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ON NONNEGATIVE SOLUTIONS OF SECOND ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. On the interval $[a,b]$, we consider the boundary value problems

$$u''(t) = \ell(u)(t) + q(t); \qquad u(a) = 0, \quad u(b) = 0$$

and

$$u''(t) = \ell(u)(t) + q(t); \qquad u(a) = 0, \quad u'(b) = 0,$$

where $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ and $q \in L([a,b];\mathbb{R})$.

The existence and uniqueness of nonnegative solutions of these problems are studied.

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რეზიუმე. ნაშრომში განხილულია სასაზღვრო ამოცანები

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შესწავლილია ამ ამოცანების არაუარყოფითი ამონახსნების არსებობის და ერთადერთობის საკითხები.

Large Introduction

The following notation is used throughout the paper.

$\mathbb{N}$ is the set of all natural numbers.

$\mathbb{R}$ is the set of all real numbers. $\mathbb{R}_+ = [0, +\infty[$.

If $x \in \mathbb{R}$, then $[x]_+ = \frac{1}{2}(|x| + x)$.

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$.

$C([a,b];\mathbb{R}_+) = \{u \in C([a,b];\mathbb{R}) : u(t) \geq 0 \text{ for } t \in [a,b]\}$.

$C_{t_0}([a,b];\mathbb{R}_+) = \{v \in C([a,b];\mathbb{R}_+) : v(t_0) = 0\}$, where $t_0 \in [a,b]$.

$\widetilde{C}([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$.

$\widetilde{C}'([a,b];\mathbb{R})$ is the set of functions $u \in \widetilde{C}([a,b];\mathbb{R})$, such that $u' \in \widetilde{C}([a,b];\mathbb{R})$.

$\widetilde{C}'_{loc}(D;\mathbb{R})$, where $D \subset \mathbb{R}$, is the set of functions $\gamma \in \widetilde{C}(\bar{D};\mathbb{R})$ such that $\gamma' \in \widetilde{C}([\alpha,\beta];\mathbb{R})$ for every $[\alpha,\beta] \subseteq D$.

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\, ds$.

$L([a,b];\mathbb{R}_+) = \{p \in L([a,b];\mathbb{R}) : p(t) \geq 0 \text{ for } t \in [a,b]\}$.

M_{ab} is the set of measurable functions $f : [a,b] \to [a,b]$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.

We will say that $\ell \in \mathcal{L}_{ab}$ is a t_0–Volterra operator, where $t_0 \in [a,b]$, if for every $a_1 \in [a,t_0]$, $b_1 \in [t_0,b]$, $a_1 \neq b_1$ and $v \in C([a,b];\mathbb{R})$ satisfying the condition

$$v(t) = 0 \quad \text{for } t \in [a_1,b_1],$$

we have

$$\ell(v)(t) = 0 \quad \text{for } t \in [a_1,b_1].$$

By a solution of the equation

$$u''(t) = \ell(u)(t) + q(t), \tag{0.1}$$

where $\ell \in \mathcal{L}_{ab}$ and $q \in L([a,b];\mathbb{R})$, we understand a function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying equality (0.1) almost everywhere in $[a,b]$.

Consider the problem on the existence and uniqueness of a solution of equation (0.1) satisfying one of the following boundary conditions

$$u(a) = c_1, \quad u(b) = c_2, \tag{0.2}$$

$$u(a) = c_1, \quad u'(b) = c_2, \tag{0.3}$$

where $c_1, c_2 \in \mathbb{R}$. Along with problems (0.1), (0.2) and (0.1), (0.3), consider the corresponding homogeneous problems

$$u''(t) = \ell(u)(t), \tag{0.1$_0$}$$

$$u(a) = 0, \quad u(b) = 0, \tag{0.2$_0$}$$

$$u(a) = 0, \quad u'(b) = 0. \tag{0.3$_0$}$$

The following result is well-known from the general theory of boundary value problems for functional differential equations (see e.g. [1, 2, 3, 11, 16]).

Theorem 0.1. *The problem* (0.1), (0.2) (*resp.* (0.1), (0.3)) *is uniquely solvable iff the corresponding homogeneous problem* (0.1$_0$), (0.2$_0$) (*resp.* (0.1$_0$), (0.3$_0$)) *has only the trivial solution.*

Definition 0.1. We will say that an operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $V([a,b])$ (resp. $V'([a,b])$) if for every function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying (0.2$_0$) (resp. (0.3$_0$)) and

$$u''(t) \geq \ell(u)(t) \quad \text{for } t \in [a,b], \tag{0.4}$$

the inequality

$$u(t) \leq 0 \quad \text{for } t \in [a,b] \tag{0.5}$$

holds.

Remark 0.1. It follows from Definition 0.1 and Theorem 0.1 that if $\ell \in V([a,b])$ (resp. $\ell \in V'([a,b])$) then the problem (0.1), (0.2) (resp. (0.1), (0.3)) is uniquely solvable. Moreover, if $q \in L([a,b];\mathbb{R}_+)$, then the unique solution of the problem (0.1), (0.2$_0$) (resp. (0.1), (0.3$_0$)) is nonpositive.

In the present paper we establish sufficient conditions guaranteeing the inclusion $\ell \in V([a,b])$, resp. $\ell \in V'([a,b])$. The results obtained here generalize and make more complete the previously known ones of analogous character (see e.g., [1, 2, 13, 14, 17, 18] and references therein) The related results for another type of equations can be found in [4, 5, 6, 7, 9, 10, 12, 15].

The paper is organized as follow. The main results are formulated in §1. In §2, some auxiliary propositions are proved. The proofs of main results are contained in §3. §4 deals with the special case of operator ℓ, with so-called operator with deviating arguments. The last §5 is devoted to the examples verifying the optimality of some obtained results.

1. Main Results

In this section we formulate the main results. Theorems 1.1 and 1.2, and Corollaries 1.1 and 1.2 concern the case $-\ell \in \mathcal{P}_{ab}$. The case when $\ell \in \mathcal{P}_{ab}$ is considered in Theorems 1.3–1.6 and Corollaries 1.3–1.7. Finally, Theorem 1.7 deals with the case when the operator $\ell \in \mathcal{L}_{ab}$ admits the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$.

Theorem 1.1. *Let* $-\ell \in \mathcal{P}_{ab}$. *Then* $\ell \in V([a,b])$ *iff there exists a function*
$\gamma \in \widetilde{C}'_{loc}(]a,b[;\mathbb{R})$ *satisfying*

$$\gamma''(t) \leq \ell(\gamma)(t) \quad \textit{for } t \in [a,b], \tag{1.1}$$

$$\gamma(t) > 0 \quad \textit{for } t \in]a,b[, \tag{1.2}$$

$$\gamma(a) + \gamma(b) + \operatorname{mes}\{t \in [a,b] : \gamma''(t) < \ell(\gamma)(t)\} > 0. \tag{1.3}$$

Theorem 1.2. *Let $-\ell \in \mathcal{P}_{ab}$. Then $\ell \in V'([a,b])$ iff there exists a function*
$\gamma \in \widetilde{C}'_{loc}(]a,b];\mathbb{R})$ *satisfying* (1.1), (1.2) *and*

$$\gamma'(b) \geq 0, \tag{1.4}$$

$$\gamma(a) + \gamma'(b) + \operatorname{mes}\{t \in [a,b] : \gamma''(t) < \ell(\gamma)(t)\} > 0. \tag{1.5}$$

Corollary 1.1. *Let $-\ell \in \mathcal{P}_{ab}$ and*

$$(b-t)\int_a^t (s-a)|\ell(1)(s)|\,ds + (t-a)\int_t^b (b-s)|\ell(1)(s)|\,ds < b-a \quad \textit{for } t \in [a,b]. \tag{1.6}$$

Then $\ell \in V([a,b])$.

Corollary 1.2. *Let $-\ell \in \mathcal{P}_{ab}$ and*

$$\int_a^b (s-a)|\ell(1)(s)|\,ds < 1. \tag{1.7}$$

Then $\ell \in V'([a,b])$.

