carathéodory conditions, are established.

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რეზიუმე. დადგენილია არაგაუმჯობესებადი ეფექტური საკმარისი პირობები

$$u'(t) = f(u)(t),$$

ამოცანის ω-პერიოდული ამონახსნის არსებობისა და ერთადერთობისათვის, სადაც $f : C_\omega(R) \to L_\omega(R)$ უწყვეტი ოპერატორია და აკმაყოფილებს კარათეოდორის პირობებს.

Introduction

Consider the problem on the existence and uniqueness of an ω-periodic solution of the scalar functional differential equation

$$u'(t) = f(u)(t), \tag{0.1}$$

where $f : C_\omega(R) \to L_\omega(R)$ is a continuous operator. In the case, where f is a Volterra operator, the problem has already been studied enough (see [1–3,6–25] and references therein). There are also a lot of interesting results concerning the general boundary value problems even in the case, where f is not a Volterra operator (see, e.g., [6–10,14,16–21,24]). However, in that case only a few effective sufficient conditions of the existence and uniqueness of ω-periodic solutions are known. In the present paper, we try to fill this gap in a certain way. More precisely, below we establish the nonimprovable in some sense existence and uniqueness conditions.

Along with (0.1) we will consider an important special case, where (0.1) is the equation with deviating arguments, i.e.,

$$u'(t) = g(t, u(t), u(\mu_1(t)), \dots, u(\mu_n(t))), \tag{0.2}$$

where the function $g : R \times R^{n+1} \to R$ is ω-periodic with respect to the first variable and satisfies the Carathéodory conditions, and $\mu_k : R \to R$ $k = \overline{1,n}$ are measurable functions.

Throughout the paper, the following notation and terms are used.

R is the set of real numbers, $R_+ = [0, +\infty[$.

$[x]_+ = \frac{1}{2}(|x| + x)$, $[x]_- = \frac{1}{2}(|x| - x)$.

$C([a, a+\omega]; R)$ is the Banach space of continuous functions $u : [a, a+\omega] \to R$ with the norm

$$\|u\|_C = \max\{|u(t)| : a \le t \le a+\omega\}.$$

$C_\omega(R)$ is the Banach space of continuous ω-periodic functions $u : R \to R$ with the norm

$$\|u\|_{C_\omega} = \max\{|u(t)| : 0 \le t \le \omega\}.$$

$C_\omega(R_+) = \{u \in C_\omega(R) : u(t) \ge 0 \text{ for } t \in R\}$.

$\widetilde{C}_\omega(R)$ is the set of absolutely continuous ω-periodic functions $u : R \to R$.

$\widetilde{C}(I; D)$, where $I \subset R$, $D \subseteq R$, is the set of absolutely continuous functions $u : I \to D$.

$L(]a, a+\omega[; R)$ is the Banach space of Lebesgue integrable functions $p :]a, a+\omega[\to R$ with the norm

$$\|p\|_L = \int_a^{a+\omega} |p(s)|ds.$$

$L_\omega(R)$ is the Banach space of ω-periodic Lebesgue integrable functions $p: R \to R$ with the norm

$$\|p\|_{L_\omega} = \int_0^\omega |p(s)|ds.$$

$L_\omega(R_+) = \{p \in L_\omega(R) : p(t) \geq 0 \ \text{ for almost all } \ t \in R\}$.

$\mathcal{L}_\omega(R)$ is the set of linear bounded operators $\ell : C_\omega(R) \to L_\omega(R)$ such that

$$\sup\{|\ell(v)(\cdot)| : \|v\|_{C_\omega} = 1\} \in L_\omega(R_+).$$

$\mathcal{P}_\omega(R)$ is the set of linear operators $\ell \in \mathcal{L}_\omega(R)$ transforming $C_\omega(R_+)$ into $L_\omega(R_+)$.

It is obvious that for any $x \in [0, \omega[$ the operator $\ell \in \mathcal{P}_\omega(R)$ uniquely defines the corresponding operator

$$\widetilde{\ell}_x : \{u \in C([x, x+\omega]; R) : u(x) = u(x+\omega)\} \to L(]x, x+\omega[; R).$$

In the sequel, we will assume that the linear bounded operator

$$\ell_x : C([x, x+\omega]; R) \to L(]x, x+\omega[; R)$$

is the extension of the operator $\widetilde{\ell}_x$. Furthermore we will assume that ℓ_x is a nonnegative operator, i.e., it transforms $C([x, x+\omega]; R_+)$ into $L(]x, x+\omega[; R_+)$.

In the case

$$\ell(v)(t) \stackrel{\text{def}}{=} \sum_{k=0}^n p_k(t) v(\tau_k(t)),$$

we will assume that $\sum\limits_{k=0}^n |p_k(t)| \not\equiv 0$ and

$$\ell_{1x}(v)(t) \stackrel{\text{def}}{=} \sum_{k=0}^n [p_k(t)]_+ v(\tau_{kx}(t)), \qquad \ell_{2x}(v)(t) \stackrel{\text{def}}{=} \sum_{k=0}^n [p_k(t)]_- v(\tau_{kx}(t)),$$

where $\tau_{kx}(t) = \tau_k(t) - \omega\eta_{kx}(t)$ for $t \in]x, x+\omega[$, and $\eta_{kx}(t)$ is the integer part of the number $\frac{1}{\omega}(\tau_k(t) - x)$.

We say that the operator $f : C_\omega(R) \to L_\omega(R)$ satisfies the Carathéodory conditions if it is continuous and

$$f_r^*(\cdot) = \sup\{|f(u)(\cdot)| : \|u\|_C \leq r\} \in L_\omega(R_+) \qquad \text{for } r > 0.$$

We say that the function $g : R \times R^{n+1} \to R$ satisfies the Carathéodory conditions if $g(\cdot, x_0, x_1, \ldots, x_n) : R \to R$ is measurable for all $(x_0, x_1, \cdots, x_n) \in R^{n+1}$, $g(t, \cdot, \ldots, \cdot) : R^{n+1} \to R$ is continuous for almost all $t \in R$, and

$$g_r^*(\cdot) = \sup\{|g(\cdot, x_0, x_1, \ldots, x_n)| : |x_i| \leq r, i = \overline{0, n}\} \in L_\omega(R_+) \quad \text{for } r > 0.$$

An absolutely continuous function $u : R \to R$ is said to be an ω-periodic solution of the equation (0.1) if it satisfies this equation almost everywhere in R and

$$u(t) = u(t+\omega) \qquad \text{for} \quad t \in R.$$

Everywhere in what follows, we will assume that the operator $f : C_\omega(R) \to L_\omega(R)$ and the function $g : R \times R^{n+1} \to R$ satisfy the Carathéodory conditions.

The paper is organized in the following way. In the Section 1 there are established the main results. Sections 2, 3, and 4 are devoted to auxiliary propositions, proofs of the main results, and examples verifying their optimality, respectively.

1. Existence and Uniqueness Theorems

Theorem 1.1. *Let $i, j \in \{1,2\}$, $i \neq j$, and on the set $C_\omega(R)$ the inequality*

$$(-1)^j [f(u)(t) + \ell_1(u)(t) - \ell_2(u)(t)] \operatorname{sgn} u(t) \le q(t) \qquad \text{for} \quad t \in R \tag{1.1}$$

hold, where $q \in L_\omega(R_+)$, $\ell_1, \ell_2 \in \mathcal{P}_\omega(R)$, and

$$\|\ell_j(1)\|_{L_\omega} < 1, \tag{1.2$_j$}$$

$$\frac{\|\ell_j(1)\|_{L_\omega}}{1 - \|\ell_j(1)\|_{L_\omega}} < \|\ell_i(1)\|_{L_\omega}. \tag{1.3$_j$}$$

Let, moreover, for any $x \in [0, \omega[$ there exist $\gamma_x \in \widetilde{C}([x, x+\omega];]0, +\infty[)$ satisfying the conditions

$$(-1)^j \gamma_x'(t) \ge \ell_{jx}(\gamma_x)(t) + \ell_{ix}(1)(t) \qquad \text{for} \quad t \in]x, x+\omega[, \tag{1.4}$$

$$\gamma_x(x + (2-i)\omega) - \gamma_x(x + (i-1)\omega) < 2. \tag{1.5}$$

Then the equation (0.1) has at least one ω-periodic solution.

