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ON SIGN CONSTANT AND MONOTONE SOLUTIONS OF SECOND ORDER LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. In this paper, the question on the existence and uniqueness of a constant sign solution of the initial value problem

$$u''(t) = \ell(u)(t) + q(t), \quad u(a) = c_1, \quad u'(a) = c_2$$

is studied. More precisely, the nonimprovable effective sufficient conditions for a linear operator $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ are established guaranteeing that the considered problem with $q \in L([a,b];\mathbb{R}_+)$ and $c_1, c_2 \in \mathbb{R}_+$ has a unique solution and this solution is nonnegative. The question on the existence and uniqueness of a monotone solution of the same problem is discussed, as well.

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$$u''(t) = \ell(u)(t) + q(t), \quad u(a) = c_1, \quad u'(a) = c_2$$

კოშის ამოცანის ნიშანმუდმივი ამონახსნის არსებობისა და ერთადერთობის საკითხი. სახელდობრ, დადგენილია წრფივი $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ ოპერატორის მიერ დასაკმაყოფილებელი გაუუმჯობესებადი ეფექტური საკმარისი პირობები, რომლებიც უზრუნველყოფს, რომ ზემოთხსენებულ ამოცანას ნებისმიერი $q \in L([a,b];\mathbb{R}_+)$ და $c_1, c_2 \in \mathbb{R}_+$-სთვის აქვს ერთადერთი ამონახსნი და ეს ამონახსნი არაუარყოფითია. აგრეთვე განხილულია იმავე ამოცანის მონოტონური ამონახსნის არსებობის და ერთადერთობის საკითხი.

Introduction

The following notation is used throughout the paper.

$\mathbb{N}$ is the set of natural numbers.

$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+ = [0, +\infty[$.

If $x \in \mathbb{R}$, then $[x]_+ = \frac{1}{2}(|x| + x)$ and $[x]_- = \frac{1}{2}(|x| - x)$.

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : t \in [a,b]\}$.

$C([a,b];\mathbb{R}_+) = \{u \in C([a,b];\mathbb{R}) : u(t) \geq 0 \text{ for } t \in [a,b]\}$.

$C_a([a,b];\mathbb{R}_+) = \{u \in C([a,b];\mathbb{R}_+) : \ u(a) = 0\}$.

$\widetilde{C}([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$.

$\widetilde{C}'([a,b];\mathbb{R})$ is the set of functions $u \in \widetilde{C}([a,b];\mathbb{R})$ such that $u' \in \widetilde{C}([a,b];\mathbb{R})$.

$\widetilde{C}'_{loc}([a,b[\,;\mathbb{R})$ is the set of functions $u \in \widetilde{C}([a,b];\mathbb{R})$ such that $u' \in \widetilde{C}([a,\beta];\mathbb{R})$ for every $\beta \in]a,b[$.

$\widetilde{C}'_{loc}(]a,b[\,;\mathbb{R})$ is the set of functions $u \in \widetilde{C}([a,b];\mathbb{R})$ such that $u' \in \widetilde{C}([\alpha,\beta];\mathbb{R})$ for every $[\alpha,\beta] \subset]a,b[$.

M_{ab} is the set of measurable functions $\tau : [a,b] \to [a,b]$.

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int_a^b |p(s)|ds$.

$L([a,b];\mathbb{R}_+) = \{p \in L([a,b];\mathbb{R}) : p(t) \geq 0 \text{ for } t \in [a,b]\}$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

P_{ab} is the set of operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.

We will say that $\ell \in \mathcal{L}_{ab}$ is an $a-$Volterra operator if for arbitrary $a_0 \in]a,b]$ and $v \in C([a,b];\mathbb{R})$ satisfying the condition

$$v(t) = 0 \quad \text{for} \quad t \in [a, a_0]$$

we have

$$\ell(v)(t) = 0 \quad \text{for almost all} \quad t \in [a, a_0].$$

The equalities and inequalities with integrable functions are understood almost everywhere.

Consider the problem on the existence and uniqueness of a solution of the equation

$$u''(t) = \ell(u)(t) + q(t) \tag{0.1}$$

satisfying the initial conditions

$$u(a) = c_0, \quad u'(a) = c_1, \tag{0.2}$$

where $\ell \in \mathcal{L}_{ab}$, $q \in L([a,b];\mathbb{R})$ and $c_0, c_1 \in \mathbb{R}$. By a solution of the equation (0.1) we understand a function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying this equation (almost everywhere) in $[a,b]$.

Along with the problem (0.1), (0.2) consider the corresponding homogeneous problem

$$u''(t) = \ell(u)(t), \tag{0.1_0}$$

$$u(a) = 0, \quad u'(a) = 0. \tag{0.2_0}$$

The following result is well-known from the general theory of boundary value problems for functional differential equations (see, e.g., [9]).

Theorem 0.1. *The problem* (0.1), (0.2) *is uniquely solvable iff the corresponding homogeneous problem* (0.1_0), (0.2_0) *has only the trivial solution.*

Introduce the following definitions.

Definition 0.1. An operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $H_{ab}(a)$ if for every function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying

$$u''(t) \geq \ell(u)(t) \quad \text{for} \quad t \in [a,b], \tag{0.3}$$

$$u(a) \geq 0, \quad u'(a) \geq 0, \tag{0.4}$$

the inequality

$$u(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{0.5}$$

holds.

Definition 0.2. An operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $\widetilde{H}_{ab}(a)$ if for every function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying (0.2_0) and (0.3), the inequality (0.5) holds.

Definition 0.3. An operator $\ell \in \mathcal{L}_{ab}$ belongs to the set $H'_{ab}(a)$ if for every function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying (0.3) and

$$u(a) = 0, \quad u'(a) \geq 0, \tag{0.6}$$

the inequalities

$$u(t) \geq 0, \quad u'(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{0.7}$$

hold.

Remark 0.1. From Definitions 0.1–0.3 it immediately follows that

$$H_{ab}(a) \subseteq \widetilde{H}_{ab}(a), \quad H'_{ab}(a) \subseteq \widetilde{H}_{ab}(a). \tag{0.8}$$

It is not difficult to verify that

$$P_{ab} \cap H_{ab}(a) = P_{ab} \cap \widetilde{H}_{ab}(a) \quad \text{and} \quad P_{ab} \cap H'_{ab}(a) = P_{ab} \cap \widetilde{H}_{ab}(a). \tag{0.9}$$

Nevertheless, in general

$$H_{ab}(a) \neq \widetilde{H}_{ab}(a) \quad \text{and} \quad H'_{ab}(a) \neq \widetilde{H}_{ab}(a).$$

Indeed, let

$$\ell(v)(t) \stackrel{def}{=} -\frac{\pi^2}{(b-a)^2} v(t).$$

By virtue of Sturm's comparison theorem (see, e.g., [7]), it is not difficult to verify that $\ell \in \widetilde{H}_{ab}(a)$. On the other hand, the functions

$$u_1(t) = \sin\frac{\pi(t-a)}{b-a}, \quad u_2(t) = \cos\frac{\pi(t-a)}{b-a} \quad \text{for} \quad t \in [a,b]$$

satisfy

$$u_1''(t) = \ell(u_1)(t) \text{ for } t \in [a,b], \quad u_1(a) = 0, \quad u_1'(a) = \frac{\pi}{b-a}, \quad u_1'(b) < 0,$$
$$u_2''(t) = \ell(u_2)(t) \text{ for } t \in [a,b], \quad u_2(a) = 1, \quad u_2'(a) = 0, \qquad u_2(b) < 0.$$

Therefore, $\ell \notin H'_{ab}(a)$ and $\ell \notin H_{ab}(a)$.

Remark 0.2. As it follows from (0.9), $P_{ab} \cap H_{ab}(a) = P_{ab} \cap H'_{ab}(a)$. Nevertheless, in general

$$H_{ab}(a) \not\subseteq H'_{ab}(a) \quad \text{and} \quad H'_{ab}(a) \not\subseteq H_{ab}(a).$$

First, let $\ell(v)(t) \overset{def}{=} -g(t)v(a)$, where $g \in L([a,b];\mathbb{R}_+)$. Evidently, $\ell \in H'_{ab}(a)$. By a direct calculation, one can easily verify that $\ell \in H_{ab}(a)$ if and only if

$$\int_a^b (b-s)g(s)ds \leq 1. \tag{0.10}$$

Therefore, in general $H'_{ab}(a) \not\subseteq H_{ab}(a)$.