Remark 1.1. Example 5.1, resp. Example 5.2, below shows that condition (1.6), resp. condition (1.7), cannot be replaced by the condition

$$(b-t)\int_a^t (s-a)|\ell(1)(s)|\,ds + (t-a)\int_t^b (b-s)|\ell(1)(s)|\,ds \leq b-a \quad \text{for } t \in [a,b], \tag{1.8}$$

resp.

$$\int_a^b (s-a)|\ell(1)(s)|\,ds \leq 1. \tag{1.9}$$

Before the formulation of next theorems we introduce the following notation. Let $\ell \in \mathcal{L}_{ab}$ be a b–Volterra (resp. a–Volterra) operator and $\xi \in]a,b[$. Let $\ell_{\xi b}$ (resp. $\ell_{a\xi}$) denote the restriction of the operator ℓ to the space $C([\xi,b];\mathbb{R})$ (resp. $C([a,\xi];\mathbb{R})$). Put

$$A_\ell \overset{\text{def}}{=} \{t \in [a,b] : \ell(1)(x) = 0 \text{ for } x \in [a,t]\}, \tag{1.10}$$

$$B_\ell \overset{\text{def}}{=} \{t \in [a,b] : \ell(1)(x) = 0 \text{ for } x \in [t,b]\}, \tag{1.11}$$

$$a_\ell = \sup A_\ell, \tag{1.12}$$

$$b_\ell = \inf B_\ell. \tag{1.13}$$

Theorem 1.3. *Let $\ell \in \mathcal{P}_{ab}$ be a b–Volterra operator and let there exist a function $\gamma \in \widetilde{C}'_{loc}([a_\ell, b[;\mathbb{R})$ satisfying the conditions*

$$\gamma''(t) \geq \ell_{a_\ell b}(\gamma)(t) \quad \textit{for } t \in [a_\ell, b], \tag{1.14}$$

$$\gamma(t) > 0 \quad \textit{for } t \in \,]a_\ell, b], \tag{1.15}$$

$$\gamma'(a_\ell) \geq 0. \tag{1.16}$$

Then $\ell \in V([a,b])$.

Theorem 1.4. *Let $\ell \in \mathcal{P}_{ab}$ be an a–Volterra operator and let there exist a function $\gamma \in \widetilde{C}'_{loc}(]a, b_\ell];\mathbb{R})$ satisfying the conditions*

$$\gamma''(t) \geq \ell_{ab_\ell}(\gamma)(t) \quad \textit{for } t \in [a, b_\ell], \tag{1.14$'$}$$

$$\gamma(t) > 0 \quad \textit{for } t \in [a, b_\ell[, \tag{1.15$'$}$$

$$\gamma'(b_\ell) \leq 0. \tag{1.16$'$}$$

Then $\ell \in V([a,b])$.

Corollary 1.3. *Let $\ell \in \mathcal{P}_{ab}$ be a b–Volterra operator and let there exist $m, k \in \mathbb{N}, m > k$, such that*

$$\varphi_m(t) \leq \varphi_k(t) \quad \textit{for } t \in [a,b], \tag{1.17}$$

where $\varphi_1 \in \widetilde{C}'([a,b];\mathbb{R})$ satisfies

$$\varphi_1(t) > 0, \ \varphi_1'(t) \geq 0 \quad \textit{for } t \in \,]a,b] \tag{1.18}$$

and

$$\varphi_{i+1}(t) \overset{def}{=} \int_a^t (t-s)\ell(\varphi_i)(s)\,ds \quad \textit{for } t \in [a,b].$$

Then $\ell \in V([a,b])$.

Corollary 1.4. *Let $\ell \in \mathcal{P}_{ab}$ be an a–Volterra operator and there exist $m, k \in \mathbb{N}, m > k$, such that*

$$\psi_m(t) \leq \psi_k(t) \quad \textit{for } t \in [a,b],$$

where $\psi_1 \in \widetilde{C}'([a,b];\mathbb{R})$ satisfies

$$\psi_1(t) > 0, \ \psi_1'(t) \leq 0 \quad \textit{for } t \in [a,b[$$

and

$$\psi_{i+1}(t) \overset{def}{=} \int_t^b (s-t)\ell(\psi_i)(s)\,ds \quad \textit{for } t \in [a,b].$$

Then $\ell \in V([a,b])$.

Corollary 1.5. *Let $\ell \in \mathcal{P}_{ab}$ be a b–Volterra operator and let there exist an $\bar{f} \in \mathcal{P}_{ab}$ such that*

$$\int_a^b \bar{f}(1)(s) \exp\left[\int_s^b f(1)(\xi)\, d\xi\right] ds < 1, \tag{1.19}$$

where the operator $f \in \mathcal{L}_{ab}$ is defined by the formulas

$$f(v)(t) \overset{def}{=} \ell(\theta(v))(t), \tag{1.20}$$

$$\theta(v)(t) \overset{def}{=} \int_a^t v(s)\, ds. \tag{1.21}$$

Let, moreover, the inequality

$$f(\varphi(v))(t) - f(1)(t)\varphi(v)(t) \leq \bar{f}(v)(t) \quad \text{for } t \in [a,b] \tag{1.22}$$

holds on the set $C_a([a,b];\mathbb{R}_+)$, where

$$\varphi(v)(t) \overset{def}{=} \int_a^t f(v)(s)\, ds \quad \text{for } t \in [a,b]. \tag{1.23}$$

Then $\ell \in V([a,b])$.

Corollary 1.6. *Let $\ell \in \mathcal{P}_{ab}$ be an a–Volterra operator and let there exists $\bar{f} \in \mathcal{P}_{ab}$ such that*

$$\int_a^b \bar{f}(1)(s) \exp\left[\int_a^s |f(1)(\xi)|\, d\xi\right] ds < 1,$$

where the operator $f \in \mathcal{L}_{ab}$ defined by formulas

$$f(v)(t) \overset{def}{=} -\ell(\theta(v))(t),$$

$$\theta(v)(t) \overset{def}{=} \int_t^b v(s)\, ds.$$

Moreover, assume that on the set $C_b([a,b];\mathbb{R}_+)$, the inequality (1.22) *holds, where*

$$\varphi(v)(t) \overset{def}{=} \int_t^b f(v)(s)\, ds \quad \text{for } t \in [a,b].$$

Then $\ell \in V([a,b])$.

Theorem 1.5. *Let the conditions of Theorem* 1.3 *be fulfilled. Then $\ell \in V'([a,b])$.*

Theorem 1.6. *Let the conditions of Theorem* 1.4 *be fulfilled. Then $\ell \in V'([a,b])$.*

Corollary 1.7. *Let $\ell \in \mathcal{P}_{ab}$ be a b–Volterra (resp. a–Volterra) operator and conditions of Corollary 1.3 or 1.5 (resp. Corollary 1.4 or 1.6) hold. Then $\ell \in V'([a,b])$.*

Theorem 1.7. *Let the operator $\ell \in \mathcal{L}_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and*

$$\ell_0 \in V([a,b]), \quad -\ell_1 \in V([a,b]) \tag{1.24}$$

$$\Big(\ell_0 \in V'([a,b]), \quad -\ell_1 \in V'([a,b])\Big). \tag{1.25}$$

Then $\ell \in V([a,b])$ $\big(\ell \in V'([a,b])\big)$.

2. Auxiliary Propositions

Lemma 2.1. *Let $v \in \widetilde{C}'_{loc}(]a,b[;\mathbb{R})$ be a nontrivial function satisfying the relations*

$$v(t) \geq 0 \quad \text{for } t \in [a,b], \tag{2.1}$$

$$v''(t) \leq 0 \quad \text{for } t \in [a,b], \tag{2.2}$$

$$v(a) = 0 \quad (\text{resp. } v(b) = 0). \tag{2.3}$$

Then there exists a finite or infinite limit

$$\lim_{t\to a+} \frac{v(t)}{t-a} > 0 \qquad \left(\text{resp. } \lim_{t\to b-} \frac{v(t)}{b-t} > 0\right).$$

Proof. By virtue of (2.2), there exists a finite or infinite limit

$$\lim_{t\to a+} v'(t) = r \qquad \left(\text{resp. } \lim_{t\to b-} v'(t) = r\right). \tag{2.4}$$

Suppose that $r \leq 0$ (resp. $r \geq 0$). From (2.2) and (2.4), we then easily get

$$v'(t) \leq 0 \quad (\text{resp. } v'(t) \geq 0) \quad \text{for } t \in]a,b[.$$

Integrating the above inequality from a to t (resp. from t to b) and taking into account (2.3), we obtain

$$v(t) = \int_a^t v'(s)\,ds \leq 0 \quad \text{for } t \in [a,b]$$

$$\left(\text{resp. } v(t) = -\int_t^b v'(s)\,ds \leq 0 \quad \text{for } t \in [a,b]\right),$$

which, together with (2.1), contradicts the assumption of the lemma. Therefore $r > 0$ (resp. $r < 0$).