Remark 1.1. Theorem 1.1 is nonimprovable in this sense that the inequality (1.4) cannot be replaced by the inequality

$$\gamma_x(x + (2-i)\omega) - \gamma_x(x + (i-1)\omega) \le 2 + \varepsilon, \tag{1.6}$$

no matter how small $\varepsilon > 0$ would be (see Example 4.1).

Furthermore, neither one of the strict inequalities in (1.2$_j$) and (1.3$_j$) can be replaced by the nonstrict one (see Examples 3.2 and 3.3 in [8]).

Corollary 1.1. *Let $\sigma \in \{-1, 1\}$ and on the set R^{n+1} the inequality*

$$\sigma\Big[g(t, x_0, x_1, \dots, x_n) + \sum_{k=0}^{n} p_k(t) x_k\Big] \operatorname{sgn} x_0 \le q(t) \qquad \text{for} \quad t \in R \tag{1.7}$$

hold, where $p_k \in L_\omega(R)$, $k = \overline{0, n}$. Let, moreover, one of the following items be fulfilled:

(H_1)

$$\sigma p_k(t) \geq 0, \quad k = \overline{1,n}, \quad \sigma p_0(t) \leq 0, \quad \textit{for} \;\; t \in R, \tag{1.8}$$

$$\int_0^\omega |p_0(s)|ds < 1, \quad \frac{\int_0^\omega |p_0(s)|ds}{1 - \int_0^\omega |p_0(s)|ds} < \sum_{k=1}^n \int_0^\omega |p_k(s)|ds, \tag{1.9}$$

$$\sum_{k=1}^n \int_0^\omega |p_k(s)| \exp\left(\sigma \int_s^{\frac{1+\sigma}{2}\omega} |p_0(\xi)|d\xi\right) ds < 2; \tag{1.10}$$

(H_2)

$$\sigma p_k(t) \leq 0, \quad k = \overline{1,n}, \quad \sigma p_0(t) \geq 0, \quad \textit{for} \;\; t \in R, \tag{1.11}$$

and either

$$\begin{gathered} \sum_{k=1}^n \int_0^\omega |p_k(s)|ds < 1, \\ \frac{\sum\limits_{k=1}^n \int_0^\omega |p_k(s)|ds}{1 - \sum\limits_{k=1}^n \int_0^\omega |p_k(s)|ds} < \int_0^\omega |p_0(s)|ds < 2\left(1 - \sum_{k=1}^n \int_0^\omega |p_k(s)|ds\right) \end{gathered} \tag{1.12}$$

or

$$\begin{gathered} \int_0^\omega |p_0(s)|ds < 1, \\ \frac{\int_0^\omega |p_0(s)|ds}{1 - \int_0^\omega |p_0(s)|ds} < \sum_{k=1}^n \int_0^\omega |p_k(s)|ds < 2\left(1 - \int_0^\omega |p_0(s)|ds\right). \end{gathered} \tag{1.11$'$}$$

Then the equation (0.2) *has at least one ω-periodic solution.*

Theorem 1.2. *Let $i, j \in \{1,2\}$, $i \neq j$ and on the sets $\{u \in C_\omega(R) : u(0) = 0\}$ and $C_\omega(R)$ the inequalities* (1.1) *and*

$$(-1)^j[f(u)(t) + \ell_1(u)(t) - \ell_2(u)(t)] \operatorname{sgn} u(0) \geq -q(t) \quad \textit{for} \;\; t \in R \tag{1.13}$$

be fulfilled, respectively, where $q \in L_\omega(R_+)$, and $\ell_1, \ell_2 \in \mathcal{P}_\omega(R)$ satisfy the conditions (1.2_i) *and* (1.3_i). *Let, moreover, there exist $\gamma \in \widetilde{C}([0,\omega];]0,+\infty[)$ such that*

$$(-1)^j \gamma'(t) \geq \ell_{j0}(\gamma)(t) + \ell_{i0}(1)(t) \quad \textit{for} \;\; t \in]0,\omega[, \tag{1.14}$$

$$\gamma((2-i)\omega) - \gamma((i-1)\omega) < 1. \tag{1.15}$$

Then the equation (0.1) *has at least one ω-periodic solution.*

Remark 1.2. Theorem 1.2 is nonimprovable in this sense that the inequality (1.15) cannot be replaced by the inequality

$$\gamma((2-i)\omega)-\gamma((i-1)\omega)\le 1+\varepsilon, \tag{1.16}$$

no matter how small $\varepsilon>0$ would be (see Example 4.2).

Furthermore, neither one of the strict inequalities in (1.2_i) and (1.3_i) can be replaced by the nonstrict one (see Examples 3.2 and 3.3 in [8]).

Corollary 1.2. *Let $\sigma\in\{-1,1\}$ and on the set R^{n+1} the inequalities*

$$\sigma[g(t,x_0,x_1,\dots,x_n)+\sum_{k=2}^{n}p_k(t)x_k]\operatorname{sgn}x_1\ge -q(t)\quad \textit{for}\ \ t\in R, \tag{1.17}$$

$$\sigma[g(t,x_0,0,\dots,x_n)+\sum_{k=2}^{n}p_k(t)x_k]\operatorname{sgn}x_0\le q(t)\quad \textit{for}\ \ t\in R \tag{1.18}$$

hold, where $p_k\in L_\omega(R)$, $k=\overline{2,n}$. Let, moreover, either

(H_3)

$$\sum_{k=2}^{n}\int\limits_0^{\omega}[p_k(s)]_+ds<1,\qquad \frac{\sum\limits_{k=2}^{n}\int_0^{\omega}[p_k(s)]_+ds}{1-\sum\limits_{k=2}^{n}\int_0^{\omega}[p_0(s)]_+ds}<\sum_{k=2}^{n}\int\limits_0^{\omega}[p_k(s)]_-ds, \tag{1.19}$$

$$\sum_{k=2}^{n}\int\limits_0^{\omega}[p_k(s)]_+\exp\left(\sum_{m=2}^{n}\int\limits_s^{\omega}[p_m(\xi)]_-d\xi\right)ds<1, \tag{1.20}$$

$$(t-\mu_{k0}(t))[p_k(t)]_-\ge 0\qquad k=\overline{2,n}\qquad \textit{for}\ \ t\in]0,\omega[, \tag{1.21}$$

or

(H_4)

$$p_k(t)\le 0\qquad k=\overline{2,n}\qquad \textit{for}\ \ t\in R, \tag{1.22}$$

$$\sum_{k=2}^{n}\int\limits_t^{\mu_{i0}(t)}|p_k(s)|ds\le\frac{1}{e}\qquad i=\overline{2,n}\qquad \textit{for}\ \ t\in]0,\omega[. \tag{1.23}$$

Then the equation (0.2) *with $\mu_1\equiv 0$ has at least one ω-periodic solution.*

Theorem 1.3. *Let $i,j\in\{1,2\}$, $i\ne j$, and on the set $C_\omega(R)$ the inequality*

$$\begin{gathered}(-1)^j[f(u)(t)-f(v)(t)+\ell_1(u-v)(t)-\\ -\ell_2(u-v)(t)]\operatorname{sgn}(u(t)-v(t))\le 0\quad \textit{for}\ \ t\in R\end{gathered} \tag{1.24}$$

hold, where $\ell_1,\ell_2\in\mathcal{P}_\omega(R)$ satisfy the conditions (1.2_j) and (1.3_j). Let, moreover, for any $x\in[0,\omega[$ there exist $\gamma_x\in\widetilde{C}([x,x+\omega];]0,+\infty[)$ such that the inequalities (1.4) *and* (1.4) *are fulfilled. Then the equation* (0.1) *has a unique ω-periodic solution.*

Remark 1.3. Theorem 1.3 is nonimprovable in this sense that the inequality (1.4) cannot be replaced by the inequality (1.6), no matter how small $\varepsilon > 0$ would be (see Example 4.1).

