Memoirs on Differential Equations and Mathematical Physics
Volume 40, 2007, 55-65

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ON SOLVABILITY OF A THREE-POINT BOUNDARY VALUE PROBLEM FOR SECOND ORDER NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

Abstract. Sufficient conditions for the solvability of the problem

$$u''(t) = \ell(u)(t) + F(u)(t); \quad u(a) = 0, \;\; u(b) = u(t_0)$$

are established, where $t_0 \in]a,b[\,$, ℓ, $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ are continuous operators, and ℓ is linear.

2000 Mathematics Subject Classification. 34K06, 34K10.

Key words and phrases. second order nonlinear functional differential equation, three-point BVP.

რეზიუმე. დადგენილია

$$u''(t) = \ell(u)(t) + F(u)(t); \quad u(a) = 0, \;\; u(b) = u(t_0)$$

ამოცანის ამოხსნადობის საკმარისი პირობები, სადაც $t_0 \in]a,b[\,$, ℓ, F : $C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ უწყვეტი ოპერატორებია და ℓ წრფივია.

Introduction

In the present paper, for the nonlinear functional differential equation

$$u''(t) = \ell(u)(t) + F(u)(t), \tag{0.1}$$

we consider the problem on the existence of a solution satisfying the boundary conditions

$$u(a) = 0, \quad u(b) = u(t_0). \tag{0.2}$$

Here we suppose that $t_0 \in \,]a,b[$ is fixed, ℓ, $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ are continuous operators and, moreover, ℓ is linear.

Such problems for ordinary differential equations have been studied in detail even for equations with singularities (see, e.g., [1], [3], [4], [5], [6], [7] and references therein). Conditions of unique solvability of the linear problem

$$u''(t) = \ell(u)(t) + q(t); \quad u(a) = 0, \;\; u(b) = u(t_0)$$

are stated in [8] and [9]. However, the nonlinear problem (0.1), (0.2) has not been investigated sufficiently yet. Below we will establish efficient conditions for the solvability of (0.1), (0.2) and concretize them for special cases of (0.1) – for so-called equations with deviating argument and integro-differential equations.

Throughout the paper we will use the following notation.

$\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.

If $x \in \mathbb{R}$, then $[x]_+ = \frac{1}{2}(|x| + x)$ and $[x]_- = \frac{1}{2}(|x| - x)$.

$C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u : [a,b] \to \mathbb{R}$ with the norm $\|u\|_C = \max\{|u(t)| : \; t \in [a,b]\}$.

$C([a,b];\mathbb{R}_+) = \{u \in C([a,b];\mathbb{R}) : \; u(t) \geq 0 \text{ for } t \in [a,b]\}$.

$\widetilde{C}([a,b];\mathbb{R})$ is the set of absolutely continuous functions $u : [a,b] \to \mathbb{R}$.

$\widetilde{C}'([a,b];\mathbb{R})$ is the set of functions $u \in \widetilde{C}([a,b];\mathbb{R})$ such that $u' \in \widetilde{C}([a,b];\mathbb{R})$.

$L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $p : [a,b] \to \mathbb{R}$ with the norm $\|p\|_L = \int\limits_a^b |p(s)|\, ds$.

$L([a,b];\mathbb{R}_+) = \{p \in L([a,b];\mathbb{R}) : \; p(t) \geq 0 \text{ for almost all } t \in [a,b]\}$.

$L_2([a,b];\mathbb{R})$ is the Banach space of functions $v : [a,b] \to \mathbb{R}$, $v^2 \in L([a,b];\mathbb{R})$ with the norm $\|v\|_{L_2} = \sqrt{\int\limits_a^b v^2(s)\, ds}$.

M_{ab} is the set of measurable functions $f : [a,b] \to [a,b]$.

$\mathcal{L}_{ab}$ is the set of linear bounded operators $\ell : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

$\mathcal{P}_{ab}$ is the set of linear operators $\ell \in \mathcal{L}_{ab}$ transforming the set $C([a,b];\mathbb{R}_+)$ into the set $L([a,b];\mathbb{R}_+)$.