Now, put $a = 0$, $b \in \left]\frac{\pi}{4}, \frac{\pi}{2}\right[$, and

$$\ell(v)(t) \overset{def}{=} -\frac{1 + 2\sin^2\varphi^{-1}(t)}{\cos^8\varphi^{-1}(t)} \cdot v(t),$$

where

$$\varphi(t) = \sin t - \frac{1}{3}\sin^3 t \quad \text{for} \quad t \in [a,b].$$

Clearly, the function

$$\gamma(t) = \cos\varphi^{-1}(t) \quad \text{for} \quad t \in [a,b]$$

satisfy

$$\gamma''(t) = \ell(\gamma)(t) \quad \text{for} \quad t \in [a,b],$$
$$\gamma(t) > 0 \quad \text{for} \quad t \in [a,b[, \ \gamma'(a) = 0.$$

Hence, by virtue of Theorem 1.2 below, we get $\ell \in H_{ab}(a)$. On the other hand, the function

$$u(t) = \sin 2\varphi^{-1}(t) \quad \text{for} \quad t \in [a,b]$$

satisfies (0.1_0), (0.6), and $u'(b) < 0$. Therefore, $\ell \notin H'_{ab}(a)$. Thus, in general $H_{ab}(a) \not\subseteq H'_{ab}(a)$.

Remark 0.3. It follows from Definition 0.2 that if $\ell \in \widetilde{H}_{ab}(a)$, then the homogeneous problem (0.1_0), (0.2_0) has only the trivial solution. Therefore, according to Theorem 0.1, the problem (0.1), (0.2) is uniquely solvable provided $\ell \in \widetilde{H}_{ab}(a)$. Consequently, by virtue of (0.8), each of the inclusion

$\ell \in H_{ab}(a)$ and $\ell \in H'_{ab}(a)$ yields the unique solvability of the problem (0.1), (0.2). Moreover, the inclusion $\ell \in \widetilde{H}_{ab}(a)$, resp. $\ell \in H_{ab}(a)$, guarantees that if $q \in L([a,b];\mathbb{R}_+)$, then the unique solution of the problem (0.1), (0.2_0), resp. the problem (0.1), (0.2), with $c_0, c_1 \in \mathbb{R}_+$ is nonnegative. Analogously, if $\ell \in H'_{ab}(a)$, $q \in L([a,b];\mathbb{R}_+)$, and $c \geq 0$, then the unique solution of the problem

$$u''(t) = \ell(u)(t) + q(t), \quad u(a) = 0, \quad u'(a) = c$$

is nonnegative and nondecreasing.

In the present paper, we establish sufficient conditions guaranteeing the inclusions $\ell \in H_{ab}(a)$, $\ell \in \widetilde{H}_{ab}(a)$, $\ell \in H'_{ab}(a)$. The results obtained here generalize and make more complete the previously known ones of an analogous character (see, e.g., [1, 3, 8] and references therein). The related results for another type of the equations can be found in [4, 5, 6, 8].

The paper is organized as follows. The main results are formulated in Section 1. Their proofs are contained in Section 2. Section 3 deals with the special case of operator ℓ, with so-called operator with a deviating argument. Section 4 is devoted to the examples verifying the optimality of obtained results.

1. Main Results

In this section, we formulate the main results. Theorem 1.1, Corollaries 1.1 and 1.2, and Proposition 1.1 concern the case $\ell \in P_{ab}$. The case, when $-\ell \in P_{ab}$, is considered in Theorems 1.2–1.4, and Corollaries 1.3–1.5. Finally, Theorem 1.5 deals with the case, where the operator $\ell \in \mathcal{L}_{ab}$ admits the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in P_{ab}$.

Theorem 1.1. *Let $\ell \in P_{ab}$. Then $\ell \in H_{ab}(a)$ iff there exists a function $\gamma \in \widetilde{C}'_{loc}([a,b[\,;\mathbb{R})$ satisfying the inequalities*

$$\gamma''(t) \geq \ell(\gamma)(t) \quad \text{for} \quad t \in [a,b], \tag{1.1}$$

$$\gamma(t) > 0 \quad \text{for} \quad t \in [a,b], \tag{1.2}$$

$$\gamma'(a) \geq 0. \tag{1.3}$$

Corollary 1.1. *Let $\ell \in P_{ab}$ be an $a-$Volterra operator. Then $\ell \in H_{ab}(a)$.*

Corollary 1.2. *Let $\ell \in P_{ab}$ and let at least one of the following items be fulfilled:*

a) *there exist $m, k \in \mathbb{N}$ and a constant $\alpha \in]0,1[$ such that $m > k$ and*

$$\varphi_m(t) \leq \alpha \varphi_k(t) \quad \text{for} \quad t \in [a,b], \tag{1.4}$$

where

$$\varphi_1(t) \overset{def}{=} 1, \quad \varphi_{i+1}(t) \overset{def}{=} \int_a^t (t-s)\ell(\varphi_i)(s)ds \quad \text{for} \quad t \in [a,b], \ i \in \mathbb{N};$$

b) *there exists* $\overline{\ell} \in P_{ab}$ *such that*

$$\int_a^b (b-s)\overline{\ell}(1)(s)ds < \exp\left[\frac{-1}{b-a}\int_a^b (s-a)(b-s)\ell(1)(s)ds\right] \tag{1.5}$$

and on the set $C_a([a,b];\mathbb{R}_+)$ *the inequality*

$$\ell(\varphi(v))(t) - \ell(1)(t)\varphi(v)(t) \le \overline{\ell}(v)(t) \quad \textit{for} \quad t \in [a,b] \tag{1.6}$$

holds, where

$$\varphi(v)(t) \overset{def}{=} \int_a^t (t-s)\ell(v)(s)ds \quad \textit{for} \quad t \in [a,b]. \tag{1.7}$$

Then $\ell \in H_{ab}(a)$.

Remark 1.1. Example 4.1 below shows that the assumption $\alpha \in]0,1[$ in Corollary 1.2 a) cannot be replaced by the assumption $\alpha \in]0,1]$.

Remark 1.2. It follows from Corollary 1.2 a) (for $m = 2$ and $k = 1$) that if $\ell \in P_{ab}$, then $\ell \in H_{ab}(a)$ provided

$$\int_a^b (b-s)\ell(1)(s)ds < 1. \tag{1.8}$$

Example 4.1 below shows that the strict inequality (1.8) cannot be replaced by the nonstrict one. However, the following assertion is true.

Proposition 1.1. *Let* $\ell \in P_{ab}$ *and*

$$\int_a^b (b-s)\ell(1)(s)ds = 1. \tag{1.9}$$

If, moreover, the problem (0.1_0), (0.2_0) *has only the trivial solution, then* $\ell \in H_{ab}(a)$.

Theorem 1.2. *Let* $-\ell \in P_{ab}$ *be an* $a-$*Volterra operator and let there exist a function* $\gamma \in \widetilde{C}'_{loc}([a,b[\,;\mathbb{R})$ *satisfying*

$$\gamma''(t) \le \ell(\gamma)(t) \quad \textit{for} \quad t \in [a,b], \tag{1.10}$$

$$\gamma(t) > 0 \quad \textit{for} \quad t \in [a,b[\,, \tag{1.11}$$

$$\gamma'(a) \le 0. \tag{1.12}$$

Then $\ell \in H_{ab}(a)$.

Corollary 1.3. *Let* $-\ell \in P_{ab}$ *be an* $a-$*Volterra operator and*

$$\int_a^b (b-s)|\ell(1)(s)|ds \le 1. \tag{1.13}$$

Then $\ell \in H_{ab}(a)$.

Remark 1.3. Let $\ell(v)(t) \overset{def}{=} -g(t)v(a)$, where $g \in L([a,b];\mathbb{R}_+)$. As it was mentioned above, by the direct calculation, one can easily verify that $\ell \in H_{ab}(a)$ iff (0.10) holds. Therefore, the constant 1 in the right-hand side of the condition (1.13) is the best possible.