Let us first suppose that $r = +\infty$ (resp. $r = -\infty$). Then, for every $n \in \mathbb{N}$, there exists $t_n \in]a,b[$ such that

$$v'(t) \geq n \quad \text{for } t \in]a,t_n[\qquad \big(\text{resp. } v'(t) \leq -n \quad \text{for } t \in]t_n,b[\big).$$

The integration of the above inequality from a to t (from t to b), by virtue of (2.3), yields

$$v(t) \geq n(t-a) \quad \text{for } t \in [a, t_n], \quad n \in \mathbb{N},$$
$$\big(\text{resp. } v(t) \geq n(b-t) \quad \text{for } t \in [t_n, b], \quad n \in \mathbb{N}\big).$$

Therefore,

$$\lim_{t \to a+} \frac{v(t)}{t-a} = +\infty \qquad \left(\text{resp. } \lim_{t \to b-} \frac{v(t)}{b-t} = +\infty\right).$$

Suppose now that $0 < r < +\infty$ (resp. $-\infty < r < 0$). Then, for every $n \in \mathbb{N}$, there exists $t_n \in]a, b[$ such that

$$r - \frac{1}{n} \leq v'(t) \leq r + \frac{1}{n} \quad \text{for } t \in]a, t_n[\quad \big(\text{resp. for } t \in]t_n, b[\big).$$

Integrating the above inequality from a to t (from t to b) and taking into account (2.3), we get

$$(r - \frac{1}{n})(t-a) \leq v(t) \leq (r + \frac{1}{n})(t-a) \quad \text{for } t \in [a, t_n],\ n \in \mathbb{N}$$
$$\left(\text{resp. } -(r + \frac{1}{n})(b-t) \leq v(t) \leq -(r - \frac{1}{n})(b-t) \quad \text{for } t \in [t_n, b],\ n \in \mathbb{N}\right).$$

Therefore

$$\lim_{t \to a+} \frac{v(t)}{t-a} = r \qquad \left(\text{resp. } \lim_{t \to b-} \frac{v(t)}{b-t} = -r\right).$$

The lemma is proved. □

Lemma 2.2. *Assume that $t_0 \in]a, b[$ and the function $w \in \widetilde{C}'_{loc}(]a, b[; \mathbb{R})$ satisfies the equality $w(t_0) = 0$ and*

$$\begin{aligned} w(t) &\geq 0 \quad \textit{for } t \in [a, b], \\ w''(t) &\leq 0 \quad \textit{for } t \in [a, b]. \end{aligned} \tag{2.5}$$

Then $w \equiv 0$.

Proof. It is obvious that

$$w'(t_0) = 0$$

and

$$w'(t) \geq 0 \quad \text{for } t \in]a, t_0], \qquad w'(t) \leq 0 \quad \text{for } t \in [t_0, b[.$$

Hence,

$$w(t) = -\int_t^{t_0} w'(s)\, ds \leq 0 \quad \text{for } t \in]a, t_0],$$
$$w(t) = \int_{t_0}^{t} w'(s)\, ds \leq 0 \quad \text{for } t \in [t_0, b[.$$

These two inequalities, together with (2.5) yield $w \equiv 0$. □

Analogously, one can prove that the following lemma is true.

Lemma 2.3. *Let $v \in \widetilde{C}'_{loc}(]a,b];\mathbb{R})$ be a nontrivial function satisfying* (2.1), (2.2)*, and the inequality*

$$v'(b) \geq 0.$$

Then $v(b) > 0$.

Lemma 2.4. *Let $\ell \in \mathcal{P}_{ab}$, $\ell(1) \not\equiv 0$, and let the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy* (0.4) *and*

$$u(a) = 0, \tag{2.6}$$

$$u(t_1) > 0, \tag{2.7}$$

where $t_1 \in]a,b[$. Then there exists $t_0 \in]a_\ell, b[$ such that

$$u(t_0) > 0, \quad u'(t_0) > 0, \tag{2.8}$$

where a_ℓ is defined by (1.10) *and* (1.12).

Proof. Suppose that $a_\ell \in]a,b[$. (The case when $a_\ell = a$ is proved analogously.) Assume that the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfies (2.6), (0.4), and (2.7). By virtue of the condition $\ell \in \mathcal{P}_{ab}$ it follows from (0.4) and the definition of a_ℓ (see (1.10) and (1.12)) that

$$u''(t) \geq -\|u\|_C \ell(1)(t) \quad \text{for } t \in [a,b].$$

Consequently,

$$u''(t) \geq 0 \quad \text{for } t \in [a, a_\ell]. \tag{2.9}$$

It is obvious that either

$$u'(t_1) > 0 \tag{2.10}$$

or

$$u'(t_1) \leq 0. \tag{2.11}$$

Let us first suppose that (2.10) holds. If $t_1 \in]a_\ell, b[$ then obviously (2.8) is fulfilled for $t_0 = t_1$. If $t_1 \in]a, a_\ell]$, then, by virtue of (2.7) and (2.9), we also have

$$u(a_\ell) > 0, \quad u'(a_\ell) > 0.$$

These inequalities imply that there exists $t_0 \in]a_\ell, b[$ such that (2.8) hold.

Now we suppose that (2.11) is fulfilled. Obviously $t_1 \in]a_\ell, b[$, because otherwise, by virtue of (2.9) and (2.11), we get

$$u'(t) \leq 0 \quad \text{for } t \in [a, t_1],$$

which, together with the condition (2.6), contradicts (2.7). Moreover,

$$\max\{u'(t) : t \in [a_\ell, t_1]\} > 0, \tag{2.12}$$

because otherwise it follows from (2.7) that

$$u(a_\ell) > 0, \quad u'(a_\ell) \leq 0,$$

which, together with (2.9), contradicts the condition (2.6). It is clear that either

$$u(t) > 0 \quad \text{for } t \in [a_\ell, t_1] \tag{2.13}$$

or there exists $t_* \in [a_\ell, t_1[$ such that

$$u(t) > 0 \quad \text{for } t \in]t_*, t_1], \quad u(t_*) = 0. \tag{2.14}$$

If (2.13) is satisfied, then, by virtue of (2.12), there exists $t_0 \in]a_\ell, t_1[$ such that (2.8) holds. If (2.14) is fulfilled, then, evidently,

$$\max\{u'(t) : t \in [t_*, t_1]\} > 0, \tag{2.15}$$

because otherwise, it follows from the second condition in (2.14) that $u(t_1) \leq 0$, which contradicts (2.7). It follows immediately from (2.14) and (2.15) that, for some $t_0 \in]t_*, t_1[$, inequalities (2.8) are true. □

The following lemma is proved analogously.

Lemma 2.5. *Let $\ell \in \mathcal{P}_{ab}$, $\ell(1) \not\equiv 0$, and let the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy* (0.4), (2.7), *where $t_1 \in]a,b[$, and*

$$u(b) = 0. \tag{2.16}$$

Then there exists $t_0 \in]a, b_\ell[$ such that

$$u(t_0) > 0, \quad u'(t_0) < 0, \tag{2.17}$$

where b_ℓ is defined by (1.11) *and* (1.13).

Lemma 2.6. *Let $\ell \in \mathcal{P}_{ab}$, $\ell(1) \not\equiv 0$, and let the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy* (0.4), (2.7), *where $t_1 \in]a,b[$, and*

$$u'(b) = 0. \tag{2.18}$$

Then there exists $t_0 \in]a, b_\ell]$ such that

$$u(t_0) > 0, \quad u'(t_0) \leq 0, \tag{2.19}$$

where b_ℓ is defined by (1.11) *and* (1.13).

Proof. Suppose that $b_\ell \in]a,b[$. (The case when $b_\ell = b$ is proved analogously.) Let $u \in \widetilde{C}'([a,b];\mathbb{R})$ be a function satisfying (0.4), (2.7), and (2.18). It follows immediately from (0.4), (1.11), (1.13), and the condition $\ell \in \mathcal{P}_{ab}$ that

$$u''(t) \geq 0 \quad \text{for } t \in [b_\ell, b]. \tag{2.20}$$

It is clear that

$$u(b_\ell) > 0 \tag{2.21}$$

or

$$u(b_\ell) \leq 0. \tag{2.22}$$

By virtue of (2.20) and the condition (2.18), we have

$$u'(b_\ell) \leq 0.$$

Therefore, in the case where (2.21) is fulfilled, the inequalities (2.19) also hold with $t_0 = b_\ell$.