Furthermore, neither one of the strict inequalities in (1.2_j) and (1.3_j) can be replaced by the nonstrict one (see Examples 3.2 and 3.3 in [8]).

Corollary 1.3. *Let $\sigma \in \{-1, 1\}$ and on the set R^{n+1} the inequality*

$$\sigma[g(t, x_0, x_1, \dots, x_n) - g(t, y_0, y_1, \dots, y_n) + \\ + \sum_{k=0}^{n} p_k(t)(x_k - y_k)] \operatorname{sgn}(x_0 - y_0) \leq 0 \quad \text{for} \ \ t \in R, \tag{1.25}$$

be fulfilled, where the functions $p_k \in L_\omega(R)$, $k = \overline{0, n}$, satisfy either the conditions (1.8)–(1.10) *or the conditions* (1.11) *and* (1.12) ((1.11′) *resp.*). *Then the equation* (0.2) *has a unique ω-periodic solution.*

Theorem 1.4. *Let $i, j \in \{1, 2\}$, $i \neq j$, and on the sets $\{u \in C_\omega(R) : u(0) = 0\}$ and $C_\omega(R)$ the inequalities* (1.24) *and*

$$(-1)^j [f(u)(t) - f(v)(t) + \ell_1(u - v)(t) - \\ - \ell_2(u - v)(t)] \operatorname{sgn}(u(0) - v(0)) \geq 0 \quad \text{for} \ \ t \in R \tag{1.26}$$

hold, respectively, where $\ell_1, \ell_2 \in \mathcal{P}_\omega(R)$ satisfy the conditions (1.2_i) and (1.3_i). Let, moreover, there exist $\gamma \in \widetilde{C}([0, \omega];]0, +\infty[)$ such that the inequalities (1.14) *and* (1.15) *are fulfilled. Then the equation* (0.1) *has a unique ω-periodic solution.*

Remark 1.4. Theorem 1.4 is nonimprovable in this sense that the inequality (1.15) cannot be replaced by the inequality (1.16), no matter how small $\varepsilon > 0$ would be (see Example 4.2).

Furthermore, neither one of the strict inequalities in (1.2_i) and (1.3_i) can be replaced by the nonstrict one (see Examples 3.2 and 3.3 in [8]).

Corollary 1.4. *Let $\sigma \in \{-1, 1\}$ and on the set R^{n+1} the inequalities*

$$\sigma[g(t, x_0, x_1, \dots, x_n) - g(t, y_0, y_1, \dots, y_n) + \\ + \sum_{k=2}^{n} p_k(t)(x_k - y_k)] \operatorname{sgn}(x_1 - y_1) \geq 0 \quad \text{for} \ \ t \in R, \tag{1.27}$$

$$\sigma[g(t, x_0, 0, \dots, x_n) - g(t, y_0, 0, \dots, y_n) + \\ + \sum_{k=2}^{n} p_k(t)(x_k - y_k)] \operatorname{sgn}(x_0 - y_0) \leq 0 \quad \text{for} \ \ t \in R \tag{1.28}$$

be fulfilled, where the functions $p_k \in L_\omega(R)$, $k = \overline{2, n}$, satisfy either the conditions (1.19)–(1.21) *or* (1.22) *and* (1.23). *Then the equation* (0.2) *with $\mu_1 \equiv 0$ has a unique ω-periodic solution.*

2. Auxiliary Propositions

First we formulate in a suitable for us form two lemmas from [4] and [19], respectively.

Lemma 2.1. *Let $a \in [0, \omega[$, $\hat{\ell} : C([a, a+\omega]; R) \to L(]a, a+\omega[; R)$ be a linear bounded operator transforming $C([a, a+\omega]; R_+)$ into $L(]a, a+\omega[; R_+)$. Let, moreover, there exist a function $\gamma \in \widetilde{C}([a, a+\omega];]0, +\infty[)$ such that*

$$\gamma'(t) \geq \hat{\ell}(\gamma)(t) \qquad \text{for } \ t \in]a, a+\omega[.$$

Then for any $g \in L(]a, a+\omega[; R)$ the Cauchy problem

$$u'(t) = \hat{\ell}(u)(t) + g(t), \qquad u(a) = 0$$

has a unique solution[1] *and, moreover, the inequalities*

$$v(t) \geq 0, \qquad v'(t) \geq 0 \qquad \text{for } \ t \in]a, a+\omega[$$

are fulfilled whenever the function $v \in \widetilde{C}([a, a+\omega]; R)$ satisfies the conditions

$$v'(t) \geq \hat{\ell}(v)(t) \quad \text{for } \ t \in]a, a+\omega[, \qquad v(a) \geq 0.$$

Lemma 2.2. *Let there exist a positive number ρ and an operator $\ell \in \mathcal{L}_\omega(R)$ such that the homogeneous equation*

$$u'(t) + \ell(u)(t) = 0 \tag{2.1}$$

has only the trivial ω-periodic solution and for every $\lambda \in]0, 1[$ an arbitrary ω-periodic solution of the equation

$$u'(t) + \ell(u)(t) = \lambda[f(u)(t) + \ell(u)(t)] \tag{2.2}$$

admits the estimate

$$\|u\|_{C_\omega} \leq \rho. \tag{2.3}$$

Then the equation (0.1) has at least one ω-periodic solution.

Definition 2.1. *We say that the operator $\ell \in \mathcal{L}_\omega(R)$ belongs to the set A^i_ω, $i \in \{1, 2\}$, if there exists a positive number r such that for any $q \in L_\omega(R_+)$ every function $u \in \widetilde{C}_\omega(R)$ satisfying the inequality*

$$(-1)^{i+1}[u'(t) + \ell(u)(t)] \operatorname{sgn} u(t) \leq q(t) \qquad \text{for } \ t \in R, \tag{2.4}$$

admits the estimate

$$\|u\|_{C_\omega} \leq r\|q\|_{L_\omega}. \tag{2.5}$$

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[1] Under a solution of this problem we understand a function $u \in \widetilde{C}([a, a+\omega]; R)$ satisfying the corresponding equation almost everywhere in $]a, a+\omega[$ and the initial condition.

Definition 2.2. *We say that the operator $\ell \in \mathcal{L}_\omega(R)$ belongs to the set B_ω^i, $i \in \{1,2\}$, if there exists a positive number r such that for any $q \in L_\omega(R_+)$ every function $u \in \widetilde{C}_\omega(R)$ satisfying the inequalities*

$$(-1)^{i+1}[u'(t) + \ell(u)(t)] \operatorname{sgn} u(0) \geq -q(t) \qquad \textit{for } \ t \in R \quad \textit{if } \ u(0) \neq 0$$

and

$$(-1)^{i+1}[u'(t) + \ell(u)(t)] \operatorname{sgn} u(t) \leq q(t) \qquad \textit{for } \ t \in R \quad \textit{if } \ u(0) = 0,$$

admits the estimate (2.5).

Lemma 2.3. *Let $i \in \{1,2\}$ and there exist $\ell \in A_\omega^i$ and $q \in L_\omega(R_+)$ such that for any $u \in C_\omega(R)$ the inequality*

$$(-1)^{i+1}[f(u)(t) + \ell(u)(t)] \operatorname{sgn} u(t) \leq q(t) \qquad \textit{for } \ t \in R \tag{2.6}$$

is fulfilled. Then the equation (0.1) *has at least one ω-periodic solution.*

Proof. First note that due to the condition $\ell \in A_\omega^i$, the homogeneous equation (2.1) has only a trivial ω-periodic solution.

Let r be the number appearing in Definition 2.1. Put

$$\rho = r\|q\|_{L_\omega}.$$

Now assume that u is an ω-periodic solution of the equation (2.2) for some $\lambda \in]0,1[$. Then due to (2.6) u satisfies the differential inequality (2.4). Hence by the condition $\ell \in A_\omega^i$ and the definition of the number ρ we get the estimate (2.3).