Theorem 1.3. *Let $-\ell \in P_{ab}$ be an $a-$Volterra operator and let there exist a function $\gamma \in \widetilde{C}'_{loc}(]a,b[\,;\mathbb{R})$ satisfying* (1.10) *and*

$$\gamma(t) > 0 \quad \textit{for} \quad t \in]a,b[\,, \tag{1.14}$$

$$\gamma(a) + \lim_{t \to a+} \gamma'(t) \neq 0. \tag{1.15}$$

Then $\ell \in \widetilde{H}_{ab}(a)$.

Corollary 1.4. *Let $-\ell \in P_{ab}$ be an $a-$Volterra operator and*

$$(b-t)\int_a^t (s-a)|\ell(1)(s)|ds+ \\ +(t-a)\int_t^b (b-s)|\ell(1)(s)|ds \leq b-a \quad \textit{for} \quad t \in [a,b]. \tag{1.16}$$

Then $\ell \in \widetilde{H}_{ab}(a)$.

Remark 1.4. Example 4.2 below shows that the condition (1.16) in Corollary 1.4 cannot be replaced by the condition

$$(b-t)\int_a^t (s-a)|\ell(1)(s)|ds+ \\ +(t-a)\int_t^b (b-s)|\ell(1)(s)|ds \leq (1+\varepsilon)(b-a) \quad \text{for} \quad t \in [a,b], \tag{1.17}$$

no matter how small $\varepsilon > 0$ would be.

Theorem 1.4. *Let $-\ell \in P_{ab}$ be an $a-$Volterra operator. Then $\ell \in H'_{ab}(a)$ iff there exists a function $\gamma \in \widetilde{C}'_{loc}([a,b[\,;\mathbb{R})$ satisfying* (1.10)*,*

$$\gamma(t) \geq 0, \quad \gamma'(t) \geq 0 \quad \textit{for} \quad t \in]a,b[\,, \tag{1.18}$$

and

$$\lim_{t \to a+} \gamma'(t) > 0. \tag{1.19}$$

Corollary 1.5. *Let $-\ell \in P_{ab}$ be an $a-$Volterra operator and*

$$\int_a^b \ell(\varphi)(s)|ds \leq 1, \tag{1.20}$$

where $\varphi(t) = t-a$ for $t \in [a,b]$. Then $\ell \in H'_{ab}(a)$.

Remark 1.5. Example 4.3 below shows that the condition (1.20) in Corollary 1.5 cannot be replaced by the condition

$$\int_a^b |\ell(\varphi)(s)|ds \leq 1+\varepsilon, \tag{1.21}$$

no matter how small $\varepsilon > 0$ would be.

Theorem 1.5. *Let the operator $\ell \in \mathcal{L}_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$, where $\ell_0, \ell_1 \in P_{ab}$, and let*

$$\ell_0 \in H_{ab}(a), \quad -\ell_1 \in H_{ab}(a), \tag{1.22}$$

resp.

$$\ell_0 \in \widetilde{H}_{ab}(a), \quad -\ell_1 \in \widetilde{H}_{ab}(a), \tag{1.23}$$

resp.

$$\ell_0 \in H'_{ab}(a), \quad -\ell_1 \in H'_{ab}(a). \tag{1.24}$$

Then $\ell \in H_{ab}(a)$, resp. $\ell \in \widetilde{H}_{ab}(a)$, resp. $\ell \in H'_{ab}(a)$.

2. Proof of the Main Results

Proof of Theorem 1.1. Let $\ell \in H_{ab}(a)$. According to Remark 0.3, the problem

$$\gamma''(t) = \ell(\gamma)(t), \tag{2.1}$$

$$\gamma(a) = 1, \quad \gamma'(a) = 1 \tag{2.2}$$

has a unique solution γ (i.e., $\gamma \in \widetilde{C}'([a,b];\mathbb{R})$) and

$$\gamma(t) \geq 0 \quad \text{for} \quad t \in [a,b]. \tag{2.3}$$

It follows from (2.1), by virtue of (2.3) and the assumption $\ell \in P_{ab}$, that

$$\gamma''(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

Hence, on account of (2.2), the inequality (1.2) holds, as well. Therefore, the function γ satisfies (1.1)–(1.3).

Now suppose that $\gamma \in \widetilde{C}'_{loc}([a,b[\,;\mathbb{R})$ is a function satisfying (1.1)–(1.3) and $\ell \notin H_{ab}(a)$. Then there exists a function $u \in \widetilde{C}'([a,b];\mathbb{R})$ and $t_0 \in \,]a,b[$ such that (0.3), (0.4) hold and

$$u(t_0) < 0. \tag{2.4}$$

Put

$$w(t) = \lambda\gamma(t) + u(t) \quad \text{for} \quad t \in [a,b],$$

where

$$\lambda = \max\left\{-\frac{u(t)}{\gamma(t)} : \; t \in [a,b]\right\}.$$

Obviously

$$w(t) \geq 0 \quad \text{for} \quad t \in [a,b] \tag{2.5}$$

and there exists $t_* \in]a,b]$ such that

$$w(t_*) = 0. \tag{2.6}$$

On account of (2.4), it is clear that

$$\lambda > 0. \tag{2.7}$$

By (1.1)–(1.3), (0.3), (0.4), and (2.7), we get

$$w''(t) \geq \ell(w)(t) \quad \text{for} \quad t \in [a,b], \tag{2.8}$$

$$w(a) > 0, \; w'(a) \geq 0. \tag{2.9}$$

It follows from (2.8), by virtue of (2.5) and the assumption $\ell \in P_{ab}$, that

$$w''(t) \geq 0 \quad \text{for} \quad t \in [a,b].$$

Hence, on account of (2.9), we get

$$w(t) > 0 \quad \text{for} \quad t \in [a,b],$$

which contradicts (2.6). □

Proof of Corollary 1.1. Let γ be a solution of the problem

$$\gamma''(t) = \ell(1)(t)\gamma(t), \tag{2.10}$$
$$\gamma(a) = 1, \quad \gamma'(a) = 1.$$

It follows from (2.10), in view of the assumption $\ell(1) \in L([a,b];\mathbb{R}_+)$, that the inequality (1.2) holds. Moreover, $\gamma \in \widetilde{C}'([a,b];\mathbb{R})$ and

$$\gamma'(t) \geq 0 \quad \text{for} \quad t \in [a,b],$$

i.e., the function γ is nondecreasing. Since ℓ is an $a-$Volterra operator and $\ell \in P_{ab}$, we easily conclude that

$$\ell(\gamma)(t) \leq \ell(1)(t)\gamma(t) \quad \text{for} \quad t \in [a,b].$$

Hence, on account of (2.10), the inequality (1.1) is fulfilled. Therefore, the function γ satisfies all the assumptions of Theorem 1.1. □

Proof of Corollary 1.2. a) It is not difficult to verify that the function

$$\gamma(t) \stackrel{def}{=} (1-\alpha)\sum_{i=1}^{k} \varphi_i(t) + \sum_{i=k+1}^{m} \varphi_i(t) \quad \text{for} \quad t \in [a,b]$$

satisfies the assumptions of Theorem 1.1.

b) Denote by $v_0, \; v_1$, and v_2 the solutions of the problems

$$v_0''(t) = \ell(1)(t)v_0(t), \quad v_0(b) = 0, \quad v_0'(b) = -1,$$
$$v_1''(t) = \ell(1)(t)v_1(t), \quad v_1(a) = 0, \quad v_1'(a) = 1,$$
$$v_2''(t) = \ell(1)(t)v_2(t), \quad v_2(a) = 1, \quad v_2'(a) = 0.$$

It is not difficult to verify that

$$v_0(t) = v_1(b)v_2(t) - v_2(b)v_1(t) \quad \text{for} \quad t \in [a,b] \tag{2.11}$$

and

$$0 \leq v_0(t) \leq (b-t)r_0 \quad \text{for} \quad t \in [a,b], \tag{2.12}$$

where

$$r_0 = \exp\left[\frac{1}{b-a}\int_a^b (s-a)(b-s)\ell(1)(s)ds\right]. \tag{2.13}$$

On account of (1.5), there exists $\varepsilon > 0$ such that

$$r_0 \int_a^b (b-s)\overline{\ell}(1)(s)ds + \varepsilon \|v_2\|_C \le 1. \tag{2.14}$$

Let γ be a solution of the problem

$$\begin{aligned} &\gamma''(t) = \ell(1)(t)\gamma(t) + \overline{\ell}(1)(t), \\ &\gamma(a) = \varepsilon, \quad \gamma'(a) = 0. \end{aligned} \tag{2.15}$$