Suppose now that (2.22) is fulfilled. Then (2.20) and the condition (2.18) yield

$$u(t) \le 0 \quad \text{for } t \in [b_\ell, b].$$

Hence, in view of (2.7),

$$t_1 \in]a, b_\ell[. \tag{2.23}$$

It follows from (2.7), (2.22), and (2.23) that there exists $t_0 \in]t_1, b_\ell[$ such that (2.19) hold. □

3. Proof of the Main Results

Proof of Theorem 1.1. Let $-\ell \in \mathcal{P}_{ab}$ and $\ell \in V([a,b])$. Let γ denote the solution of the problem

$$\gamma''(t) = \ell(\gamma)(t); \quad \gamma(a) = 1, \ \gamma(b) = 1 \tag{3.1}$$

(see Remark 0.1). Clearly, (1.1) and (1.3) hold. Put

$$u_0(t) = 1 - \gamma(t) \quad \text{for } t \in [a,b].$$

By virtue of (3.1) and the condition $-\ell \in \mathcal{P}_{ab}$, we get

$$u_0''(t) \ge \ell(u_0)(t) \quad \text{for } t \in [a,b]; \quad u_0(a) = 0, \ u_0(b) = 0.$$

Hence, by virtue of the condition $\ell \in V([a,b])$, we have

$$u_0(t) \le 0 \quad \text{for } t \in [a,b],$$

i.e.,

$$\gamma(t) \ge 1 \quad \text{for } t \in [a,b].$$

Therefore, (1.2) holds as well.

Assume now that the function $\gamma \in \widetilde{C}'_{loc}(]a,b[;\mathbb{R})$ satisfies (1.1)–(1.3). By virtue of (1.1), (1.2), and the condition $-\ell \in \mathcal{P}_{ab}$, it follows that

$$\gamma''(t) \le 0 \quad \text{for } t \in [a,b]. \tag{3.2}$$

Let the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy (0.2_0), (0.4), and the condition

$$\max\{u(t) : t \in [a,b]\} > 0. \tag{3.3}$$

Integrating (0.4) from t to $\frac{a+b}{2}$ and from $\frac{a+b}{2}$ to t, we get

$$u'(t) \le M \quad \text{for } t \in]a, \tfrac{a+b}{2}], \tag{3.4}$$

$$u'(t) \ge -M \quad \text{for } t \in [\tfrac{a+b}{2}, b[, \tag{3.5}$$

where

$$M = \left| u'\left(\frac{a+b}{2}\right)\right| + \int_a^b |\ell(u)(s)|\, ds. \tag{3.6}$$

The integration of (3.4) and (3.5), by virtue of (0.2_0), yields

$$u(t) \le M(t-a) \quad \text{for } t \in [a, \tfrac{a+b}{2}], \tag{3.7}$$

$$u(t) \le M(b-t) \quad \text{for } t \in [\tfrac{a+b}{2}, b]. \tag{3.8}$$

Let us set

$$\lambda = \sup\left\{ \frac{u(t)}{\gamma(t)} \,:\, t \in \,]a,b[\right\}. \tag{3.9}$$

On account of (1.2), (3.2), (3.7), (3.8), and Lemma 2.1, we have

$$\lambda < +\infty.$$

On the other hand, by virtue of (1.2) and (3.3),

$$\lambda > 0. \tag{3.10}$$

Put

$$w(t) = \lambda\gamma(t) - u(t) \quad \text{for } t \in [a,b]. \tag{3.11}$$

Obviously,

$$w(t) \geq 0 \quad \text{for } t \in [a,b]. \tag{3.12}$$

It follows immediately from (0.4), (1.1), (1.3), and (3.10) that

$$w''(t) \leq \ell(w)(t) \quad \text{for } t \in [a,b], \tag{3.13}$$

$$w \not\equiv 0. \tag{3.14}$$

By virtue of the condition $-\ell \in \mathcal{P}_{ab}$, it follows from (3.12) and (3.13) that

$$w''(t) \leq 0 \quad \text{for } t \in [a,b]. \tag{3.15}$$

Now we will show that

$$\limsup_{t\to a+} \frac{u(t)}{\gamma(t)} < \lambda. \tag{3.16}$$

Indeed, if $\gamma(a) \neq 0$, then, by virtue of (0.2_0) and (3.10), inequality (3.16) is fulfilled. Suppose that

$$\gamma(a) = 0. \tag{3.17}$$

Then on account of (1.2), (3.2), (3.17), and Lemma 2.1, there exists (a finite or infinite) limit

$$\lim_{t\to a+} \frac{\gamma(t)}{t-a}.$$

If $\lim\limits_{t\to a+} \frac{\gamma(t)}{t-a} = +\infty$, then, by virtue of (3.7), we get

$$\limsup_{t\to a+} \frac{u(t)}{\gamma(t)} \leq M \lim_{t\to a+} \frac{t-a}{\gamma(t)} = 0$$

and, therefore, on account of (3.10), inequality (3.16) holds. Assume that

$$\lim_{t\to a+} \frac{\gamma(t)}{t-a} = r < +\infty. \tag{3.18}$$

Clearly,

$$r > 0 \tag{3.19}$$

(see Lemma 2.1). On account of (0.2_0), (3.12), (3.14), (3.15), (3.17), and Lemma 2.1, there exist $\varepsilon_0 > 0$ and $a_0 \in \,]a,b[$ such that

$$w(t) \geq \varepsilon_0(t-a) \quad \text{for } t \in \,]a, a_0[,$$

i.e.,

$$\frac{u(t)}{\gamma(t)} \le \lambda - \frac{\varepsilon_0(t-a)}{\gamma(t)} \quad \text{for } t \in]a, a_0[.$$

Hence, by virtue of (3.18) and (3.19), the inequality (3.16) holds. Analogously, one can show that

$$\limsup_{t \to b-} \frac{u(t)}{\gamma(t)} < \lambda. \tag{3.20}$$

By virtue of (3.9), (3.10), (3.16), and (3.20), it is clear that there exists $t_0 \in]a, b[$ such that

$$w(t_0) = 0.$$

Hence, on account of (3.12), (3.15), and Lemma 2.2, we get $w \equiv 0$, which contradicts (3.14). □

Proof of Theorem 1.2. Let $-\ell \in \mathcal{P}_{ab}$ and $\ell \in V'([a, b])$. Let γ denote the solution of the problem

$$\gamma''(t) = \ell(\gamma)(t); \quad \gamma(a) = 1, \ \gamma'(b) = 0. \tag{3.21}$$

Clearly (1.1), (1.4), and (1.5) are fulfilled. Put

$$u_0(t) = 1 - \gamma(t) \quad \text{for } t \in [a, b].$$

By virtue of (3.21) and the condition $-\ell \in \mathcal{P}_{ab}$, we get

$$u_0''(t) \ge \ell(u_0)(t) \quad \text{for } t \in [a, b]; \quad u_0(a) = 0, \ u_0'(b) = 0.$$

Hence, by virtue of condition $\ell \in V'([a, b])$, we easily get that (1.2) is fulfilled.

Now assume that a function $\gamma \in \widetilde{C}'_{loc}(]a, b]; \mathbb{R})$ satisfies (1.1), (1.2), (1.4), and (1.5). By virtue of (1.1), (1.2), and the condition $-\ell \in \mathcal{P}_{ab}$, it is clear that (3.2) holds. On account of (1.2), (1.4), (3.2), and Lemma 2.3, we get

$$\gamma(b) > 0. \tag{3.22}$$

Let a function $u \in \widetilde{C}'([a, b]; \mathbb{R})$ satisfy (0.3_0), (0.4), and (3.3). The integration of (0.4) from t to $\frac{a+b}{2}$ yields (3.4), where M is defined by (3.6). Integrating (3.4) and taking the first condition of (0.3_0) into account, we arrive at (3.7). Put

$$\lambda = \sup \left\{ \frac{u(t)}{\gamma(t)} \, : \, t \in]a, b] \right\}. \tag{3.23}$$

By virtue of (1.2), (3.2), (3.7), (3.22), and Lemma 2.1, we have

$$\lambda < +\infty.$$

On the other hand, on account of (1.2) and (3.3), inequality (3.10) holds. Define the function w by (3.11). Clearly, (3.12) is fulfilled. On account of (0.4), (1.1), (1.5), and (3.10), inequalities (3.13) and (3.14) are fulfilled. By virtue of (3.12) and the condition $-\ell \in \mathcal{P}_{ab}$, from (3.13) we get that (3.15) holds. By virtue of (0.3_0), (1.4), and (3.10),

$$w'(b) \ge 0. \tag{3.24}$$

Conditions (3.12), (3.14), and (3.15), on account of Lemma 2.3, imply

$$w(b) > 0. \tag{3.25}$$

Arguing similarly to the proof of Theorem 1.1, one can easily verify that (3.16) is fulfilled. On account of (3.16), (3.23), and (3.25), there exists $t_0 \in \,]a,b[$ such that

$$w(t_0) = 0. \tag{3.26}$$

By virtue of (3.12), (3.15), and (3.26), it follows from Lemma 2.2 that $w \equiv 0$, which contradicts (3.14). □