$\mathcal{K}_{ab}$ is the set of continuous operators $F : C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ satisfying the Carathéodory conditions, i.e., for every $r > 0$ there exists $q_r \in L([a,b];\mathbb{R}_+)$ such that

$$|F(v)(t)| \leq q_r(t) \;\text{ for almost all }\; t \in [a,b], \;\; \|v\|_C \leq r.$$

$K([a,b]\times A;B)$, where $A\subseteq\mathbb{R}$, $B\subseteq\mathbb{R}$, is the set of functions $f:[a,b]\times A\to B$ satisfying the Carathéodory conditions, i.e., $f(\cdot,x):[a,b]\to B$ is a measurable function for all $x\in A$, $f(t,\cdot):A\to B$ is a continuous function for almost all $t\in[a,b]$, and for every $r>0$ there exists $q_r\in L([a,b];\mathbb{R}_+)$ such that

$$|f(t,x)|\le q_r(t) \ \text{ for almost all } \ t\in[a,b], \ \ x\in A, \ \ |x|\le r.$$

By a solution to the equation (0.1), where $\ell\in\mathcal{L}_{ab}$ and $F\in\mathcal{K}_{ab}$, we understand a function $u\in\widetilde{C}'([a,b];\mathbb{R})$ satisfying the equality (0.1) almost everywhere in $[a,b]$.

1. Main Results

Before the formulation of the main result, introduce the following notation:

$$\varphi(t)\stackrel{\text{def}}{=}\sqrt{t-a} \ \text{ for } \ t\in[a,b].$$

If $v\in L([a,b];\mathbb{R})$, then $\theta(v)(t)\stackrel{\text{def}}{=}\int\limits_a^t v(s)\,ds$ for $t\in[a,b]$.

Definition 1.1. We will say that an operator $\ell\in\mathcal{L}_{ab}$ belongs to the set $\mathcal{A}$, if ℓ admits the representation $\ell=\ell_0-\ell_1$, where ℓ_0, $\ell_1\in\mathcal{P}_{ab}$, and there exists an operator $\widetilde{\ell}:L_2([a,b];\mathbb{R})\to L_2([a,b];\mathbb{R})$ such that on the set $\widetilde{C}([a,b];\mathbb{R})$ the inequality

$$\big|\ell_0(\theta(v))(t)-\ell_0(1)(t)\theta(v)(t)\big|\le|\widetilde{\ell}(|v|)(t)|\sqrt{\ell_0(1)(t)} \ \text{ for } \ t\in[a,b] \tag{1.1}$$

holds and

$$\|\widetilde{\ell}\|^2<4\Bigg(1-\int\limits_a^b\varphi(t)\ell_1(\varphi)(t)\,dt- -\frac{\varphi(t_0)}{b-t_0}\int\limits_{t_0}^b(t-t_0)\big(\ell_0(\varphi)(t)+\ell_1(\varphi)(t)\big)\,dt\Bigg). \tag{1.2}$$

Theorem 1.1. *Let $\ell\in\mathcal{A}$ and on the set*

$$\big\{u\in\widetilde{C}'([a,b];\mathbb{R}):\ u(a)=0,\ u(b)=u(t_0)\big\}$$

the inequalities

$$F(u)(t)\operatorname{sgn}u(t)\ge-q(t,\|u\|_C) \ \textit{ for } \ t\in[a,b], \tag{1.3}$$

$$(t-t_0)|F(u)(t)|\le q(t,\|u\|_C) \ \textit{ for } \ t\in[t_0,b] \tag{1.4}$$

hold, where $q\in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ satisfies the condition

$$\lim_{x\to+\infty}\frac{1}{x}\int\limits_a^b q(t,x)\,dt=0. \tag{1.5}$$

Then the problem (0.1), (0.2) *has at least one solution.*

As an example, consider the equation

$$u''(t) = p(t)u(\tau(t)) - g(t)u(\mu(t)) + f\big(t, u(t), u(\sigma(t))\big), \tag{1.6}$$

where p, $g \in L([a,b];\mathbb{R}_+)$, τ, μ, $\sigma \in M_{ab}$, and $f \in K([a,b]\times\mathbb{R}^2;\mathbb{R})$.
Theorem 1.1 implies

Corollary 1.1. *Let*

$$\int_a^b p(t)|\tau(t)-t|\,dt < 4\bigg(1 - \int_a^b g(t)\sqrt{(\mu(t)-a)(t-a)}\,dt -$$
$$- \frac{\sqrt{t_0-a}}{b-t_0}\int_{t_0}^b (t-t_0)\big(p(t)\sqrt{\tau(t)-a} + g(t)\sqrt{\mu(t)-a}\big)\,dt\bigg). \tag{1.7}$$

Let, moreover,

$$\begin{gathered} f(t,x,y)\operatorname{sgn} x \geq -q(t,|x|) \ \ \textit{for} \ \ t \in [a,b], \ \ x,\, y \in \mathbb{R}, \\ (t-t_0)|f(t,x,y)| \leq q(t,|x|) \ \ \textit{for} \ \ t \in [t_0,b], \ x,\, y \in \mathbb{R}, \end{gathered} \tag{1.8}$$

where $q \in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ *satisfies* (1.5). *Then the problem* (1.6), (0.2) *has at least one solution.*

As another example, consider the equation

$$u''(t) = \int_a^b h(t,s)u(s)\,ds + f\big(t, u(t), u(\sigma(t))\big), \tag{1.9}$$

where $f \in K([a,b]\times\mathbb{R}^2;\mathbb{R})$ and $h : [a,b]\times[a,b] \to \mathbb{R}$ is integrable on the rectangle $[a,b]\times[a,b]$.