Obviously, $\gamma \in \widetilde{C}'([a,b];\mathbb{R})$,

$$\gamma(t) > 0, \quad \gamma'(t) \ge 0 \quad \text{for} \quad t \in [a,b], \tag{2.16}$$

and

$$\begin{aligned} \gamma(t) = \varepsilon v_2(t) + \\ + \int_a^t [v_1(t)v_2(s) - v_2(t)v_1(s)]\overline{\ell}(1)(s)ds \quad \text{for} \quad t \in [a,b]. \end{aligned} \tag{2.17}$$

By virtue of (2.11) and (2.16), it follows from (2.17) that

$$\gamma(t) \le \gamma(b) \le \varepsilon \|v_2\|_C + \int_a^b v_0(s)\overline{\ell}(1)(s)ds \quad \text{for} \quad t \in [a,b].$$

The latter inequality, together with (2.12)–(2.14), implies

$$\gamma(t) \le 1 \quad \text{for} \quad t \in [a,b].$$

Hence, we get from (2.15), on account the assumption $\overline{\ell} \in P_{ab}$,

$$\gamma''(t) \ge \ell(1)(t)\gamma(t) + \overline{\ell}(\gamma)(t) \quad \text{for} \quad t \in [a,b].$$

Therefore, according to Theorem 1.1, we find

$$\widetilde{\ell} \in H_{ab}(a), \tag{2.18}$$

where

$$\widetilde{\ell}(v)(t) \overset{def}{=} \ell(1)(t)v(t) + \overline{\ell}(v)(t) \quad \text{for} \quad t \in [a,b].$$

Now assume that the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfies (0.3) and (0.4). It is not difficult to verify that

$$[u(t)]_- \le \int_a^t (t-s)\ell([u]_-)(s)ds \quad \text{for} \quad t \in [a,b]. \tag{2.19}$$

Put

$$w(t) = \varphi([u]_-)(t) \quad \text{for} \quad t \in [a,b], \tag{2.20}$$

where φ is the operator defined by (1.7). Clearly,

$$w(a) = 0, \quad w'(a) = 0, \tag{2.21}$$

$$w(t) \ge 0 \quad \text{for} \quad t \in [a,b]. \tag{2.22}$$

By virtue of (1.7), (2.19), (2.20), and the assumption $\ell \in P_{ab}$, it is also evident that

$$w''(t) = \ell([u]_-)(t) \le \ell(w)(t) = \ell(1)(t)w(t) + \\ + [\ell(w)(t) - \ell(1)(t)w(t)] \quad \text{for} \quad t \in [a,b]. \tag{2.23}$$

Hence, on account of (1.6), (2.19), (2.20), and the condition $\overline{\ell} \in P_{ab}$, we get

$$w''(t) \le \ell(1)(t)w(t) + \overline{\ell}([u]_-)(t) \le \ell(1)(t)w(t) + \\ + \overline{\ell}(w)(t) = \widetilde{\ell}(w)(t) \quad \text{for} \quad t \in [a,b].$$

The latter inequality, together with (2.21), (2.22), and the condition (2.18), implies $w \equiv 0$. Therefore, it follows from (1.7), (2.19), and (2.20) that $[u]_- \equiv 0$, i.e., (0.5) holds. □

Proof of Proposition 1.1. Let γ be a solution of the problem (2.1), (2.2). Put

$$\gamma_* = -\min\{\gamma(t): \ t \in [a,b]\} \tag{2.24}$$

and choose $t_* \in]a,b]$ such that

$$\gamma(t_*) = -\gamma_* \,. \tag{2.25}$$

Suppose that

$$\gamma_* \ge 0. \tag{2.26}$$

The integration of (2.1) from a to t, by virtue of (2.2), yields

$$\gamma'(t) = 1 + \int_a^t \ell(\gamma)(s)ds \quad \text{for} \quad t \in [a,b]. \tag{2.27}$$

Integrating (2.27) from a to t_* and taking into account (1.9), (2.2), (2.24)–(2.26), and the condition $\ell \in P_{ab}$, we get the contradiction

$$\gamma_* + 1 = -(t_* - a) - \int_a^{t_*}\int_a^s \ell(\gamma)(\xi)d\xi ds \le \gamma_* \int_a^b (b-s)\ell(1)(s)ds = \gamma_* \,.$$

Thus, $\gamma_* < 0$, i.e., the inequality (1.2) holds. Therefore, the function γ satisfies the assumptions of Theorem 1.1. □

Proof of Theorem 1.2. Assume the contrary, let $\ell \notin H_{ab}(a)$. Then there exist $u \in \widetilde{C}'([a,b];\mathbb{R})$ and $t_0 \in]a,b[$ such that (0.3), (0.4), and (2.4) hold. Denote by ℓ_{t_0} the restriction of the operator ℓ to the space $C([a,t_0];\mathbb{R})$. By virtue of (0.3) and (1.10), we have

$$u''(t) \ge \ell_{t_0}(u)(t) \quad \text{for} \quad t \in [a,t_0], \tag{2.28}$$

$$\gamma''(t) \le \ell_{t_0}(\gamma)(t) \quad \text{for} \quad t \in [a,t_0]. \tag{2.29}$$

Taking now into account (0.4) and the assumption $-\ell_{t_0} \in P_{at_0}$, it follows from (2.28) that

$$\max\{u(t): \ t \in [a,t_0]\} > 0. \tag{2.30}$$

Put

$$\lambda = \max\left\{\frac{u(t)}{\gamma(t)} : \ t \in [a, t_0]\right\} \tag{2.31}$$

and

$$w(t) = \lambda\gamma(t) - u(t) \quad \text{for} \quad t \in [a, t_0]. \tag{2.32}$$

By (1.11) and (2.30) we get

$$\lambda > 0. \tag{2.33}$$

On account of (1.11), (2.4), and (2.33), we obtain

$$w(t_0) > 0. \tag{2.34}$$

In view of (2.31), clearly

$$w(t) \geq 0 \quad \text{for} \quad t \in [a, t_0], \tag{2.35}$$

and there exists $t_* \in [a, t_0[$ such that

$$w(t_*) = 0. \tag{2.36}$$

It follows from (2.28), (2.29), and (2.33) that

$$w''(t) \leq \ell_{t_0}(w)(t) \quad \text{for} \quad t \in [a, t_0]. \tag{2.37}$$

Hence, on account of (2.35) and the condition $-\ell_{t_0} \in P_{at_0}$, we get

$$w''(t) \leq 0 \quad \text{for} \quad t \in [a, t_0]. \tag{2.38}$$

On the other hand, it follows from (0.4), (1.12), and (2.33) that

$$w'(a) \leq 0 \quad \text{for} \quad t \in [a, t_0],$$

which, together with (2.35), (2.36), and (2.38), contradicts (2.34). □

Proof of Corollary 1.3. Assume that $\ell(1) \not\equiv 0$ (if $\ell(1) \equiv 0$ then Corollary 1.3 is trivial). Put

$$\gamma(t) = (b - t)\int_a^t |\ell(1)(s)|ds + \int_t^b (b - s)|\ell(1)(s)|ds \quad \text{for} \quad t \in [a, b]. \tag{2.39}$$

Obviusly, (1.11) holds and $\gamma'(a) = 0$. Moreover, $\gamma \in \widetilde{C}'([a, b]; \mathbb{R})$,

$$\gamma'(t) \leq 0 \quad \text{for} \quad t \in [a, b], \tag{2.40}$$

and

$$\gamma''(t) = \ell(1)(t) \quad \text{for} \quad t \in [a, b]. \tag{2.41}$$

By virtue of (1.13), (2.39), and (2.40), we get

$$\gamma(t) \leq 1 \quad \text{for} \quad t \in [a, b]. \tag{2.42}$$

On account of (2.42) and the assumption $-\ell \in P_{ab}$, it follows from (2.41) that (1.10) holds. Therefore, the function γ satisfies all the conditions of Theorem 1.2. □

Proof of Theorem 1.3. Assume the contrary, let $\ell \not\in \widetilde{H}_{ab}(a)$. Then there exist $u \in \widetilde{C}'([a,b];\mathbb{R})$ and $t_0 \in]a,b[$ such that (0.3), (0.2_0), and (2.4) hold. Denote by ℓ_{t_0} the restriction of the operator ℓ to the space $C([a,t_0];\mathbb{R})$. By virtue of (0.3) and (1.10), the inequalities (2.28) and (2.29) are fulfilled. In view of (0.2_0) and the assumption $-\ell_{t_0} \in P_{at_0}$, it follows from (2.28) that (2.30) holds.