Since ρ depends neither on u nor on λ, from Lemma 2.2 it follows that the equation (0.1) has at least one ω-periodic solution. $\square$

Lemma 2.4. *Let $i \in \{1,2\}$ and there exist $\ell \in B_\omega^i$ and $q \in L_\omega(R_+)$ such that on the sets $\{u \in C_\omega(R) : u(0) = 0\}$ and $C_\omega(R)$ the inequalities*

$$(-1)^{i+1}[f(u)(t) + \ell(u)(t)] \operatorname{sgn} u(t) \leq q(t) \qquad \textit{for } \ t \in R$$

and

$$(-1)^{i+1}[f(u)(t) + \ell(u)(t)] \operatorname{sgn} u(0) \geq -q(t) \qquad \textit{for } \ t \in R$$

are fulfilled, respectively. Then the equation (0.1) *has at least one ω-periodic solution.*

This lemma can be proved analogously to Lemma 2.3.

Lemma 2.5. *Let $i \in \{1,2\}$ and there exist $\ell \in A_\omega^i$ and $q \in L_\omega(R_+)$ such that for any $u_k \in C_\omega(R)$, $k = 1,2$, the inequality*

$$\begin{aligned} &(-1)^{i+1}[f(u_1)(t) - f(u_2)(t) + \\ &+\ell(u_1 - u_2)(t)] \operatorname{sgn}(u_1(t) - u_2(t)) \leq 0 \qquad \textit{for } \ t \in R \end{aligned} \tag{2.7}$$

is fulfilled. Then the equation (0.1) *has a unique ω-periodic solution.*

Proof. From (2.7) it follows that the operator f for any $u \in C_\omega(R)$ satisfies the inequality (2.6), where $q(t) = |f(0)(t)|$ for $t \in R$. Consequently, the assumptions of Lemma 2.3 are satisfied and this guarantees that the equation (0.1) has at least one ω-periodic solution. It remains to show that the equation (0.1) has at most one ω-periodic solution.

Let u_1 and u_2 be arbitrary ω-periodic solutions of the equation (0.1). Put $u(t) = u_1(t) - u_2(t)$ for $t \in R$. Then by (2.7) we get

$$(-1)^{i+1}(u'(t) + \ell(u)(t))\operatorname{sgn} u(t) \le 0 \qquad \text{for } \ t \in R.$$

This together with the condition $\ell \in A^i_\omega$ results in $u \equiv 0$. Consequently, $u_1 \equiv u_2$. $\square$

The following lemma can be proved analogously.

Lemma 2.6. *Let $i \in \{1,2\}$ and there exist $\ell \in B^i_\omega$ and $q \in L_\omega(R_+)$ such that for any $u_k \in C_\omega(R)$, $k = 1,2$, the inequalities*

$$\begin{gathered}(-1)^{i+1}[f(u_1)(t) - f(u_2)(t) + \\ +\ell(u_1 - u_2)(t)]\operatorname{sgn}(u_1(t) - u_2(t)) \le 0 \quad \textit{for } \ t \in R \quad \textit{if } \ u_1(0) = u_2(0)\end{gathered}$$

and

$$\begin{gathered}(-1)^{i+1}[f(u_1)(t) - f(u_2)(t) + \\ +\ell(u_1 - u_2)(t)]\operatorname{sgn}(u_1(0) - u_2(0)) \ge 0 \quad \textit{for } \ t \in R \quad \textit{if } \ u_1(0) \ne u_2(0)\end{gathered}$$

are fulfilled. Then the equation (0.1) *has a unique ω-periodic solution.*

2.1. On the sets $\mathbf{A}^{\mathbf{i}}_{\boldsymbol{\omega}}$ and $\mathbf{B}^{\mathbf{i}}_{\boldsymbol{\omega}}$, $\mathbf{i = 1, 2}$.

Lemma 2.7. *Let $i,j \in \{1,2\}$, $i \ne j$, $\ell_1, \ell_2 \in \mathcal{P}_\omega(R)$, and the conditions* (1.2_j) *and* (1.3_j) *be fulfilled. Then there exists a number $r_0 > 0$ such that for any $\sigma \in \{-1,1\}$ and $q \in L_\omega(R_+)$ every function $u \in \widetilde{C}_\omega(R)$ satisfying the inequality* (2.4) *with $\ell = \ell_1 - \ell_2$ and the condition*

$$\sigma u(t) \ge 0 \qquad \textit{for } \ t \in R$$

admits the estimate

$$\|u\|_{C_\omega} \le r_0 \|q\|_{L_\omega}. \tag{2.8}$$

Proof. We will prove the lemma for $i = 1$ and $j = 2$. The case $i = 2$ and $j = 1$ can be proved analogously.

Let $u \in \widetilde{C}_\omega(R)$ satisfy the assumptions of the lemma. Put

$$m = \min\{|u(t)| : 0 \le t \le \omega\} \tag{2.9}$$

and choose $t_1 \in [0,\omega[$, $t_2 \in]t_1, t_1+\omega[$ such that

$$|u(t_1)| = m, \qquad |u(t_2)| = \|u\|_{C_\omega}. \tag{2.10}$$

Taking into account that $\ell_1, \ell_2 \in \mathcal{P}_\omega(R)$, from (2.4) it immediately follows that

$$|u(t)|' \leq \|u\|_{C_\omega} \ell_2(1)(t) - m\ell_1(1)(t) + q(t) \qquad \text{for} \quad t \in R. \tag{2.11}$$

The integration of this inequality from t_1 to t_2 in view of (2.9) and (2.10) yields

$$\|u\|_{C_\omega} - m \leq \|u\|_{C_\omega} \|\ell_2(1)(t)\|_{L_\omega} + \|q\|_{L_\omega}. \tag{2.12}$$

On the other hand, integrating (2.11) from 0 to ω, we obtain

$$m\|\ell_1(1)\|_{L_\omega} \leq \|u\|_{C_\omega} \|\ell_2(1)\|_{L_\omega} + \|q\|_{L_\omega}. \tag{2.13}$$

Now from (2.12) and (2.13), on account of (1.3_2), we get the estimate (2.8), where

$$r_0 = \frac{1 + \|\ell_1(1)\|_{L_\omega}}{\|\ell_1(1)\|_{L_\omega}(1 - \|\ell_2(1)\|_{L_\omega}) - \|\ell_2(1)\|_{L_\omega}}. \qquad \square$$

Lemma 2.8. *Let $i, j \in \{1, 2\}$, $i \neq j$, and $\ell_1, \ell_2 \in \mathcal{P}_\omega(R)$ satisfy the conditions (1.2_j) and (1.3_j). Let, moreover, for any $x \in [0, \omega[$ there exist $\gamma_x \in \widetilde{C}([x, x+\omega];]0, +\infty[)$ such that the inequalities* (1.4) *and* (1.4) *are fulfilled. Then*

$$\ell_1 - \ell_2 \in A^i_\omega.$$

Proof. We will prove the lemma for $i = 1$. The case $i = 2$ can be proved analogously.

Let $q \in L_\omega(R_+)$, and $u \in \widetilde{C}_\omega(R)$ satisfy the inequality

$$[u'(t) + \ell_1(u)(t) - \ell_2(u)(t)] \operatorname{sgn} u(t) \leq q(t) \qquad \text{for} \quad t \in R. \tag{2.14}$$

Show that the estimate (2.5) is valid with

$$r = r_0 + \widetilde{r}\Big(1 - \|\ell_2(1)\|_{L_\omega}\Big)^{-1}, \tag{2.15}$$

where r_0 is the number appearing in Lemma 2.7 and

$$\widetilde{r} = \max\left\{ \frac{3}{1 - \frac{1}{4}(\gamma_x(x+\omega) - \gamma_x(x))^2} : 0 \leq x \leq \omega \right\}. \tag{2.16}$$

If the function u does not change sign, then the validity of the estimate (2.5) follows from Lemma 2.7.