Proof of Corollary 1.1. If $\ell(1) \equiv 0$, then Corollary 1.1 is trivial. Therefore, we will assume that

$$\ell(1) \not\equiv 0. \tag{3.27}$$

Put

$$\gamma(t) = \frac{1}{b-a}\left[(b-t)\int_a^t (s-a)|\ell(1)(s)|\,ds + \right.$$

$$\left. + (t-a)\int_t^b (b-s)|\ell(1)(s)|\,ds\right] \quad \text{for } t \in [a,b]. \tag{3.28}$$

It follows from (1.6) that there exists $\varepsilon \in \,]0,1[$ such that

$$\gamma(t) \le \varepsilon \quad \text{for } t \in [a,b]. \tag{3.29}$$

On account of (3.27), (3.29), and the condition $-\ell \in \mathcal{P}_{ab}$, it is clear that

$$\ell(\gamma)(t) \ge \ell(1)(t) \quad \text{for } t \in [a,b], \tag{3.30}$$

$$\operatorname{mes}\{t \in [a,b] : \ell(\gamma)(t) > \ell(1)(t)\} > 0. \tag{3.31}$$

On the other hand, (3.27), (3.28), (3.30), and (3.31) imply that (1.1)–(1.3) are fulfilled. Therefore, the function γ satisfies all the conditions of Theorem 1.1. □

Proof of Corollary 1.2. As in the proof of Corollary 1.1, one can easily verify that the function

$$\gamma(t) = \int_a^t (s-a)|\ell(1)(s)|\,ds + (t-a)\int_t^b |\ell(1)(s)|\,ds \quad \text{for } t \in [a,b]$$

satisfies all the conditions of Theorem 1.2. □

Proof of Theorem 1.3 and 1.5. We first note that, by virtue of (1.14)–(1.16) and the condition $\ell \in \mathcal{P}_{ab}$,

$$\gamma'(t) \ge 0 \quad \text{for } t \in [a_\ell, b]. \tag{3.32}$$

Let a function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy (0.2_0) (resp. (0.3_0)), (0.4), and (3.3). According to Lemma 2.4 there exists

$$t_0 \in \,]a_\ell, b[\tag{3.33}$$

such that
$$u(t_0) > 0, \quad u'(t_0) > 0. \tag{3.34}$$
From (0.4), (1.14), and (3.33), we get
$$u''(t) \geq \ell_{t_0 b}(u)(t) \quad \text{for } t \in [t_0, b], \tag{3.35}$$
$$\gamma''(t) \geq \ell_{t_0 b}(\gamma)(t) \quad \text{for } t \in [t_0, b]. \tag{3.36}$$
Obviously
$$\min\{u(t) : t \in [t_0, b]\} < 0, \tag{3.37}$$
because otherwise by virtue of the condition $\ell \in \mathcal{P}_{ab}$, it follows from (3.35), that
$$u''(t) \geq 0 \quad \text{for } t \in [t_0, b],$$
which, together with (3.34), contradicts the second condition in (0.2_0) (resp. (0.3_0)). Put
$$\lambda = \max\left\{\frac{-u(t)}{\gamma(t)} : t \in [t_0, b]\right\} \tag{3.38}$$
and
$$w(t) = \lambda\gamma(t) + u(t) \quad \text{for } t \in [t_0, b]. \tag{3.39}$$
Clearly,
$$w(t) \geq 0 \quad \text{for } t \in [t_0, b]. \tag{3.40}$$
On account of (3.37) and (1.15), we also have
$$\lambda > 0. \tag{3.41}$$
By virtue of (3.35), (3.36), (3.40), (3.41), and the condition $\ell \in \mathcal{P}_{ab}$, we get
$$w''(t) \geq 0 \quad \text{for } t \in [t_0, b]. \tag{3.42}$$
It follows from (1.15), (3.32), (3.34), and (3.41) that
$$w(t_0) > 0, \quad w'(t_0) > 0. \tag{3.43}$$
By virtue of (3.38), (3.39), and the first inequality in (3.43), there exists $t_* \in]t_0, b]$ such that
$$w(t_*) = 0,$$
which contradicts (3.42) and (3.43). □

Proof of Theorems 1.4 *and* 1.6. The proof of Theorem 1.4, resp. Theorem 1.6, is analogous of Theorem 1.3 and 1.5. Lemma 2.5, resp. Lemma 2.6, should be used instead of Lemma 2.4. □

Proof of Corollary 1.3. Suppose that $a_\ell \in [a, b[$ (if $a_\ell = b$, then $\ell \equiv 0$ and, therefore, $\ell \in V([a, b])$). By virtue of (1.18), it is easy to verify that
$$\varphi_i(t) > 0 \quad \text{for } t \in]a_\ell, b], \ i \in \mathbb{N}. \tag{3.44}$$
Let us put
$$\gamma(t) = \sum_{i=k+1}^{m} \varphi_i(t) \quad \text{for } t \in [a, b].$$

Evidently $\gamma \in \widetilde{C}'([a,b];\mathbb{R})$,

$$\gamma(t) > 0 \quad \text{for } t \in]a_\ell, b]; \quad \gamma'(a_\ell) \geq 0.$$

On the other hand, by virtue of (1.17) and the condition $\ell \in \mathcal{P}_{ab}$, we get

$$\gamma''(t) = \sum_{i=k}^{m-1} \ell(\varphi_i)(t) = \ell(\gamma)(t) + \ell(\varphi_k - \varphi_m)(t) \geq \ell(\gamma)(t) \quad \text{for } t \in [a,b].$$

Therefore, the function γ satisfies all the conditions of Theorem 1.3. □

Corollary 1.4 is proved similarly.

Proof of Corollary 1.5. According to Corollary 1.1 in [8] and the conditions (1.19) and (1.22), there exists $w \in \widetilde{C}([a,b];\mathbb{R})$ such that

$$w'(t) = f(w)(t) \quad \text{for } t \in [a,b], \tag{3.45}$$

$$w(t) > 0 \quad \text{for } t \in [a,b]. \tag{3.46}$$

Put

$$\gamma(t) = \theta(w)(t) \quad \text{for } t \in [a,b], \tag{3.47}$$

where θ is defined by (1.21). Clearly, $\gamma \in \widetilde{C}'([a,b];\mathbb{R})$. It follows immediately from (1.20), (1.21), (3.45), (3.46), and (3.47) that

$$\gamma''(t) = w'(t) = f(w)(t) = \ell(\gamma)(t) \quad \text{for } t \in [a,b],$$

$$\gamma(t) > 0 \quad \text{for } t \in]a,b],$$

$$\gamma'(t) > 0 \quad \text{for } t \in [a,b].$$

Therefore function γ satisfies all the conditions of Theorem 1.3. □

Proof of Corollary 1.6 is similar.

Proof of Theorem 1.7. Let $u \in \widetilde{C}'([a,b];\mathbb{R})$ be a function satisfying (0.2_0) (resp. (0.3_0)) and (0.4). According to Remark 0.1 and the condition $\ell_0 \in V([a,b])$ (resp. $\ell_0 \in V'([a,b])$), the problem

$$v''(t) = \ell_0(v)(t) - \ell_1([u]_+)(t), \tag{3.48}$$

$$v(a) = 0, \quad v(b) = 0 \qquad \big(\text{resp. } v(a) = 0, \quad v'(b) = 0\big) \tag{3.49}$$

has a unique solution v and

$$v(t) \geq 0 \quad \text{for } t \in [a,b]. \tag{3.50}$$

It follows from Remark 0.1, relations (0.2_0) (resp. (0.3_0)), (0.4), (3.48), (3.49), and the condition $\ell_1 \in \mathcal{P}_{ab}$ that

$$u(t) \leq v(t) \quad \text{for } t \in [a,b]. \tag{3.51}$$

Hence, on account of (3.50),

$$[u(t)]_+ \leq v(t) \quad \text{for } t \in [a,b]. \tag{3.52}$$

By virtue of (3.50), (3.52), and the conditions $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, it follows from (3.48) that

$$v''(t) \geq -\ell_1(v)(t) \quad \text{for } t \in [a,b]. \tag{3.53}$$

On account of the condition $-\ell_1 \in V([a,b])$ (resp. $-\ell_1 \in V'([a,b])$), (3.49), and (3.53) imply

$$v(t) \le 0 \quad \text{for } t \in [a,b].$$

Hence, by virtue of (3.51), the inequality (0.5) is fulfilled. □

4. Equations with Deviating Argument

In this section, the results from §1 will be concretized for the case, where the operator $\ell \in \mathcal{L}_{ab}$ is given by one of the following formulas.

$$\ell(v)(t) \stackrel{\text{def}}{=} -g(t)v(\mu(t)), \tag{4.1}$$

$$\ell(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)), \tag{4.2}$$

$$\ell(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)), \tag{4.3}$$

where $p, g \in L([a,b];\mathbb{R}_+)$ and $\tau, \mu \in M_{ab}$.