Put

$$\lambda = \sup\left\{\frac{u(t)}{\gamma(t)} :\ t \in]a,t_0]\right\}. \tag{2.43}$$

By (1.15) and (0.2_0), evidently

$$\lim_{t\to a+}\frac{u(t)}{\gamma(t)} = 0. \tag{2.44}$$

Therefore, $\lambda < +\infty$. On the other hand, by virtue of (2.30), the inequality (2.33) is satisfied.

Define the function w by (2.32). In view of (1.14), (2.4), and (2.33), we get (2.34). On account of (2.43), the inequality (2.35) holds. It easily follows from (2.35), (2.43), and (2.44) that there exists $t_* \in]a,t_0[$ such that

$$w(t_*) = 0, \quad w'(t_*) = 0. \tag{2.45}$$

The inequalities (2.28), (2.29), and (2.33) imply (2.37). Hence, on account of (2.35) and the condition $-\ell_{t_0} \in P_{at_0}$, we get (2.38). It follows from (2.38) and (2.45) that $w(t_0) \le 0$, which contradicts (2.34). □

Proof of Corollary 1.4. Assume that $\ell(1) \not\equiv 0$ (if $\ell(1) \equiv 0$ then Corollary 1.4 is trivial). By the same arguments as in the proof of Corollary 1.3 one can easily verify that the function

$$\gamma(t) = \frac{1}{b-a}\left[(b-t)\int_a^t (s-a)|\ell(1)(s)|ds + \right.$$
$$\left. +(t-a)\int_t^b (b-s)|\ell(1)(s)|ds\right] \quad \text{for} \quad t \in [a,b]$$

satisfies the assumption of Theorem 1.3. □

Proof of Theorem 1.4. Let $\ell \in H'_{ab}(a)$. According to Remark 0.3, the problem

$$\gamma''(t) = \ell(\gamma)(t), \quad \gamma(a) = 0, \quad \gamma'(a) = 1$$

has a unique solution γ (i.e., $\gamma \in \widetilde{C}'([a,b];\mathbb{R})$) and

$$\gamma(t) \ge 0, \qquad \gamma'(t) \ge 0 \quad \text{for} \quad t \in [a,b].$$

Therefore, the function γ satisfies (1.10), (1.18), and (1.19).

Now suppose that a function $\gamma \in \widetilde{C}'_{loc}(]a,b[;\mathbb{R})$ satisfies (1.10), (1.18), and (1.19). Put

$$A = \big\{x \in]a,b[: \gamma'(t) > 0 \text{ for } t \in]a,x]\big\} \tag{2.46}$$

and

$$b_0 = \sup A. \tag{2.47}$$

By virtue of (1.19), we have $b_0 \in \,]a,b]$. It is clear that

$$\gamma'(t) > 0 \quad \text{for} \quad t \in \,]a, b_0[\,. \tag{2.48}$$

Let $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfies (0.3) and (0.6). First we will show that

$$u'(t) \geq 0 \quad \text{for} \quad t \in [a, b_0]. \tag{2.49}$$

Assume the contrary, let (2.49) do not hold. Then there exists $t_0 \in]a, b_0[$ such that

$$u'(t_0) < 0. \tag{2.50}$$

Denote by ℓ_{t_0} the restriction of the operator ℓ to the space $C([a,t_0];\mathbb{R})$. Clearly, (2.28) and (2.29) are fulfilled. It is not difficult to verify that

$$\max\{u'(t) :\ t \in [a, t_0]\} > 0. \tag{2.51}$$

Indeed, if (2.51) does not hold, then, by virtue of (0.6), the inequality

$$u(t) \leq 0 \quad \text{for} \quad t \in [a, t_0].$$

Is satisfied. Hence, on account of (2.28) and the assumption $-\ell_{t_0} \in P_{at_0}$, we get

$$u''(t) \geq 0 \quad \text{for} \quad t \in [a, t_0].$$

which, together with (0.6), contradicts (2.50).

Put

$$\lambda = \sup\left\{ \frac{u'(t)}{\gamma'(t)} :\ t \in \,]a, t_0] \right\} \tag{2.52}$$

and

$$w(t) = \lambda\gamma(t) - u(t) - \lambda\gamma(a) \quad \text{for} \quad t \in [a, t_0]. \tag{2.53}$$

By (2.51), evidently (2.33) holds. The inequalities (2.28), (2.29), and (2.33) imply (2.37). On the other hand, (1.18), (2.33), and (2.50) yield

$$w'(t_0) > 0. \tag{2.54}$$

It easily follows from (1.18), (2.33), (2.52), and (2.53) that

$$w'(t) \geq 0 \quad \text{for} \quad t \in \,]a, t_0], \tag{2.55}$$

$$w(a) = 0, \tag{2.56}$$

and there exists $t_* \in [a, t_0\,[$ such that

$$w'(t_*) = 0. \tag{2.57}$$

On account of (2.55), (2.56), and the condition $-\ell_{t_0} \in P_{at_0}$, it follows from (2.37) that (2.38) holds. Hence, by (2.57), we get $w'(t_0) \leq 0$, which contradicts (2.54).

Thus, we have proved that (2.49) is fulfilled. Consequently, if $b_0 = b$, then the theorem is proved. Therefore, we will suppose that $b_0 < b$.

By virtue of (1.10), (1.18), and the assumption $-\ell \in P_{ab}$, it is clear that

$$\gamma''(t) \le 0 \quad \text{for} \quad t \in [a,b].$$

Hence, on account of (1.18), (1.19), and (2.46)–(2.48), we get

$$\gamma(t) > 0 \quad \text{for} \quad t \in]a,b_0], \tag{2.58}$$

$$\gamma(t) = \gamma(b_0) \quad \text{for} \quad t \in [b_0,b], \tag{2.59}$$

On the other hand, (1.10) and (2.59) yield

$$\ell(\gamma)(t) = 0 \quad \text{for} \quad t \in [b_0,b]. \tag{2.60}$$

It easily follows from (0.6), (1.19), and (2.58) that

$$\lim_{t\to a+} \frac{u(t)}{\gamma(t)} < +\infty.$$

Hence, by virtue of (2.58), there exists $M > 0$ such that

$$u(t) \le M\gamma(t) \quad \text{for} \quad t \in [a,b]. \tag{2.61}$$

On account of (2.60), (2.61), and the condition $-\ell \in P_{ab}$, it follows from (0.3) that

$$u''(t) \ge 0 \quad \text{for} \quad t \in [b_0,b].$$

Hence, by virtue of (2.49), we get

$$u'(t) \ge u'(b_0) \ge 0 \quad \text{for} \quad t \in [b_0,b],$$
$$u(t) \ge u(b_0) \ge 0 \quad \text{for} \quad t \in [b_0,b].$$

Therefore, (0.7) holds. □

Proof of Corollary 1.5. If $\ell(\varphi) \equiv 0$, then Corollary 1.5 is trivial. Indeed, let the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfies (0.3) and (0.6). Obviously,

$$u(t) \le (t-a)\|u'\|_C = \varphi(t)\|u'\|_C \quad \text{for} \quad t \in [a,b]. \tag{2.62}$$

It follows from (0.3), by virtue of (2.62) and the assumption $-\ell \in P_{ab}$, that

$$u''(t) \ge \ell(\varphi)(t)\|u'\|_C = 0 \quad \text{for} \quad t \in [a,b].$$

The latter inequality and (0.6) yield (0.7).