Suppose now that u changes sign. Put

$$m = -\min\{u(t) : 0 \leq t \leq \omega\}, \quad M = \max\{u(t) : 0 \leq t \leq \omega\} \tag{2.17}$$

and choose $t_m, t_M, x \in [0, \omega[$ such that

$$u(t_m) = -m, \qquad u(t_M) = M, \qquad u(x) = 0. \tag{2.18}$$

According to (2.14), it is clear that

$$u'(t) = \ell_{2x}(u)(t) - \ell_{1x}(u)(t) + h_x(t) \qquad \text{for} \ \ x < t < x+\omega, \tag{2.19}$$

$$h_x(t) \operatorname{sgn} u(t) \le q(t) \qquad \text{for} \ \ x < t < x+\omega, \tag{2.20}$$

where

$$h_x(t) = u'(t) + \ell_{1x}(u)(t) - \ell_{2x}(u)(t) \qquad \text{for} \ \ x < t < x+\omega.$$

From (2.19), in view of (2.20) we obtain

$$[u(t)]'_+ \le \ell_{2x}([u]_+)(t) + \ell_{1x}([u]_-)(t) + q(t) \quad \text{for} \ \ x < t < x+\omega, \tag{2.21}$$

$$[u(t)]'_- \le \ell_{2x}([u]_-)(t) + \ell_{1x}([u]_+)(t) + q(t) \quad \text{for} \ \ x < t < x+\omega. \tag{2.22}$$

Denote by α_x, β_x, and v_x, respectively, the solutions of the problems

$$\alpha'_x(t) = \ell_{2x}(\alpha_x)(t) + \frac{1}{M}\ell_{1x}([u]_+)(t), \qquad \alpha_x(x) = 0, \tag{2.23}$$

$$\beta'_x(t) = \ell_{2x}(\beta_x)(t) + \frac{1}{m}\ell_{1x}([u]_-)(t), \qquad \beta_x(x) = 0, \tag{2.24}$$

$$v'_x(t) = \ell_{2x}(v_x)(t) + q(t), \qquad v_x(x) = 0. \tag{2.25}$$

The existence and uniqueness of these solutions are guaranteed by the condition (1.4) and Lemma 2.1. From (1.4), (2.23)–(2.25) we get

$$\begin{aligned}
(m\beta_x(t) + v_x(t) - [u(t)]_+)' \ &\ge\ \ell_{2x}(m\beta_x + v_x - [u]_+)(t) \text{ for } \ x<t<x+\omega,\\
(M\alpha_x(t) + v_x(t) - [u(t)]_-)' \ &\ge\ \ell_{2x}(M\alpha_x + v_x - [u]_-)(t) \ \text{ for } \ \ x<t<x+\omega,\\
(\gamma_x(t) - \alpha_x(t) - \beta_x(t))' \ &\ge\ \ell_{2x}(\gamma_x - \alpha_x - \beta_x)(t) \ \text{ for } \ \ x<t<x+\omega.
\end{aligned}$$

The last inequalities, according to Lemma 2.1, imply

$$[u(t)]_+ \ \le\ m\beta_x(t) + v_x(t) \quad \text{for} \ \ x < t < x+\omega, \tag{2.26}$$

$$[u(t)]_- \ \le\ M\alpha_x(t) + v_x(t) \quad \text{for} \ \ x < t < x+\omega, \tag{2.27}$$

$$\gamma'_x(t) \ \ge\ (\alpha_x(t) + \beta_x(t))' \quad \text{for} \ \ x < t < x+\omega. \tag{2.28}$$

From (2.26) and (2.27), in view of (2.18) and the fact that the functions α_x, β_x and v_x are nondecreasing it immediately follows

$$\begin{aligned}
M = [u(t_M)]_+ &\le m\beta_x(x+\omega) + v_x(x+\omega),\\
m = [u(t_m)]_- &\le \alpha_x(x+\omega) + v_x(x+\omega).
\end{aligned}$$

Hence we get

$$M(1 - \alpha_x(x+\omega)\beta_x(x+\omega)) \ \le\ (\beta_x(x+\omega) + 1)v_x(x+\omega), \tag{2.29}$$

$$m(1 - \alpha_x(x+\omega)\beta_x(x+\omega)) \ \le\ (\alpha_x(x+\omega) + 1)v_x(x+\omega). \tag{2.30}$$

On the other hand, the integration of (2.28) from x to $x+\omega$ yields

$$\alpha_x(x+\omega) + \beta_x(x+\omega) \le \gamma_x(x+\omega) - \gamma_x(x).$$

From (2.29) and (2.30), on account of the last inequality, (1.4), and the fact that $4\alpha_x(x+\omega)\beta_x(x+\omega) \le (\alpha_x(x+\omega)+\beta_x(x+\omega))^2$ we obtain

$$\begin{aligned} M &\le [1-\frac{1}{4}(\gamma_x(x+\omega)-\gamma_x(x))^2]^{-1}(1+\gamma_x(x+\omega))v_x(x+\omega), \\ m &\le [1-\frac{1}{4}(\gamma_x(x+\omega)-\gamma_x(x))^2]^{-1}(1+\gamma_x(x+\omega))v_x(x+\omega). \end{aligned}$$

Consequently,

$$\|u\|_{C_\omega} \le \widetilde{r} v_x(x+\omega), \tag{2.31}$$

where $\widetilde{r}$ is defined by (2.16).

Now the integration of (2.25) from x to $x+\omega$ together with $\|v_x\|_C = v_x(x+\omega)$ and $\|\ell_{2x}(1)\|_L = \|\ell_2(1)\|_{L_\omega} < 1$ results in

$$v_x(x+\omega) < (1-\|\ell_2(1)\|_{L_\omega})^{-1}\|q\|_{L_\omega}. \tag{2.32}$$

According to (2.32), (2.31), (2.15), and (2.16), we have the estimate (2.5). □

Lemma 2.9. *Let $i,j \in \{1,2\}$, $i \ne j$, and $\ell_1, \ell_2 \in \mathcal{P}_\omega(R)$ satisfy the conditions* (1.2_j) *and* (1.3_j)*. Let, moreover, there exist $\gamma \in \widetilde{C}([0,\omega];]0,+\infty[)$ such that the inequalities* (1.14) *and* (1.15) *are fulfilled. Then*

$$\ell_1 - \ell_2 \in B^i_\omega.$$

Proof. We will prove the lemma for $i=1$ and $j=2$. The case $i=2$ and $j=1$ can be proved analogously.

First assume that $u \in \widetilde{C}_\omega(R)$ is such that $u(0)=0$ and the inequality (2.14) is fulfilled. Obviously, u satisfies also the inequality

$$[u'(t)+\ell_{10}(u)(t)-\ell_{20}(u)(t)]\operatorname{sgn} u(t) \le q(t) \qquad \text{for } \ 0<t<\omega. \tag{2.33}$$

By virtue of Lemma 2.1, (1.14), and (1.15), without loss of generality we can assume that

$$\gamma(\omega) \le 2. \tag{2.34}$$

Therefore, according to (1.14), (2.33), (2.34), and Lemma 2.9 and Definition 2.1 in [5], there exists a number ρ_0, not depending on u, such that

$$\|u\|_{C_\omega} \le \rho_0\|q\|_{L_\omega}.$$

Assume now that $u(0) \ne 0$ and

$$[u'(t)+\ell_1(u)(t)-\ell_2(u)(t)]\operatorname{sgn} u(0) \ge -q(t) \qquad \text{for } \ t \in R. \tag{2.35}$$

Put $\widetilde{u}(t) = u(t)\operatorname{sgn} u(0)$ for $t \in R$. Due to (2.35), it is clear that

$$\widetilde{u}'(t) \ge \ell_{20}(\widetilde{u})(t) - \ell_{10}(\widetilde{u})(t) - q(t) \qquad \text{for } \ 0<t<\omega. \tag{2.36}$$