Corollary 1.2. *Let*

$$\int_a^b \bigg(\int_a^b |s-t|\,[h(t,s)]_+\,ds\bigg)\,dt <$$
$$< 4\bigg[1 - \int_a^b \bigg(\sqrt{t-a}\int_a^b \sqrt{s-a}\,[h(t,s)]_-\,ds\bigg)\,dt -$$
$$- \frac{\sqrt{t_0-a}}{b-t_0}\int_{t_0}^b \bigg((t-t_0)\int_a^b \sqrt{s-a}\,|h(t,s)|\,ds\bigg)\,dt\bigg]. \tag{1.10}$$

Let, moreover, the inequalities (1.8) *be fulfilled, where* $q \in K([a,b]\times\mathbb{R}_+;\mathbb{R}_+)$ *satisfies* (1.5). *Then the problem* (1.9), (0.2) *has at least one solution.*

2. Proofs

To prove the main results, we will need the following lemma which is a special case of the so-called principle of a priori estimate established in [2] (see [2, Theorem 1]).

Lemma 2.1. *Let the problem*

$$u''(t) = \ell(u)(t); \quad u(a) = 0, \ \ u(b) = u(t_0) \tag{2.1}$$

have only the trivial solution. Let, moreover, there exist $\rho > 0$ such that for each $\delta \in]0,1[$ and each $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying (0.2) and

$$u''(t) = \ell(u)(t) + \delta F(u)(t) \ \ \text{for} \ \ t \in [a,b], \tag{2.2}$$

the estimate

$$\|u\|_C \leq \rho \tag{2.3}$$

holds. Then the problem (0.1), (0.2) has at least one solution.

Lemma 2.2. *Let $\ell \in \mathcal{A}$. Then there exists $r > 0$ such that for each function $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfying (0.2) and*

$$\big(u''(t) - \ell(u)(t)\big)\operatorname{sgn} u(t) \geq -q(t, \|u\|_C) \ \ \text{for} \ \ t \in [a,b], \tag{2.4}$$

$$(t - t_0)|u''(t) - \ell(u)(t)| \leq q(t, \|u\|_C) \ \ \text{for} \ \ t \in [t_0,b], \tag{2.5}$$

where $q \in K([a,b] \times \mathbb{R}_+; \mathbb{R}_+)$ satisfies (1.5), the estimate

$$\|u\|_C \leq r \cdot \|q(\cdot, \|u\|_C)\|_L \tag{2.6}$$

holds.

Proof. Let ℓ_0, ℓ_1 and $\widetilde{\ell}$ be the operators appearing in Definition 1.1. Let, moreover, $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy the conditions (0.2), (2.4), and (2.5). It is clear that

$$u''(t) = \ell(u)(t) + h(t) \ \ \text{for} \ \ t \in [a,b], \tag{2.7}$$

where

$$h(t) \overset{\text{def}}{=} u''(t) - \ell(u)(t) \ \ \text{for} \ \ t \in [a,b]. \tag{2.8}$$

Moreover, in view of (2.4) and (2.5), we have

$$h(t)\operatorname{sgn} u(t) \geq -q(t, \|u\|_C) \ \ \text{for} \ \ t \in [a,b], \tag{2.9}$$

$$(t - t_0)|h(t)| \leq q(t, \|u\|_C) \ \ \text{for} \ \ t \in [t_0,b]. \tag{2.10}$$

The condition (1.1) implies

$$\begin{aligned} \big|\ell_0(u)(t) - \ell_0(1)(t)u(t)\big| &= \big|\ell_0(\theta(u'))(t) - \ell_0(1)(t)\theta(u')(t)\big| \leq \\ &\leq \big|\widetilde{\ell}(|u'|)(t)\big| \cdot \sqrt{\ell_0(1)(t)} \ \ \text{for} \ \ t \in [a,b]. \end{aligned} \tag{2.11}$$