Suppose that $\ell(\varphi) \not\equiv 0$. It is easy to verify that the function

$$\gamma(t) = \int_a^t (s-a)|\ell(1)(s)|ds + (t-a)\int_t^b |\ell(1)(s)|ds \quad \text{for} \quad t \in [a,b]$$

satisfies the assumptions of Theorem 1.4. □

Proof of Theorem 1.5. Assume that (1.22) (resp. (1.23)) holds and the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfies (0.3) and (0.4) (resp. (0.3) and (0.2_0)). By virtue of the assumption $-\ell_1 \in H_{ab}(a)$ (resp. $-\ell_1 \in \widetilde{H}_{ab}(a)$), the problem

$$\alpha''(t) = -\ell_1(\alpha)(t) - \ell_0([u]_-)(t), \tag{2.63}$$

$$\alpha(a) = 0, \quad \alpha'(a) = 0 \tag{2.64}$$

has a unique solution α (see Remark 0.3) and

$$\alpha(t) \le 0 \quad \text{for} \quad t \in [a,b]. \tag{2.65}$$

It follows from (0.3), (2.63), and the assumption $\ell_0 \in P_{ab}$ that

$$(u(t)-\alpha(t))'' \ge -\ell_1(u-\alpha)(t) + \ell_0([u]_+)(t) \ge \\ \ge -\ell_1(u-\alpha)(t) \quad \text{for} \quad t \in [a,b].$$

Hence, by virtue of (0.4), (2.64), and the condition $-\ell_1 \in H_{ab}(a)$ (resp. $-\ell_1 \in \widetilde{H}_{ab}(a)$), we get

$$u(t) \ge \alpha(t) \quad \text{for} \quad t \in [a,b]. \tag{2.66}$$

On account of (2.65), we get from (2.66) that

$$-[u(t)]_- \ge \alpha(t) \quad \text{for} \quad t \in [a,b]. \tag{2.67}$$

By virtue of (2.65), (2.67), and the assumptions $\ell_0, \ell_1 \in P_{ab}$, the equality (2.63) results in

$$\alpha''(t) \ge \ell_0(\alpha)(t) \quad \text{for} \quad t \in [a,b].$$

Hence, on account of (2.64) and the condition $\ell_0 \in H_{ab}(a)$ (resp. $\ell_0 \in \widetilde{H}_{ab}(a)$), we get

$$\alpha(t) \ge 0 \quad \text{for} \quad t \in [a,b].$$

The latter inequality, (2.65), and (2.66) yield (0.5). Therefore, $\ell \in H_{ab}(a)$ (resp. $\ell \in \widetilde{H}_{ab}(a)$).

Suppose now that (1.24) holds and the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfies (0.3) and (0.6). By the same arguments as above we get that (0.5) is fulfilled. On account of (0.5) and the condition $\ell_0 \in P_{ab}$, it follows from (0.3) that

$$u''(t) \ge -\ell_1(u)(t) \quad \text{for} \quad t \in [a,b].$$

Hence, by virtue of the condition $-\ell_1 \in H'_{ab}(a)$, we get

$$u'(t) \ge 0 \quad \text{for} \quad t \in [a,b]. \qquad \square$$

3. Corollaries for Equation with Deviating Argument

In this section, the results from Section 1 will be concretized for the case, when the operator $\ell \in \mathcal{L}_{ab}$ has one of the following forms:

$$\ell(v)(t) \stackrel{def}{=} p(t)v(\tau(t)), \tag{3.1}$$

$$\ell(v)(t) \stackrel{def}{=} -g(t)v(\mu(t)), \tag{3.2}$$

$$\ell(v)(t) \stackrel{def}{=} p(t)v(\tau(t)) - g(t)v(\mu(t)), \tag{3.3}$$

where $p, g \in L([a,b];\mathbb{R}_+)$ and $\tau, \mu \in M_{ab}$. In the sequel, we will use the notation

$$\tau^* = \operatorname{ess\,sup}\{\tau(t) : \ t \in [a,b]\}.$$

Theorem 3.1. *Let at least one of the following items be fulfilled:*

a) *there exists* $\alpha \in]0,1[$ *such that*

$$\int_a^t (t-s)p(s)\left(\int_a^{\tau(s)}(\tau(s)-\xi)p(\xi)d\xi\right)ds \leq$$
$$\leq \alpha \int_a^t (t-s)p(s)ds \quad \text{for} \quad t\in[a,b]; \quad (3.4)$$

b)

$$r\int_a^b (b-s)p(s)\sigma(s)\left[\int_s^{\tau(s)}\int_a^{\xi} p(\eta)d\eta d\xi\right]ds < 1, \quad (3.5)$$

where

$$\sigma(t) = \frac{1}{2}(1+\operatorname{sgn}(\tau(t)-t)) \quad \text{for} \quad t\in[a,b], \quad (3.6)$$
$$r = \exp\left(\frac{1}{b-a}\int_a^b (s-a)(b-s)p(s)ds\right);$$

c) $\int_a^{\tau^*}(\tau^*-s)p(s)ds \neq 0$ *and*

$$\operatorname{ess\,sup}\left\{\int_t^{\tau(t)}\int_a^s p(\xi)d\xi ds \ : \ t\in[a,b]\right\} < \lambda^*, \quad (3.7)$$

where

$$\lambda^* = \sup\left\{\frac{1}{\lambda}\ln\left[\frac{\lambda\exp\left[\lambda\int_a^{\tau^*}(\tau^*-s)p(s)ds\right]}{\exp\left[\lambda\int_a^{\tau^*}(\tau^*-s)p(s)ds\right]-1}\right] \ : \ \lambda>0\right\}. \quad (3.8)$$

Then the operator ℓ *defined by* (3.1) *belongs to the set* $H_{ab}(a)$ (*and therefore to the sets* $\widetilde{H}_{ab}(a)$ *and* $H'_{ab}(a)$).

From Theorem 3.1 a) and c) it immediately follows

Corollary 3.1. *Let either*

$$\int_a^{\tau^*}(\tau^*-s)p(s)ds < 1,$$

or

$$\int_a^{\tau^*}(\tau^*-s)p(s)ds > 1$$

and

$$\operatorname{ess\,sup}\left\{\int_t^{\tau(t)}\int_a^s p(\xi)d\xi ds : \ t\in[a,b]\right\} \leq \frac{1}{e}\,.$$

Then the operator ℓ *defined by* (3.1) *belongs to the set* $H_{ab}(a)$.

The next theorem is, in a certain sense, a complement of Corollary 3.1.

Theorem 3.2. *Let*

$$\int_a^{\tau^*} (\tau^* - s)p(s)ds = 1. \tag{3.9}$$

Then the operator ℓ *defined by* (3.1) *belongs to the set* $H_{ab}(a)$ *if and only if*

$$\int_a^{\tau^*} (\tau^* - t)p(t) \left(\int_{\tau(t)}^{\tau^*} \int_a^s p(\xi)d\xi ds \right) dt \neq 0. \tag{3.10}$$

Theorem 3.3. *Let*

$$\mu(t) \leq t \quad \text{for} \quad t \in [a,b] \tag{3.11}$$

and

$$\int_a^b (b-s)g(s)ds \leq 1. \tag{3.12}$$

Then the operator ℓ *defined by* (3.2) *belongs to the set* $H_{ab}(a)$.

Remark 3.1. The constant 1 in (3.12) is the best possible and cannot be replaced by $1+\varepsilon$, no matter how small $\varepsilon > 0$ would be (see Remark 1.3).

Theorem 3.4. *Let* (3.11) *hold and let*

$$(b-\mu(t)) \int_a^{\mu(t)} (s-a)g(s)ds + (\mu(t)-a) \int_{\mu(t)}^b (b-s)g(s)ds \leq \\ \leq b-a \quad \textit{for} \quad t \in [a,b]. \tag{3.13}$$

Then the operator ℓ *defined by* (3.2) *belongs to the set* $\widetilde{H}_{ab}(a)$.

Remark 3.2. Example 4.2 below shows that the condition (3.13) in Theorem 3.4 cannot be replaced by the condition

$$(b-\mu(t)) \int_a^{\mu(t)} (s-a)g(s)ds + (\mu(t)-a) \int_{\mu(t)}^b (b-s)g(s)ds \leq \\ \leq (1+\varepsilon)(b-a) \quad \text{for} \quad t \in [a,b],$$

no matter how small $\varepsilon > 0$ would be.

Theorem 3.5. *Let* (3.11) *hold and let*

$$\int_a^b (\mu(s)-a)g(s)ds \leq 1. \tag{3.14}$$

Then the operator ℓ *defined by* (3.2) *belongs to the set* $H'_{ab}(a)$.

Remark 3.3. Example 4.3 below shows that the condition (3.14) cannot be replaced by the condition

$$\int_a^b (\mu(s)-a)g(s)ds \leq 1+\varepsilon,$$

no matter how small $\varepsilon > 0$ would be.