Denote by v the solution of the problem

$$v'(t) = \ell_{20}(v)(t) + q(t), \qquad v(0)=0. \tag{2.37}$$

The existence and uniqueness of v is guaranteed by (1.14) and Lemma 2.1. Using Lemma 2.4 in [5], there exists a number ρ_1, not depending on v, such that

$$\|v\|_C \leq \rho_1 \|q\|_{L_\omega}. \tag{2.38}$$

From (1.14), (2.36), and (2.37) we find

$$(\|\widetilde{u}\|_{C_\omega}\gamma(t) + v(t) + \widetilde{u}(t))' \geq \ell_{20}(\|\widetilde{u}\|_{C_\omega}\gamma + v + \widetilde{u})(t) \quad \text{for } 0<t<\omega. \tag{2.39}$$

Since $\|\widetilde{u}\|_{C_\omega}\gamma(0) + v(0) + \widetilde{u}(0) > 0$, in view of (2.39) and Lemma 2.1 we have

$$\widetilde{u}'(t) \geq -\|\widetilde{u}\|_{C_\omega}\gamma'(t) - v'(t) \qquad \text{for } 0 < t < \omega. \tag{2.40}$$

Put

$$m = -\min\{\widetilde{u}(t) : 0 \leq t \leq \omega\}, \qquad M = \max\{\widetilde{u}(t) : 0 \leq t \leq \omega\}. \tag{2.41}$$

Obviously, $M > 0$. Choose $t_1, t_2 \in [0, \omega[$, $t_1 \neq t_2$ such that

$$\widetilde{u}(t_1) = -m, \qquad \widetilde{u}(t_2) = M. \tag{2.42}$$

First assume that $m > 0$. Then either $M = \|\widetilde{u}\|_{C_\omega}$ or $m = \|\widetilde{u}\|_{C_\omega}$. If $t_2 < t_1$, then the integration of (2.40) from t_2 to t_1 in view of the fact that the functions γ and v are nondecreasing (see Lemma 2.1) yields

$$M + m \leq \|\widetilde{u}\|_{C_\omega}(\gamma(\omega) - \gamma(0)) + v(\omega) \tag{2.43}$$

and, consequently, on account of (1.15) and (2.38),

$$\|\widetilde{u}\|_{C_\omega} \leq [1 - \gamma(\omega) + \gamma(0)]^{-1} v(\omega) \leq [1 - \gamma(\omega) + \gamma(0)]^{-1} \rho_1 \|q\|_{L_\omega}. \tag{2.44}$$

If $t_1 < t_2$, then the integration of (2.40) from 0 to t_1 and from t_2 to ω, respectively, results in

$$\begin{aligned} m + \widetilde{u}(0) &\leq \|\widetilde{u}\|_{C_\omega}(\gamma(t_1) - \gamma(0)) + v(t_1), \\ M - \widetilde{u}(\omega) &\leq \|\widetilde{u}\|_{C_\omega}(\gamma(\omega) - \gamma(t_2)) + v(\omega) - v(t_2). \end{aligned}$$

Summing the last two inequalities and taking into account the monotonicity of the functions γ and v, we get (2.43) and, consequently, in view of (1.15) and (2.38) we obtain the inequality (2.44).

Assume now that $m < 0$. Then $\operatorname{sgn} u(t) = \operatorname{sgn} u(0)$ and due to (2.35) the assumptions of Lemma 2.7 are fulfilled with $i = 2$, $j = 1$ and $\sigma = \operatorname{sgn} u(0)$. Consequently, there exists a number r_0, not depending on u, such that the estimate (2.8) holds. By virtue of all the estimates obtained above, it can be easily seen that the estimate (2.5) holds, where $r = \rho_0 + r_0 + \rho_1(1 - \gamma(\omega) + \gamma(0))^{-1}$ does not depend on u. $\square$

3. Proof of the Main Results

Theorem 1.1 follows from Lemmas 2.3 and 2.8, Theorem 1.2 follows from Lemmas 2.4 and 2.9, Theorem 1.3 follows from Lemmas 2.5 and 2.8, and Theorem 1.4 follows from Lemmas 2.6 and 2.9.

Proof of Corollaries 1.1 and 1.3. We will prove the corollaries for the case $\sigma = 1$. The case $\sigma = -1$ can be proved analogously.

Put

$$f(u)(t) = g(t, u(t), u(\mu_1(t)), u(\mu_2(t)), \dots, u(\mu_n(t))) \tag{3.1}$$

and

$$\ell_1(u)(t) = \sum_{k=1}^{n} p_k(t) u(\mu_k(t)), \qquad \ell_2(u)(t) = -p_0(t) u(t) \tag{3.2}$$

if the hypothesis H_1 are fulfilled, and put

$$\ell_1(u)(t) = p_0(t) u(t), \qquad \ell_2(u)(t) = -\sum_{k=1}^{n} p_k(t) u(\mu_k(t)) \tag{3.3}$$

if the hypothesis H_2 are fulfilled. Then the condition (1.7) (the condition (1.25)) can be written as (1.1) (as (1.24)), and the conditions (1.9) and (1.12) can be written as (1.2_2), (1.3_2). By Theorem 1.1 (Theorem 1.3), to prove Corollary 1.1 (Corollary 1.3) it is sufficient to show that for each $x \in [0, \omega[$ there exists a function $\gamma_x \in \widetilde{C}([x, x+\omega];]0, +\infty[)$ satisfying the inequalities (1.4) and (1.4) with $i = 1$ and $j = 2$.

Let the hypothesis H_1 be fulfilled. Put

$$\gamma_x(t) = \left(\varepsilon + \int_x^t \exp\left(-\int_x^s |p_0(\xi)| d\xi \right) \sum_{k=1}^{n} p_k(s) ds \right) \times \\ \times \exp\left(\int_x^t |p_0(s)| ds \right) \quad \text{for } t \in]x, x+\omega[, \tag{3.4}$$

where $\varepsilon > 0$ is such that

$$\int_0^\omega \exp\left(\int_s^\omega |p_0(\xi)| d\xi \right) \sum_{k=1}^{n} p_k(s) ds < 2 - \varepsilon \left(\exp\left(\int_0^\omega |p_0(s)| ds \right) - 1 \right). \tag{3.5}$$

By (3.2), (3.4), and (3.5), it is clear that $\gamma_x(x+\omega) - \gamma_x(x) < 2$ and

$$\gamma_x'(t) = |p_0(t)| \gamma_x(t) + \sum_{k=1}^{n} p_k(t) = \ell_{2x}(\gamma_x)(t) + \ell_{1x}(1)(t).$$

Therefore the conditions (1.4) and (1.4) with $i = 1$ and $j = 2$ are fulfilled.

Now suppose that the hypothesis H_2 hold. By the latter inequality in (1.12), we can choose $\varepsilon > 0$ such that

$$\int_0^\omega |p_0(s)|ds \le 2\left(1-\sum_{k=1}^n \int_0^\omega |p_k(s)|ds\right) - \varepsilon. \tag{3.6}$$

Put

$$\gamma_x(t) = 2\sum_{k=1}^n \int_x^t |p_k(s)|ds + \int_x^t |p_0(s)|ds + \varepsilon. \tag{3.7}$$

Due to (3.6) and (3.7), it is evident that γ_x is a nondecreasing function, $\gamma_x(x+\omega) - \gamma_x(x) < 2$, and

$$\gamma_x'(t) = 2\sum_{k=1}^n |p_k(t)| + |p_0(t)| \ge$$

$$\ge \sum_{k=1}^n |p_k(t)|\gamma_x(\mu_{kx}(t)) + |p_0(t)| = \ell_{2x}(\gamma_x)(t) + \ell_{1x}(1)(t).$$

Therefore the conditions (1.4) and (1.4) with $i = 1$ and $j = 2$ are fulfilled. $\square$

Proof of Corollaries 1.2 and 1.4. We will prove the corollaries for the case $\sigma = 1$. Let f be the operator defined by the equality (3.1), where $\mu_1 \equiv 0$. Put

$$\ell_1(u)(t) = \sum_{k=2}^n [p_k(t)]_+ u(\mu_k(t)), \qquad \ell_2(u)(t) = \sum_{k=2}^n [p_k(t)]_- u(\mu_k(t)) \tag{3.8}$$

if the hypothesis H_3 are fulfilled, and put

$$\ell_1(u)(t) \equiv 0, \qquad \ell_2(u)(t) = \sum_{k=2}^n |p_k(t)|u(\mu_k(t)) \tag{3.9}$$

if the hypothesis H_4 are fulfilled. Then the conditions (1.17) and (1.18) (the conditions (1.27) and (1.27)) can be written as (1.13) and (1.1), resp. (as (1.26) and (1.24), resp.) with $i = 1$, $j = 2$. By Theorem 1.2 (Theorem 1.4), to prove Corollary 1.2 (Corollary 1.4) it is sufficient to show that there exists a function $\gamma \in \widetilde{C}([0,\omega];]0,+\infty[)$ satisfying the inequalities (1.14) and (1.15) with $i = 1$ and $j = 2$.