Define the function h by the equality

$$h(t) \stackrel{\text{def}}{=} (b-\mu(t)) \int\limits_a^{\mu(t)} (s-a)(\mu(s)-a)(b-\mu(s))g(s)\,ds +$$

$$+ (\mu(t)-a) \int\limits_{\mu(t)}^{b} (b-s)(\mu(s)-a)(b-\mu(s))g(s)\,ds -$$

$$- (b-a)(\mu(t)-a)(b-\mu(t)) \quad \text{for } t \in [a,b].$$

Theorem 4.1. *Let*

$$gh \not\equiv 0 \tag{4.4}$$

and

$$h(t) \le 0 \quad \textit{for } t \in [a,b]. \tag{4.5}$$

Then the operator ℓ defined by (4.1) *belongs to the set $V([a,b])$.*

Remark 4.1. Example 5.1 below shows that condition (4.4) is essential and cannot be omited.

Corollary 4.1. *Let*

$$\int\limits_a^b (\mu(s)-a)(b-\mu(s))g(s)\,ds < b-a. \tag{4.6}$$

Then the operator ℓ defined by (4.1) *belongs to the set $V([a,b])$.*

Define the function h_1 by the equality

$$h_1(t) \stackrel{\text{def}}{=} \int\limits_a^{\mu(t)} (s-a)(\mu(s)-a)g(s)\,ds +$$

$$+ (\mu(t) - a) \int_{\mu(t)}^{b} (\mu(s) - a) g(s)\, ds - (\mu(t) - a) \quad \text{for } t \in [a, b].$$

Theorem 4.2. *Let*

$$gh_1 \not\equiv 0 \tag{4.7}$$

and

$$h_1(t) \le 0 \quad \text{for } t \in [a, b].$$

Then the operator ℓ defined by (4.1) *belongs to the set* $V'([a, b])$.

Remark 4.2. Example 5.2 below shows that condition (4.7) is essential and cannot be omited.

Corollary 4.2. *Let*

$$\int_a^b (\mu(s) - a) g(s)\, ds < 1.$$

Then the operator ℓ defined by (4.1) *belongs to the set* $V'([a, b])$.

Theorem 4.3. *Let there exist numbers $\lambda_i \in [0, 1[$, $\alpha_{ij} \in \mathbb{R}_+$ $(i, j = 1, 2)$, and $c \in [a, b]$ such that*

$$\begin{gathered} \int_0^{+\infty} \frac{ds}{\alpha_{11} + \alpha_{12} s + s^2} \ge \frac{(c - a)^{1 - \lambda_1}}{1 - \lambda_1}, \\ \int_0^{+\infty} \frac{ds}{\alpha_{21} + \alpha_{22} s + s^2} \ge \frac{(b - c)^{1 - \lambda_2}}{1 - \lambda_2} \end{gathered} \tag{4.8}$$

and

$$\begin{gathered} (t - a)^{2\lambda_1} g(t) \le \alpha_{11}, \\ (t - a)^{\lambda_1} \left[\frac{\lambda_1}{t - a} + (t - \mu(t)) g(t) \right] \ge -\alpha_{12} \ \textit{for } t \in]a, c[, \\ (b - t)^{2\lambda_2} g(t) \le \alpha_{21}, \\ (b - t)^{\lambda_2} \left[-\frac{\lambda_2}{b - t} + (t - \mu(t)) g(t) \right] \le \alpha_{22} \ \textit{for } t \in]c, b[. \end{gathered} \tag{4.9}$$

Let, moreover, at least one of the inequalities in (4.9) *holds in the strict sense on a set of positive measure, or at least one of the inequalities in* (4.8) *is strict. Then the operator ℓ defined by* (4.1) *belongs to the set* $V([a, b])$.

Remark 4.3. Example 5.3 below shows that condition, at least one of the inequalities in (4.9) holds in the strict sense on a set of positive measure, or at least one of the inequalities in (4.8) is strict, is essential and cannot be omited.

Theorem 4.4. *Let there exist numbers $\lambda \in [0,1[$ and $\alpha_i \in \mathbb{R}_+$ $(i=1,2)$ such that*

$$\int_0^{+\infty} \frac{ds}{\alpha_1+\alpha_2 s+s^2} \geq \frac{(b-a)^{1-\lambda}}{1-\lambda} \tag{4.10}$$

and

$$\begin{gathered} (t-a)^{2\lambda} g(t) \leq \alpha_1, \\ (t-a)^{\lambda}\left[\frac{\lambda}{t-a}+(t-\mu(t))g(t)\right] \geq -\alpha_2 \quad \textit{for } t \in [a,b]. \end{gathered} \tag{4.11}$$

Let, moreover, the inequality (4.10) *is strict or one of the inequalities in* (4.11) *hold in the strict sense on a set of positive measure. Then the operator ℓ defined by* (4.1) *belongs to the set $V'([a,b])$.*

Remark 4.4. Example 5.4 below shows that condition, the inequality (4.10) is strict or one of the inequalities in (4.11) hold in the strict sense on a set of positive measure, is essential and cannot be omited.

Theorem 4.5. *Let*

$$\tau(t) \geq t \quad \textit{for } t \in [a,b] \tag{4.12}$$

and

$$\sup\left\{\frac{1}{t-a}\int_a^t (t-s)(\tau(s)-a)p(s)\,ds : t \in\,]a,b]\right\} \leq 1. \tag{4.13}$$

Then the operator ℓ defined by (4.2) *belongs to both sets $V([a,b])$ and $V'([a,b])$.*

Corollary 4.3. *Let* (4.12) *holds and*

$$\int_a^b (\tau(s)-a)p(s)\,ds \leq 1. \tag{4.14}$$

Then the operator ℓ defined by (4.2) *belongs to both sets $V([a,b])$ and $V'([a,b])$.*

Theorem 4.6. *Let*

$$\tau(t) \leq t \quad \textit{for } t \in [a,b] \tag{4.15}$$

and

$$\sup\left\{\frac{1}{b-t}\int_t^b (b-\tau(s))(s-t)p(s)\,ds : t \in [a,b[\,\right\} \leq 1.$$

Then the operator ℓ defined by (4.2) *belongs to both sets $V([a,b])$ and $V'([a,b])$.*

Corollary 4.4. *Let* (4.15) *holds and*

$$\int_a^b (b-\tau(s))p(s)\,ds \le 1.$$

Then the operator ℓ *defined by* (4.2) *belongs to both sets* $V([a,b])$ *and* $V'([a,b])$.

Theorem 4.7. *Assume that* (4.12) *holds and either*

$$\int_a^b (b-s)p(s)\,ds \le 1 \tag{4.16}$$

or the following two conditions are satisfied:

$$\int_a^b (b-s)p(s)\,ds > 1 \tag{4.17}$$

and

$$\int_t^{\tau(t)} \int_a^s p(\xi)\,d\xi\,ds \le \eta^*, \tag{4.18}$$

where

$$\eta^* = \sup\{\eta(x) : x > 0\},$$

$$\eta(x) = \frac{1}{x}\ln\left[\frac{x\exp\left[x\int_a^b (b-s)p(s)\,ds\right]}{\exp\left[x\int_a^b (b-s)p(s)\,ds\right]-1}\right] \qquad \textit{for } x>0.$$

Then the operator ℓ *defined by* (4.2) *belongs to the sets* $V([a,b])$ *and* $V'([a,b])$.