Theorem 3.6. *Let ℓ be the operator defined by* (3.3)*. Let, moreover,* (3.11) *be fulfilled and at least one of the conditions of Theorem* 3.1 *hold. Then the condition* (3.12) *implies the inclusion $\ell \in H_{ab}(a)$, the condition* (3.13) *implies the inclusion $\ell \in \widetilde{H}_{ab}(a)$, and the condition* (3.14) *implies the inclusion $\ell \in H'_{ab}(a)$.*

Proof of Theorem 3.1. a) It is easy to verify that (3.4) implies

$$\varphi_3(t) \le \alpha\varphi_2(t) \quad \text{for} \quad t \in [a,b],$$

where

$$\varphi_2(t) = \int_a^t (t-s)p(s)ds \quad \text{for} \quad t \in [a,b],$$

$$\varphi_3(t) = \int_a^t (t-s)p(s)\varphi_2(\tau(s))ds \quad \text{for} \quad t \in [a,b].$$

Thus, the inequality (1.4) holds for $m = 3$ and $k = 2$. Therefore, by virtue of Corollary 1.2 a), the operator ℓ given by (3.1) belongs to the set $H_{ab}(a)$.

b) Let $\overline{\ell} \in \mathcal{L}_{ab}$ be the operator defined by

$$\overline{\ell}(v)(t) \overset{def}{=} p(t)\sigma(t)\int_t^{\tau(t)}\int_a^s p(\xi)v(\tau(\xi))d\xi ds \quad \text{for} \quad t \in [a,b].$$

Obviously, the inequality

$$\ell(\varphi(v))(t) - \ell(1)(t)\varphi(v)(t) = p(t)\int_t^{\tau(t)}\int_a^s p(\xi)v(\tau(\xi))d\xi ds \le$$
$$\le \overline{\ell}(v)(t) \quad \text{for} \quad t \in [a,b]$$

holds on the set $C_a([a,b];\mathbb{R}_+)$, where

$$\varphi(v)(t) \overset{def}{=} \int_a^t\int_a^s p(\xi)v(\tau(\xi))d\xi ds.$$

On the other hand, it follows from (3.5) that (1.5) holds. Therefore, the Assumptions of Corollary 1.2 b) are fulfilled.

c) On account of (3.7), there exists $\varepsilon_0 \in]0, \lambda^*[$ such that

$$\int_t^{\tau(t)}\int_a^s p(\xi)d\xi ds \le \lambda^* - \varepsilon_0 \quad \text{for} \quad t \in [a,b]. \tag{3.15}$$

By virtue of (3.8), there exist $\delta > 0$, $\lambda_0 > 0$, and $\varepsilon > 0$ such that

$$\varepsilon < 1 \tag{3.16}$$

and

$$\lambda^* - \varepsilon_0 \le \frac{1}{\lambda_0}\ln\left[\frac{\lambda_0\exp\left[\lambda_0\int_a^{\tau^*}(\tau^*-s)p(s)ds\right]}{\exp\left[\lambda_0\int_a^{\tau^*}(\tau^*-s)p(s)ds\right] + \delta(b-a) - \varepsilon}\right]. \tag{3.17}$$

It follows from (3.15) and (3.17) that

$$\exp\left[\lambda_0 \int_t^{\tau(t)} \int_a^s p(\xi)d\xi ds\right] \leq$$
$$\leq \frac{\lambda_0 \exp\left[\lambda_0 \int_a^{\tau(t)} (\tau(t)-s)p(s)ds\right]}{\exp\left[\lambda_0 \int_a^{\tau(t)} (\tau(t)-s)p(s)ds\right] + \delta(\tau(t)-a) - \varepsilon} \quad \text{for} \quad t \in [a,b].$$

Hence,

$$\lambda_0 \exp\left[\lambda_0 \int_a^t (t-s)p(s)ds\right] \geq \exp\left[\lambda_0 \int_a^{\tau(t)} (\tau(t)-s)p(s)ds\right] +$$
$$+\delta(\tau(t)-a) - \varepsilon \quad \text{for} \quad t \in [a,b]. \tag{3.18}$$

Put

$$\gamma(t) = \exp\left[\lambda_0 \int_a^t (t-s)p(s)ds\right] + \delta(t-a) - \varepsilon \quad \text{for} \quad t \in [a,b].$$

On account of (3.18), it is not difficult to verify that

$$\gamma''(t) \geq p(t)\gamma(\tau(t)) \quad \text{for} \quad t \in [a,b].$$

On the other hand, evidently (1.2) and (1.3) hold. Thus, the function γ satisfies the assumptions of Theorem 1.1. ☐

Proof of Corollary 3.1. Corollary 3.1 immediately follows from Theorem 3.1 a) and c). ☐

To prove Theorem 3.2 we need the following lemma.

Lemma 3.1. *Let ℓ be the operator defined by* (3.1) *and let* (3.9) *hold. Then every function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying* (0.1_0) *and* (0.2_0) *is either nonnegative or nonpositive.*

Proof. Let the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy (0.1_0) and (0.2_0). It is sufficient to show that the function u is either nonnegative or nonpositive in $[a,\tau^*]$. Put

$$M = \max\{u(t):\ t \in [a,\tau^*]\}, \quad -m = \min\{u(t):\ t \in [a,\tau^*]\} \tag{3.19}$$

and choose $t_M, t_m \in]a,\tau^*]$ such that

$$u(t_M) = M, \quad u(t_m) = -m. \tag{3.20}$$

Without loss of generality we can assume that $t_m < t_M$. Suppose that

$$M > 0 \quad \text{and} \quad m > 0. \tag{3.21}$$

Integrating (0.1_0) from a to t and taking into account (0.2_0), we get

$$u'(t) = \int_a^t p(s)u(\tau(s))ds \quad \text{for} \quad t \in [a,b]. \tag{3.22}$$

The integration of (3.22) from t_m to t_M, by virtue of (3.19)–(3.21) and (3.9), yields the contradiction

$$M + m = \int_{t_m}^{t_M} \int_a^s p(\xi)u(\tau(\xi))d\xi ds \le M \int_a^{\tau^*} (\tau^* - s)p(s)ds = M.$$

Therefore, (3.21) does not hold, i.e., the function u is either nonnegative or nonpositive. $\square$

Proof of Theorem 3.2. Let (3.9) and (3.10) hold. According to Proposition 1.1, it is sufficient to show that the problem (0.1_0), (0.2_0) has only the trivial solution. Let the function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy (0.1_0) and (0.2_0). By virtue of Lemma 3.1, without loss of generality we can assume that

$$u(t) \ge 0 \quad \text{for} \quad t \in [a,b]. \tag{3.23}$$

It follows from (0.1_0), on account of (0.2_0) and (3.23), that

$$u(\tau(t)) \le u(\tau^*) \quad \text{for} \quad t \in [a,b]. \tag{3.24}$$

The integration of (0.1_0) from a to t, in view of (0.2_0), yields (3.22). Integrating (3.22) from t to τ^* and taking into account (3.24), we get

$$u(t) \ge u(\tau^*) \left(1 - \int_t^{\tau^*} \int_a^s p(\xi)d\xi ds\right) \quad \text{for} \quad t \in [a,b]. \tag{3.25}$$

The latter inequality, by virtue of (3.9), results in

$$u(t) \ge u(\tau^*) \int_a^t \int_a^s p(\xi)d\xi ds \quad \text{for} \quad t \in [a,b]. \tag{3.26}$$

On the other hand, integrating (3.22) from a to t and taking into account (3.24) and (0.2_0), we get

$$u(t) \le u(\tau^*) \int_a^t \int_a^s p(\xi)d\xi ds \quad \text{for} \quad t \in [a,b]. \tag{3.27}$$

Thus, it follows from (3.26) and (3.27) that

$$u(t) = u(\tau^*) f(t) \quad \text{for} \quad t \in [a,b], \tag{3.28}$$

where

$$f(t) \stackrel{def}{=} \int_a^t \int_a^s p(\xi)d\xi ds \quad \text{for} \quad t \in [a,b]. \tag{3.29}$$

On account of (3.28), the equality (3.22) results in

$$u'(t) = u(\tau^*) \int_a^t p(s)f(\tau(s))ds \quad \text{for} \quad t \in [a,b]. \tag{3.30}$$

The integration of (3.30) from a to τ^*, on account of (0.2_0), implies

$$u(\tau^*) = u(\tau^*) \int_a^{\tau^*} (\tau^* - s)p(s)f(\tau(s))ds. \tag{3.31}$$

On account of (3.9) and (3.29), from (3.10) we obtain

$$\int_a^{\tau^*} (\tau^* - s)p(s)f(\tau(s))ds \neq 1.$$

Thus, it follows from (3.31) that

$$u(\tau^*) = 0.$$

Taking now into account (3.23) and (3.24), from (0.1_0) we get

$$u''(t) = 0 \quad \text{for} \quad t \in [a, b],$$

which, together with (0.2_0), yields $u \equiv 0$.