Let the hypothesis H_3 be fulfilled. Put

$$\gamma(t) = \left(\varepsilon + \sum_{k=2}^{n} \int_0^t [p_k(s)]_+ \exp\left(- \sum_{m=2}^{n} \int_0^s [p_m(\xi)]_- d\xi \right) ds \right) \times$$

$$\times \exp\left(\sum_{k=2}^{n} \int_0^t [p_k(s)]_- ds \right) \qquad \text{for} \quad t \in [0, \omega], \tag{3.10}$$

where $\varepsilon > 0$ is such that

$$\sum_{k=2}^{n} \int_0^\omega [p_k(s)]_+ \exp\left(\sum_{m=2}^{n} \int_s^\omega [p_m(\xi)]_- d\xi \right) ds <$$

$$< 1 - \varepsilon \left(\exp\left(\sum_{k=2}^{n} \int_0^\omega [p_k(s)]_- ds \right) - 1 \right). \tag{3.11}$$

By (3.8), (3.10), (3.11), and (1.21) it is clear that $\gamma(\omega) - \gamma(0) < 1$ and

$$\gamma'(t) \geq \sum_{k=2}^{n} [p_k(t)]_- \gamma(\mu_{k0}(t)) + \sum_{k=2}^{n} [p_k(t)]_+ = \ell_{20}(\gamma)(t) + \ell_{10}(1)(t).$$

Therefore the conditions (1.14) and (1.15) with $i = 1$ and $j = 2$ are fulfilled.

Let now the hypothesis H_4 hold. Put

$$\gamma(t) = \exp\left(e \sum_{k=2}^{n} \int_t^\omega p_k(s) ds \right). \tag{3.12}$$

By (1.22), (1.23), (3.9), and (3.12) we obtain $\gamma(\omega) - \gamma(0) < 1$ and

$$\gamma'(t) \geq \sum_{k=2}^{n} |p_k(t)| \gamma(\mu_{k0}(t)) = \ell_{20}(\gamma)(t).$$

Therefore the conditions (1.14) and (1.15) with $i = 1$ and $j = 2$ are fulfilled. □

4. Examples

Example 4.1. Let $\varepsilon > 0$, $\delta \in]0,1[$ be such that $\delta < \varepsilon$, $\omega = 4$,

$$\tau(t) = \begin{cases} 3 & \text{for} \quad t \in [4\nu, 4\nu + 2 - \frac{\delta}{2}] \cup [4\nu + 3, 4\nu + 4] \\ 1 & \text{for} \quad t \in]4\nu + 2 - \frac{\delta}{2}, 4\nu + 3[\end{cases},$$

$$p(t) = \begin{cases} 1 & \text{for} \quad t \in]4\nu, 4\nu + 1[\cup]4\nu + 2 - \frac{\delta}{2}, 4\nu + 3[\cup]4\nu + 4 - \frac{\delta}{2}, 4\nu + 4[\\ 0 & \text{for} \quad t \in]4\nu + 1, 4\nu + 2 - \frac{\delta}{2}[\cup]4\nu + 3, 4\nu + 4 - \frac{\delta}{2}[\end{cases},$$

$$h(t) = \begin{cases} 0 & \text{for} \quad t \in]4\nu, 4\nu + 1[\cup]4\nu + 2 - \frac{\delta}{2}, 4\nu + 3[\cup]4\nu + 4 - \frac{\delta}{2}, 4\nu + 4[\\ \frac{1}{2-t} & \text{for} \quad t \in]4\nu + 1, 4\nu + 2 - \frac{\delta}{2}[\\ \frac{1}{4-t} & \text{for} \quad t \in]4\nu + 3, 4\nu + 4 - \frac{\delta}{2}[\end{cases},$$

where ν is an integer, and put

$$\ell_2(v)(t) \overset{\text{def}}{=} 0, \qquad \ell_1(v)(t) \overset{\text{def}}{=} p(t)v(\tau(t)) \qquad \text{for} \quad t \in R,$$

$$\gamma_x(t) \overset{\text{def}}{=} \delta + \frac{2+\varepsilon}{2+\delta} \int_x^t p(s)ds \qquad \text{for} \quad x \in [0, \omega[, \quad t \in [x, x + \omega].$$

Then, obviously, the conditions (1.2_j), (1.3_j), and (1.4) are fulfilled for $i = 1$, $j = 2$, and

$$\gamma_x(x + \omega) - \gamma_x(x) = 2 + \varepsilon.$$

On the other hand, the problem

$$u'(t) = -p(t)u(\tau(t)) - h(t)u(t), \qquad u(0) = u(\omega)$$

has a nontrivial solution

$$u(t) = \begin{cases} t & \text{for} \quad t \in [0, 1[\\ 2 - t & \text{for} \quad t \in [1, 3[\\ t - 4 & \text{for} \quad t \in [3, 4] \end{cases}.$$

Therefore, according to the Riesz–Schauder theory there exists $q_0 \in L_\omega(R)$ such that on the interval $[0, \omega]$ the problem

$$u'(t) = -p(t)u(\tau(t)) - h(t)u(t) + q_0(t), \qquad u(0) = u(\omega)$$

has no solution. In other words, the equation (0.1) with

$$f(v)(t) \overset{\text{def}}{=} -p(t)v(\tau(t)) - h(t)v(t) + q_0(t) \qquad \text{for} \quad t \in R$$

has no ω-periodic solution although the operator f satisfies the condition (1.1) with $q \equiv |q_0|$.

This example shows that the condition (1.4) in Theorems 1.1 and 1.3 cannot be replaced by the condition

$$\gamma_x(x+(2-i)\omega) - \gamma_x(x+(i-1)\omega) \le 2+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Example 4.2. Let $\varepsilon > 0$ be an arbitrarily fixed number. Choose an integer $n > 1$ and $\varepsilon_0 \in]0, \frac{n-1}{n(n+1)}[$ such that

$$\frac{1}{n^2} < \frac{\varepsilon}{2}, \qquad (n+1)\varepsilon_0 < \frac{\varepsilon}{2}$$

and put

$$\delta = \left(1+n+\frac{1}{n^2\varepsilon_0}+\frac{1}{\varepsilon_0}\right)^{-1}\left(\varepsilon - \frac{1}{n^2} - (n+1)\varepsilon_0\right), \qquad t_1 = (n+1)\varepsilon_0,$$
$$t_2 = 1+(2n+1)\varepsilon_0, \qquad t_3 = 1+(3n+1)\varepsilon_0 + \frac{1}{n}, \qquad \omega = 2+2n\varepsilon_0,$$
$$c_1 = \left(1-\frac{\delta}{\varepsilon_0}\right)t_1+\delta, \qquad c_2 = \left(\frac{\delta}{\varepsilon_0} - \frac{\delta+(n+1)\varepsilon_0}{n\varepsilon_0}\right)t_2 + c_1,$$
$$c_3 = \left(\frac{\delta+(n+1)\varepsilon_0}{n\varepsilon_0} - 1\right)t_3 + c_2.$$