Theorem 4.8. *Let* (4.15) *holds and either*

$$\int_a^b (s-a)p(s)\,ds \le 1$$

or the following two conditions are satisfied:

$$\int_a^b (s-a)p(s)\,ds > 1$$

and

$$\int_{\tau(t)}^{t} \int_s^b p(\xi)\,d\xi\,ds \le \zeta^*,$$

where

$$\zeta^* = \sup\{\zeta(x) : x > 0\},$$

$$\zeta(x) = \frac{1}{x} \ln \left[\frac{x \exp\left[x \int_a^b (s-a)p(s)\,ds\right]}{\exp\left[x \int_a^b (s-a)p(s)\,ds\right] - 1} \right] \qquad \text{for } x > 0.$$

Then the operator ℓ defined by (4.2) *belongs to the sets $V([a,b])$ and $V'([a,b])$.*

Theorem 4.9. *Let* (4.12) *holds and*

$$\int_a^b p(t) \left[\int_t^{\tau(t)} (\tau(t)-s)(\tau(s)-a)p(s)\,ds \right] \times \\ \times \exp\left[\int_t^b (\tau(s)-a)p(s)\,ds \right] dt < 1. \quad (4.19)$$

Then the operator ℓ defined by (4.2) *belongs to the sets $V([a,b])$ and $V'([a,b])$.*

Corollary 4.5. *Let* (4.12) *holds, $p \not\equiv 0$, and*

$$\operatorname{ess\,sup} \left\{ \int_t^{\tau(t)} (\tau(s)-a)p(s)\,ds : t \in [a,b] \right\} < \\ < \left(\exp\left[\int_a^b (\tau(s)-a)p(s)\,ds \right] - 1 \right)^{-1}. \quad (4.20)$$

Then the operator ℓ defined by (4.2) *belongs to the sets $V([a,b])$ and $V'([a,b])$.*

Theorem 4.10. *Let* (4.15) *holds and*

$$\int_a^b p(t) \left[\int_{\tau(t)}^t (s-\tau(t))(b-\tau(s))p(s)\,ds \right] \exp\left[\int_a^t (b-\tau(s))p(s)\,ds \right] dt < 1.$$

Then the operator ℓ defined by (4.2) *belongs to the sets $V([a,b])$ and $V'([a,b])$.*

Corollary 4.6. *Let* (4.15) *holds, $p \not\equiv 0$, and*

$$\operatorname{ess\,sup} \left\{ \int_{\tau(t)}^t (b-\tau(s))p(s)\,ds : t \in [a,b] \right\} < \\ < \left(\exp\left[\int_a^b (b-\tau(s))p(s)\,ds \right] - 1 \right)^{-1}.$$

Then the operator ℓ defined by (4.2) *belongs to the sets* $V([a,b])$ *and* $V'([a,b])$*.*

Theorem 4.11. *Let $\ell \in \mathcal{L}_{ab}$ be an operator defined by* (4.3)*, where functions g and μ satisfy conditions one of the Theorem* 4.1 *or Theorem* 4.3 *or Corollary* 4.1*, while the functions p and τ satisfy conditions one of the Theorem* 4.5–4.10 *or Corollary* 4.3–4.6*. Then $\ell \in V([a,b])$.*

Theorem 4.12. *Let $\ell \in \mathcal{L}_{ab}$ be an operator defined by* (4.3)*, where functions g and μ satisfy conditions one of the Theorem* 4.2 *or Theorem* 4.4 *or Corollary* 4.2*, while the functions p and τ satisfy conditions one of the Theorem* 4.5–4.10 *or Corollary* 4.3–4.6*. Then $\ell \in V'([a,b])$.*

Proof of Theorem 4.1. Let us first suppose

$$\int_a^b (\mu(s)-a)(b-\mu(s))g(s)\,ds \neq 0. \tag{4.21}$$

Put

$$\gamma(t) = \frac{1}{b-a}\Bigg[(b-t)\int_a^t (s-a)(\mu(s)-a)(b-\mu(s))g(s)\,ds+$$
$$+(t-a)\int_t^b (b-s)(\mu(s)-a)(b-\mu(s))g(s)\,ds\Bigg] \quad \text{for } t\in[a,b].$$

By virtue of (4.21),

$$\gamma(t) > 0 \quad \text{for } t \in \,]a,b[.$$

On the other hand, by virtue of (4.4) and (4.5),

$$\gamma''(t) \le -g(t)\gamma(\mu(t)) \quad \text{for } t\in[a,b],$$
$$\text{mes}\,\{t\in[a,b] : \gamma''(t) < -g(t)\gamma(\mu(t))\} > 0.$$

Therefore, the function γ satisfies the conditions of Theorem 1.1.

Suppose now that

$$\int_a^b (\mu(s)-a)(b-\mu(s))g(s)\,ds = 0. \tag{4.22}$$

Let the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy (0.2_0) and (0.4). It is easy to verify that

$$u(t) = -\frac{1}{b-a}\Bigg[(b-t)\int_a^t (s-a)u''(s)\,ds$$
$$+(t-a)\int_t^b (b-s)u''(s)\,ds\Bigg] \quad \text{for } t\in[a,b].$$

Hence,

$$|u(t)| \le M(t-a)(b-t) \quad \text{for } t \in [a,b], \tag{4.23}$$

where

$$M = \frac{1}{b-a}\int_a^b |u''(s)|\,ds.$$

By virtue of (4.22) and (4.23), it follows from (0.4) that

$$u''(t) \ge 0 \quad \text{for } t \in [a,b].$$

Taking now (0.2_0) into account, we get that (0.5) holds. □

Proof of Corollary 4.1. Suppose that (4.21) holds (if (4.22) holds, then, obviously, $\ell \in V([a,b])$, see the proof of Theorem 4.1). Clearly for $t \in [a,b]$ the inequality

$$h(t) \le (\mu(t)-a)(b-\mu(t))\left[\int_a^b (\mu(s)-a)(b-\mu(s))g(s)\,ds - (b-a)\right] \tag{4.24}$$

holds. Hence, on account of (4.6), the inequality (4.5) holds as well. On the other hand, it follows from (4.24) and (4.21) that (4.4) is also fulfilled. □

Theorem 4.2 and Corollary 4.2 are proved analogously.

Proof of Theorem 4.3. Suppose that $c \in]a,b[$ (if $c=a$ or $c=b$, the theorem is proved analogously). Without loss of generality, assume that

$$\begin{gathered}\int_0^{+\infty} \frac{ds}{\alpha_{11}+\alpha_{12}s+s^2} = \frac{(c-a)^{1-\lambda_1}}{1-\lambda_1},\\ \int_0^{+\infty} \frac{ds}{\alpha_{21}+\alpha_{22}s+s^2} = \frac{(b-c)^{1-\lambda_2}}{1-\lambda_2},\end{gathered} \tag{4.25}$$

and that at least one of the inequalities (4.9) holds in the strict sense on a set of positive measure.

Define functions ρ_1 and ρ_2 by equalities

$$\int_{\rho_1(t)}^{+\infty} \frac{ds}{\alpha_{11}+\alpha_{12}s+s^2} = \frac{(t-a)^{1-\lambda_1}}{1-\lambda_1} \quad \text{for } t \in]a,c], \tag{4.26}$$

$$\int_{\rho_2(t)}^{+\infty} \frac{ds}{\alpha_{21}+\alpha_{22}s+s^2} = \frac{(b-t)^{1-\lambda_2}}{1-\lambda_2} \quad \text{for } t \in [c,b[. \tag{4.27}$$

By virtue of (4.25),

$$\rho_1(t) > 0 \quad \text{for } t \in]a,c[, \qquad \rho_2(t) > 0 \quad \text{for } t \in]c,b[, \tag{4.28}$$

$$\rho_1(c) = 0, \quad \rho_2(c) = 0. \tag{4.29}$$

Put

$$\gamma(t)=\begin{cases}\exp\left[-\int\limits_t^c (s-a)^{-\lambda_1}\rho_1(s)\,ds\right] & \text{for } t\in[a,c[,\\ \exp\left[-\int\limits_c^t (b-s)^{-\lambda_2}\rho_2(s)\,ds\right] & \text{for } t\in[c,b].\end{cases}$$

It follows from (4.26)–(4.29) that $\gamma\in\widetilde{C}'_{loc}(]a,b[;\mathbb{R})$,

$$\gamma''(t)\le 0 \quad \text{for } t\in[a,b], \tag{4.30}$$

$$\gamma(t)>0 \quad \text{for } t\in\,]a,b[, \tag{4.31}$$

$$\gamma'(t)>0 \quad \text{for } t\in\,]a,c[, \qquad \gamma'(t)<0 \quad \text{for } t\in\,]c,b[, \tag{4.32}$$

$$\gamma(a)\ge 0, \quad \gamma(b)\ge 0 \tag{4.33}$$

and

$$\begin{aligned}\gamma''(t)&=-\frac{\alpha_{11}}{(t-a)^{2\lambda_1}}\gamma(t)-\left[\frac{\alpha_{12}}{(t-a)^{\lambda_1}}+\frac{\lambda_1}{t-a}\right]\gamma'(t) \quad \text{for } t\in\,]a,c[,\\ \gamma''(t)&=-\frac{\alpha_{21}}{(b-t)^{2\lambda_2}}\gamma(t)+\left[\frac{\alpha_{22}}{(b-t)^{\lambda_2}}+\frac{\lambda_2}{b-t}\right]\gamma'(t) \quad \text{for } t\in\,]c,b[.\end{aligned} \tag{4.34}$$