Now suppose that (3.9) holds and

$$\int_a^{\tau^*} (\tau^* - t)p(t)\left(\int_{\tau(t)}^{\tau^*}\int_a^s p(\xi)d\xi ds\right) dt = 0. \tag{3.32}$$

By virtue of (3.9) and (3.32), we have

$$f(\tau^*) = 1 \tag{3.33}$$

and

$$\int_a^{\tau^*} (\tau^* - t)p(t)[f(\tau^*) - f(\tau(t))]dt = 0, \tag{3.34}$$

where the function f is defined by (3.29). In view of the inequality

$$f(\tau(t)) \leq f(\tau^*) \quad \text{for} \quad t \in [a, b],$$

it follows from (3.34) that

$$\begin{aligned} 0 &\leq \int_a^t \int_a^s p(\xi)[f(\tau^*) - f(\tau(\xi))]d\xi ds = \\ &= -\int_t^{\tau^*}\int_a^s p(\xi)[f(\tau^*) - f(\tau(\xi))]d\xi ds \leq 0 \quad \text{for} \quad t \in [a, \tau^*]. \end{aligned}$$

Therefore, on account of (3.33) and (3.29),

$$f(t) = \int_a^t \int_a^s p(\xi)f(\tau(\xi))d\xi ds \quad \text{for} \quad t \in [a, \tau^*]. \tag{3.35}$$

Put

$$u(t) = \begin{cases} f(t) & \text{for } t \in [a, \tau^*[\\ 1 + (t - \tau^*)\int_a^{\tau^*} p(s)ds + \int_{\tau^*}^t \int_{\tau^*}^s p(\xi)f(\tau(\xi))d\xi ds & \text{for } t \in [\tau^*, b] \end{cases}.$$

On account of (3.33), we obtain $u(\tau^*) = 1$, i.e., $u \not\equiv 0$. On the other hand, taking into account (3.35), it is not difficult to verify that

$$u''(t) = p(t)u(\tau(t)) \quad \text{for} \quad t \in [a, b].$$

Thus, u is a nontrivial solution of the problem (0.1_0), (0.2_0). Therefore, according to Remark 0.3, we have $\ell \notin H_{ab}(a)$. $\square$

The proofs of Theorems 3.3–3.5 are similar to ones of Corollaries 1.3–1.5. Theorem 3.6 immediately follows from Theorem 1.5 and Theorems 3.1–3.5.

4. Examples

Example 4.1. Let $\ell(v)(t) \stackrel{def}{=} p(t)v(b)$, where $p \in L([a,b];\mathbb{R}_+)$, and

$$\int_a^b (b-s)p(s)ds = 1.$$

Obviously,

$$\varphi_n(t) = \int_a^t (t-s)p(s)ds \quad \text{for} \quad t \in [a,b],\ n \in \mathbb{N},$$

where φ_n are functions defined in Corollary 1.2 a). It is clear that for each $m, k \in \mathbb{N}$ the inequality (1.4) holds with $\alpha = 1$. On the other hand, the function

$$u(t) = \int_a^t (t-s)p(s)ds \quad \text{for} \quad t \in [a,b]$$

is a nontrivial solution of the problem (0.1_0), (0.2_0). Therefore, according to Remark 0.3, we have $\ell \notin H_{ab}(a)$.

Example 4.2. Let $a_0 < b$, $\varepsilon \in]0,1[$, $\lambda = \frac{1}{\varepsilon}$, $t_0 = \frac{1}{2}(a_0+b)$, $\delta = \frac{1}{2}(b-a_0)$,

$$\widetilde{g}(t) = \begin{cases} \frac{(t-a_0)^{\lambda-2}}{\delta^\lambda}\left[1+\lambda-\frac{(t-a_0)^\lambda}{\delta^\lambda}\right] & \text{for} \quad t \in]a_0,t_0[\\ \frac{(b-t)^{\lambda-2}}{\delta^\lambda}\left[1+\lambda-\frac{(b-t)^\lambda}{\delta^\lambda}\right] & \text{for} \quad t \in]t_0,b[\end{cases},$$

$a = a_0 - \frac{\varepsilon}{2}\left(\int_{a_0}^b \widetilde{g}(s)ds\right)^{-1}$, and

$$g(t) = \begin{cases} 0 & \text{for} \quad t \in]a,a_0[\\ \widetilde{g}(t) & \text{for} \quad t \in]a_0,b[\end{cases}.$$

Let, moreover, $\ell(v)(t) \stackrel{def}{=} -g(t)v(t)$. Obviously, (1.17) holds. It is not difficult to verify that the function

$$v(t) = \begin{cases} (t-a_0)\exp\left[-\frac{(t-a_0)^\lambda}{\lambda\delta^\lambda}\right] & \text{for} \quad t \in [a_0,t_0[\\ (b-t)\exp\left[-\frac{(b-t)^\lambda}{\lambda\delta^\lambda}\right] & \text{for} \quad t \in [t_0,b_0] \end{cases}$$

is a solution of the problem

$$v''(t) = -g(t)v(t), \quad v(a_0) = 0, \quad v(b) = 0 \tag{4.1}$$

and

$$v(t) > 0 \quad \text{for} \quad t \in]a_0,b[. \tag{4.2}$$

Now let

$$q(t) = \begin{cases} 1 & \text{for} \quad t \in]a,a_0[\\ 0 & \text{for} \quad t \in]a_0,b[\end{cases}$$

and let the function u be a solution of the problem (0.1), (0.2_0). Obviously, (0.3) holds, as well. On the other hand, u satisfies

$$u''(t) = -g(t)u(t), \quad u(a_0) = \frac{1}{2}(a-a_0)^2, \quad u'(a_0) = a - a_0.$$

By virtue of (4.1), (4.2), and Sturm's separation theorem, we get $u(b) < 0$. Therefore, $\ell \notin \widetilde{H}_{ab}(a)$.

Example 4.3. Let $\varepsilon \in]0,1[$, $\lambda = \frac{1}{\varepsilon}$, $t_0 \in]a,b[$, $t_1 \in]t_0,b[$, $\delta = (t_0-a)^\lambda$, $m = \varepsilon^{\frac{1}{2}} \left[(t_1-a)^2 - (t_0-a)^2\right]^{-\frac{1}{2}}$ and

$$g(t) = \begin{cases} \frac{(t-a)^{\lambda-2}}{\delta^\lambda}\left[1+\lambda - \frac{(t-a)^\lambda}{\delta^\lambda}\right] & \text{for} \quad t \in]a,t_0[\\ m^2 & \text{for} \quad t \in]t_0,t_1[\\ 0 & \text{for} \quad t \in]t_1,b[\end{cases} .$$

Let, moreover, $\ell(v)(t) \stackrel{def}{=} -g(t)v(t)$. Obviously, (1.21) holds. On the other hand, the function

$$u(t) = \begin{cases} \frac{t-a}{t_0-a}\exp\left[\frac{1}{\lambda} - \frac{(t-a)^\lambda}{\lambda\delta^\lambda}\right] & \text{for} \quad t \in [a,t_0[\\ \cos[m(t-t_0)] & \text{for} \quad t \in [t_0,t_1[\\ \cos[m(t_1-t_0)] - m(t-t_1)\sin[m(t_1-t_0)] & \text{for} \quad t \in [t_1,b] \end{cases}$$

is a solution of the problem

$$u''(t) = \ell(u)(t), \quad u(a) = 0, \quad u'(a) = (t_0-a)^{-1}\exp\left(\frac{1}{\lambda}\right)$$

and

$$u'(t) < 0 \quad \text{for} \quad t_0 < t < \min\left\{t_1, t_0 + \frac{\pi}{2}\right\}.$$

Therefore, $\ell \notin H'_{ab}(a)$.

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