Consider the equation

$$u'(t) = p(t)u(\tau(t)). \tag{4.1}$$

Here

$$p(t) = \begin{cases} -1 & \text{for } \ t \in [\nu\omega, t_1+\nu\omega[\ \cup \ [t_3+\nu\omega, (\nu+1)\omega] \\ \frac{1}{\varepsilon_0} & \text{for } \ t \in [t_1+\nu\omega, t_2+\nu\omega[\\ \frac{1}{n\varepsilon_0} & \text{for } \ t \in [t_2+\nu\omega, t_3+\nu\omega[\end{cases},$$
$$\tau(t) = \begin{cases} t_2 & \text{for } \ t \in [\nu\omega, t_1+\nu\omega[\ \cup \ [t_3+\nu\omega, (\nu+1)\omega] \\ 0 & \text{for } \ t \in [t_1+\nu\omega, t_2+\nu\omega[\\ t_1 & \text{for } \ t \in [t_2+\nu\omega, t_3+\nu\omega[\end{cases},$$

where ν is an integer. Obviously, $\tau_0(t) = \tau(t)$ for $t \in]0, \omega[$ and

$$\frac{\int_0^\omega [p(s)]_- ds}{1-\int_0^\omega [p(s)]_- ds} = n-1, \qquad \int\limits_0^\omega [p(s)]_+ ds = 1+n+\frac{1}{\varepsilon_0}+\frac{1}{n^2\varepsilon_0} > n-1.$$

Further put

$$\ell_1(v)(t) \stackrel{\text{def}}{=} [p(t)]_- v(\tau(t)), \qquad \ell_2(v)(t) \stackrel{\text{def}}{=} [p(t)]_+ v(\tau(t)) \qquad \text{for } \ t \in R.$$

Define the function γ as follows:

$$\gamma(t) = \begin{cases} t+\delta & \text{for} \quad t \in [0, t_1[\\ \frac{\delta}{\varepsilon_0} t + c_1 & \text{for} \quad t \in [t_1, t_2[\\ \frac{\delta+(n+1)\varepsilon_0}{n\varepsilon_0} t + c_2 & \text{for} \quad t \in [t_2, t_3[\\ t + c_3 & \text{for} \quad t \in [t_3, \omega] \end{cases} .$$

It is easy to verify that

$$\gamma'(t) = [p(t)]_+ \gamma(\tau_0(t)) + [p(t)]_- \qquad \text{for} \quad t \in]0, \omega[,$$

i.e., the inequality (1.14) is satisfied with $i = 1$, $j = 2$, and

$$\gamma(\omega) - \gamma(0) = 1 + \varepsilon.$$

On the other hand, equation (4.1) has a nontrivial ω-periodic solution

$$u(t) = \begin{cases} t - \varepsilon_0 - \nu\omega & \text{for} \quad t \in [\nu\omega, t_1 + \nu\omega[\\ -t + (2n+1)\varepsilon_0 - \nu\omega & \text{for} \quad t \in [t_1 + \nu\omega, t_2 + \nu\omega[\\ t - 2 - (2n+1)\varepsilon_0 - \nu\omega & \text{for} \quad t \in [t_2 + \nu\omega, (\nu+1)\omega] \end{cases} ,$$

where ν is an integer. Therefore, according to the Riesz–Schauder theory there exists $q_0 \in L_\omega(R)$ such that the equation

$$u'(t) = p(t)u(\tau(t)) + q_0(t)$$

has no ω-periodic solution. In other words, the equation (0.1) with

$$f(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)) + q_0(t) \qquad \text{for} \quad t \in R$$

has no ω-periodic solution although the operator f satisfies the conditions (1.1) and (1.13) with $q \equiv |q_0|$.

This example shows that the condition (1.15) in Theorems 1.2 and 1.4 cannot be replaced by the condition

$$\gamma((2-i)\omega) - \gamma((i-1)\omega) \leq 1 + \varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

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References

1. N. V. Azbelev, V. P. Maksimov, and L. F. Rakhmatullina, Introduction to the theory of functional differential equations. (Russian) *Nauka, Moscow*, 1991.

2. N. V. Azbelev and L. F. Rakhmatullina, Theory of linear abstract functional differential equations and aplications. *Mem. Differential Equations Math. Phys.* **8**(1996), 1–102.

3. E. Bravyi, A note on the Fredholm property of boundary value problems for linear functional differential equations, *Mem. Differential Equations Math. Phys.* **20**(2000), 133–135.

4. E. Bravyi, R. Hakl, and A. Lomtatidze, Optimal conditions for unique solvability of the Cauchy problem for first order linear functional differential equations. *Czechoslovak Math. J.* (*to appear*).

5. E. Bravyi, R. Hakl, and A. Lomtatidze, On Cauchy problem for the first order nonlinear functional differential equations of non-Volterra's type. *Czechoslovak Math. J.* (*to appear*).

6. R. Hakl, On some boundary value problems for systems of linear functional differential equations. *E. J. Qualitative Theory of Diff. Equ.* 10(1999), 1–16.

7. R. Hakl, I. Kiguradze, and B. Půža, Upper and lower solutions of boundary value problems for functional differential equations and theorems on functional differential inequalities. *Georgian Math. J.* **7**(2000), No. 3, 489–512.

8. R. Hakl, A. Lomtatidze, and B. Půža, On a periodic boundary value problem for the first order scalar functional differential equation. *J. Math. Anal. Appl.* (*submitted*).

9. R. Hakl, A. Lomtatidze, and B. Půža, On periodic solutions of first order linear functional differential equations, *Nonlinear Anal.* (*to appear*).

10. R. Hakl, A. Lomtatidze, and J. Šremr, On a periodic type boundary value problem for first order linear functional differential equations, *Ann. Polon. Math.*, (*submitted*).

11. J. Hale, Theory of functional differential equations. *Springer–Verlag, New York Heidelberg Berlin*, (1977).

12. I. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Tbilisi University Press, Tbilisi*, 1975.

13. I. Kiguradze, Boundary value problems for systems of ordinary differential equations. (Russian) *Current problems in mathematics. Newest results,* 30, 3–103, *VINITI, Moscow*, 1987.

14. I. Kiguradze, On periodic solutions of first order nonlinear differential equations with deviating arguments. *Mem. Differential Equations Math. Phys.* **10**(1997), 134–137.

15. I. Kiguradze, Initial and boundary value problems for systems of ordinary differential equations, I. (Russian) *Metsniereba, Tbilisi*, 1997.

16. I. Kiguradze and B. Půža, On boundary value problems for systems of linear functional differential equations. *Czechoslovak Math. J.* **47**(1997), No. 2, 341–373.

17. I. Kiguradze and B. Půža, On periodic solutions of systems of linear functional differential equations. *Arch. Math.* **33**(1997), No. 3,

197–212.

18. I. Kiguradze and B. Půža, Conti–Opial type theorems for systems of functional differential equations. (Russian) *Differentsial'nye Uravneniya* **33**(1997), No. 2, 185–194.

19. I. Kiguradze and B. Půža, On boundary value problems for functional differential equations, *Mem. Differential Equations Math. Phys.* **12**(1997), 106–113.

20. I. Kiguradze and B. Půža, On periodic solutions of nonlinear functional differential equations, *Georgian Math. J.* **6**(1999), No. 1, 47–66.

21. I. Kiguradze and B. Půža, On periodic solutions of systems of differential equations with deviating arguments, *Nonlinear Anal.* **42**(2000), No. 2, 229–242.

22. V. Kolmanovskii and A. Myshkis, Introduction to the theory and applications of functional differential equations. *Kluwer Academic Publishers*, 1999.

23. J. Mawhin, Periodic solutions of nonlinear functional differential equations. *J. Differential Equations* **10**(1971), 240–261.

24. B. Půža, On periodic solutions of linear differential systems with deviating arguments. (Russian) *Differentsial'nye Uravneniya* **31**(1995), No. 11, 1937–1938.

25. Š. Schwabik, M. Tvrdý and O. Vejvoda, Differential and integral equations: boundary value problems and adjoints. *Academia, Praha*, 1979.

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