By virtue of (4.9), (4.31), and (4.32), equalities (4.34) imply

$$\gamma''(t)\le -g(t)\gamma(t)-(\mu(t)-t)g(t)\gamma'(t) \quad \text{for } t\in[a,b]. \tag{4.35}$$

Moreover, by virtue of the assumption that one of the inequalities (4.9) holds in the strict sense on a set of positive measure, we get

$$\operatorname{mes}\{t\in[a,b]:\gamma''(t)<-g(t)\gamma(t)-(\mu(t)-t)g(t)\gamma'(t)\}>0. \tag{4.36}$$

Taking (4.30) into account, one can easily verify that

$$(\mu(t)-t)\gamma'(t)\ge\int\limits_t^{\mu(t)}\gamma'(s)\,ds \quad \text{for } t\in[a,b].$$

This inequality, together with (4.35) and (4.36), implies

$$\gamma''(t)\le -g(t)\gamma(\mu(t)) \quad \text{for } t\in[a,b],$$
$$\operatorname{mes}\{t\in[a,b]:\gamma''(t)<-g(t)\gamma(\mu(t))\}>0.$$

Therefore, the function γ satisfies all the conditions of Theorem 1.1. □

Theorem 4.4 is proved analogously.

Proof of Theorem 4.5. Theorem 4.5 follows from Corollary 1.3 with

$$\varphi_1(t)=t-a, \quad m=2, \quad \text{and} \quad k=1.$$

□

Proof of Corollary 4.3. It is not difficult to verify that inequality (4.14) implies (4.13). □

The proof of Theorem 4.6 and Corollary 4.4 are analogous.

Proof of Theorem 4.7. If (4.16) holds, then, obviously, the conditions of Corollary 1.3 are fulfilled with $\varphi_1(t) = 1, m = 2$, and $k = 1$. Therefore, we will assume that (4.17) and (4.18) hold. On account of (4.17),

$$\lim_{x\to 0+} \eta(x) = -\infty.$$

On the other hand, clearly,

$$\lim_{x\to +\infty} \eta(x) = 0.$$

Therefore, there exists $\lambda > 0$ such that

$$\eta^* = \eta(\lambda). \tag{4.37}$$

It follows from (4.18) and (4.37) that

$$\exp\left[\lambda\int_t^{\tau(t)}\int_a^s p(\xi)\,d\xi\,ds\right] \le \frac{\lambda\exp\left[\lambda\int_a^b (b-s)p(s)\,ds\right]}{\exp\left[\lambda\int_a^b (b-s)p(s)\,ds\right]-1} \quad \text{for } t\in[a,b]. \tag{4.38}$$

Clearly,

$$\int_a^b (b-s)p(s)\,ds \ge \int_a^{\tau(t)}\int_a^s p(\xi)\,d\xi\,ds \quad \text{for } t\in[a,b]. \tag{4.39}$$

Inequalities (4.38) and (4.39) imply

$$\left(\exp\left[\lambda\int_a^{\tau(t)}\int_a^s p(\xi)\,d\xi\,ds\right]-1\right)\exp\left[\lambda\int_t^{\tau(t)}\int_a^s p(\xi)\,d\xi\,ds\right] \le$$

$$\le \lambda\exp\left[\lambda\int_a^{\tau(t)}\int_a^s p(\xi)\,d\xi\,ds\right] \quad \text{for } t\in[a,b].$$

Hence,

$$\lambda\exp\left[\lambda\int_a^t (t-s)p(s)\,ds\right] \ge$$

$$\ge \exp\left[\lambda\int_a^{\tau(t)} (\tau(t)-s)p(s)\,ds\right]-1 \quad \text{for } t\in[a,b]. \tag{4.40}$$

Put

$$\gamma(t) = \exp\left[\lambda\int_a^t (t-s)p(s)\,ds\right]-1 \quad \text{for } t\in[a,b].$$

It is easy to check that

$$\gamma(t) > 0 \quad \text{for } t \in]a_\ell, b], \quad \gamma'(a_\ell) = 0,$$

where a_ℓ is defined by (1.10), (1.12), and (4.2). On the other hand, by virtue of (4.40),

$$\gamma''(t) \geq p(t)\gamma(\tau(t)) \quad \text{for } t \in [a, b].$$

Therefore, the function γ satisfies all the conditions of Theorem 1.3. □

Theorem 4.8 is proved analogously.

Proof of Theorem 4.9. Clearly,

$$f(v)(t) = p(t) \int_a^{\tau(t)} v(s)\,ds, \quad \varphi(v)(t) = \int_a^t f(v)(s)\,ds.$$

Put

$$\bar{f}(v)(t) \stackrel{\text{def}}{=} p(t) \int_t^{\tau(t)} (\tau(t) - s) f(v)(s)\,ds.$$

It is not difficult to verify that

$$\begin{aligned} f(\varphi(v))(t) - f(1)(t)\varphi(v)(t) = p(t) \int_t^{\tau(t)} (\tau(t) - s) f(v)(s)\,ds - \\ - p(t) \int_a^t (s - a) f(v)(s)\,ds \leq \\ \leq \bar{f}(v)(t) \quad \text{for } t \in [a, b],\ v \in C_a([a, b]; \mathbb{R}_+). \end{aligned}$$

On the other hand, (4.19) implies (1.19). Therefore all the conditions of Corollary 1.5 hold. □

Proof of Corollary 4.5. It is not difficult to verify that inequality (4.20) implies (4.19). □

The proof of Theorem 4.10 and Corollary 4.6 are analogous.

Theorems 4.11 and 4.12 immediately follow from Theorems 4.1–4.10, Corollaries 4.1–4.6 and Theorem 1.7.

5. Examples

Example 5.1. Let

$$\ell(v)(t) \stackrel{\text{def}}{=} -\frac{8}{(b-a)^2}\, v\left(\frac{a+b}{2}\right).$$

Clearly, (1.8) holds. On the other hand, the function

$$u(t) = \frac{4}{(b-a)^2}(t-a)(b-t) \quad \text{for } t \in [a, b]$$

is a nontrivial solution of the problem (0.1_0), (0.2_0). Therefore, according to Remark 0.1, $\ell \notin V([a,b])$.

Example 5.2. Let

$$\ell(v)(t) \stackrel{\text{def}}{=} -\frac{2}{(b-a)^2}v(b).$$

Evidently, (1.9) holds. On the other hand, the function

$$u(t) = \frac{2}{(b-a)^2}\left(\frac{1}{2}(t-a)^2 + (t-a)(b-t)\right) \quad \text{for } t \in [a,b]$$

is a nontrivial solution of the problem (0.1_0), (0.3_0). Therefore, according to Remark 0.1, $\ell \notin V'([a,b])$.

Example 5.3. Let

$$g(t) \stackrel{\text{def}}{=} \frac{\pi^2}{(b-a)^2}, \qquad \mu(t) \stackrel{\text{def}}{=} t$$

and

$$\lambda_1 = \lambda_2 = 0, \quad \alpha_{12} = \alpha_{22} = 0, \quad \alpha_{11} = \alpha_{21} = \frac{\pi^2}{(b-a)^2}, \quad c = \frac{a+b}{2}.$$

It is not difficult to verify that all inequalities in the conditions (4.8) and (4.9) are satisfied as equalities. On the other hand, the function

$$u(t) = \sin\frac{\pi(t-a)}{b-a} \quad \text{for } t \in [a,b]$$

is a nontrivial solution of the problem (0.1_0), (0.2_0). Therefore, according to Remark 0.1, $\ell \notin V([a,b])$.

Example 5.4. Let

$$g(t) \stackrel{\text{def}}{=} \frac{\pi^2}{4(b-a)^2}, \qquad \mu(t) \stackrel{\text{def}}{=} t$$

and

$$\lambda = 0, \quad \alpha_2 = 0, \quad \alpha_1 = \frac{\pi^2}{4(b-a)^2}.$$

It is not difficult to verify that all inequalities in the conditions (4.10) and (4.11) are satisfied as equalities. On the other hand, the function

$$u(t) = \sin\frac{\pi(t-a)}{2(b-a)} \quad \text{for } t \in [a,b]$$

is a nontrivial solution of the problem (0.1_0), (0.3_0). Therefore, according to Remark 0.1, $\ell \notin V'([a,b])$.

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