Multiplying both sides of (2.7) by $u(t)$ and taking into account (2.9) and (2.11), we get

$$
\begin{gathered}
u''(t)u(t)= \\
=\ell_0(1)(t)u^2(t)+\big[\ell_0(u)(t)-\ell_0(1)(t)u(t)\big]u(t)-\ell_1(u)(t)u(t)+h(t)u(t)\geq \\
\geq\ell_0(1)(t)u^2(t)-\big|\widetilde{\ell}(|u'|)(t)\big|\sqrt{\ell_0(1)(t)}\,|u(t)|-\ell_1(u)(t)u(t)-q(t,\|u\|_C)|u(t)|\geq \\
\geq-\frac{1}{4}\big(\widetilde{\ell}(|u'|)(t)\big)^2-\ell_1(u)(t)u(t)-q(t,\|u\|_C)|u(t)| \ \text{ for } \ t\in[a,b].
\end{gathered}
$$

Integration of the last inequality from a to b results in

$$
\begin{aligned}
\|u'\|_{L_2}^2\leq u(b)u'(b)+\frac{1}{4}\,\|\widetilde{\ell}(|u'|)\|_{L_2}^2+ \\
+\int_a^b\ell_1(u)(t)u(t)\,dt+\int_a^b q(t,\|u\|_C)|u(t)|\,dt. \qquad (2.12)
\end{aligned}
$$

On account of Hölder's inequality, we get

$$
|u(t)|=\bigg|\int_a^t u'(s)\,ds\bigg|\leq\varphi(t)\|u'\|_{L_2} \ \text{ for } \ t\in[a,b]. \qquad (2.13)
$$

Hence,

$$
\|u\|_C\leq\sqrt{b-a}\,\|u'\|_{L_2}, \qquad (2.14)
$$

and

$$
|\ell_1(u)(t)\,u(t)|\leq\ell_1(|u|)(t)|u(t)|\leq\varphi(t)\ell_1(\varphi)(t)\|u'\|_{L_2}^2. \qquad (2.15)
$$

Moreover, in view of (0.2) and (2.13), we obtain

$$
|u(b)|=|u(t_0)|\leq\sqrt{t_0-a}\,\|u'\|_{L_2}. \qquad (2.16)
$$

By virtue of (2.14)–(2.16), we get from (2.12) that

$$
\begin{aligned}
\|u'\|_{L_2}^2\leq\Big[\sqrt{t_0-a}\,|u'(b)|+\sqrt{b-a}\,\|q(\cdot,\|u\|_C)\|_L\Big]\|u'\|_{L_2}+ \\
+\bigg(\frac{1}{4}\,\|\widetilde{\ell}\|^2+\int_a^b\varphi(t)\ell_1(\varphi)(t)\,dt\bigg)\|u'\|_{L_2}^2. \qquad (2.17)
\end{aligned}
$$

Now we will estimate $|u'(b)|$. First of all, let us mention that (2.7) and (2.10) imply the inequality

$$
(t-t_0)|u''(t)|\leq(t-t_0)\big(\ell_0(|u|)(t)+\ell_1(|u|)(t)\big)+q(t,\|u\|_C) \ \text{ for } \ t\in[t_0,b],
$$

whence, in view of (2.13), we get

$$
\begin{gathered}
(t-t_0)|u''(t)|\leq \\
\leq(t-t_0)\big(\ell_0(\varphi)(t)+\ell_1(\varphi)(t)\big)\|u'\|_{L_2}+q(t,\|u\|_C) \ \text{ for } \ t\in[t_0,b]. \qquad (2.18)
\end{gathered}
$$

On the other hand, one can easily verify by direct calculations that

$$|u'(b)| = \frac{1}{b-t_0}\left|\int_{t_0}^{b}(t-t_0)u''(t)\,dt\right|.$$

Hence, in view of (2.18), it holds

$$|u'(b)| \le \frac{1}{b-t_0}\int_{t_0}^{b}(t-t_0)\big(\ell_0(\varphi)(t)+\ell_1(\varphi)(t)\big)\,dt\cdot\|u'\|_{L_2}+$$
$$+\frac{1}{(b-t_0)}\|q(\cdot,\|u\|_C)\|_L.$$

Now it follows from (2.17) that

$$r_0\|u'\|_{L_2} \le \Big(\sqrt{b-a}+\frac{\sqrt{t_0-a}}{b-t_0}\Big)\|q(\cdot,\|u\|_C)\|_L, \tag{2.19}$$

where

$$r_0 = 1-\frac{1}{4}\,\|\widetilde{\ell}\|^2-\int_a^b\varphi(t)\ell_1(\varphi)(t)\,dt-\frac{\varphi(t_0)}{b-t_0}\int_{t_0}^{b}(t-t_0)\big(\ell_0(\varphi)(t)+\ell_1(\varphi)(t)\big)\,dt.$$

Note also that, on account of (1.2),

$$r_0 > 0. \tag{2.20}$$

Taking now into account (2.19) and (2.20), we get from (2.14) that (2.6) is fulfilled, where

$$r = \frac{b-a}{r_0}\Big(1+\frac{1}{b-t_0}\Big).$$ □

Proof of Theorem 1.1. To prove Theorem 1.1, it is sufficient to show that the conditions of Lemma 2.1 are fulfilled. First we will show that the homogeneous problem (2.1) has only the trivial solution. Indeed, let u be a solution of this problem. Then, evidently, (2.4) and (2.5) are fulfilled with $q \equiv 0$. Thus, by virtue of Lemma 2.2, $\|u\|_C \le 0$ and, therefore $u \equiv 0$.

Let $r > 0$ be the number appearing in Lemma 2.2. In view of (1.5), there exists $\rho > 0$ such that

$$\|q(\cdot,x)\|_L < \frac{1}{r}\,x \quad \text{for} \quad x > \rho. \tag{2.21}$$

Now let $u \in \widetilde{C}'([a,b];\mathbb{R})$ satisfy (0.2) and (2.2) for some $\delta \in]0,1[$. On account of (1.3) and (1.4), evidently (2.4) and (2.5) hold. Thus, by virtue of Lemma 2.2, we get

$$\|u\|_C \le r\|q(\cdot,\|u\|_C)\|_L.$$

The latter inequality, together with (2.21), yields (2.3). □

Proof of Corollary 1.1. Put

$$\ell_0(v)(t) \stackrel{\text{def}}{=} p(t)v(\tau(t)), \quad \ell_1(v)(t) \stackrel{\text{def}}{=} g(t)v(\mu(t)),$$

and

$$F(v)(t) \stackrel{\text{def}}{=} f(t, v(t), v(\sigma(t))), \quad \widetilde{\ell}(v)(t) \stackrel{\text{def}}{=} \sqrt{p(t)} \int\limits_t^{\tau(t)} v(s)\, ds.$$

Without loss of generality we can assume that the function q is nonincreasing with respect to the second variable. Then it is clear that the conditions (1.8) imply (1.3) and (1.4). On account of Theorem 1.1, it is sufficient to show that the inequalities (1.1) and (1.2) are fulfilled.

By virtue of Hölder's inequality, we have

$$\bigg(\int\limits_t^{\tau(t)} v(s)\, ds \bigg)^2 \leq |\tau(t) - t| \int\limits_a^b v^2(s)\, ds \quad \text{for } t \in [a,b], \quad v \in C([a,b];\mathbb{R}).$$

Hence,

$$\|\widetilde{\ell}(v)\|_{L_2}^2 = \int\limits_a^b p(t) \bigg(\int\limits_t^{\tau(t)} v(s)\, ds \bigg)^2 dt \leq$$
$$\leq \|v\|_{L_2}^2 \int\limits_a^b p(t)|\tau(t) - t|\, dt \quad \text{for } v \in C([a,b];\mathbb{R}).$$

Consequently,

$$\|\widetilde{\ell}\|^2 \leq \int\limits_a^b p(t)|\tau(t) - t|\, dt.$$

The last inequality, together with (1.7), yields (1.2). On the other hand, it is clear that (1.1) holds as well. □

Proof of Corollary 1.2. Put

$$\ell_0(v)(t) \stackrel{\text{def}}{=} \int\limits_a^b [h(t,s)]_+ v(s)\, ds, \quad \ell_1(v)(t) \stackrel{\text{def}}{=} \int\limits_a^b [h(t,s)]_- v(s)\, ds,$$

and

$$F(v)(t) \stackrel{\text{def}}{=} f(t, v(t), v(\sigma(t))), \quad \widetilde{\ell}(v)(t) \stackrel{\text{def}}{=} \sqrt{\int\limits_a^b [h(t,s)]_+ \bigg(\int\limits_t^s v(\xi)\, d\xi \bigg)^2 ds}.$$

Without loss of generality we can assume that the function q is nonincreasing with respect to the second variable. Then it is clear that the conditions (1.8) imply (1.3) and (1.4). On account of Theorem 1.1, it is sufficient to show that the inequalities (1.1) and (1.2) are fulfilled.

By virtue of Hölder's inequality,

$$\left(\int_t^s v(\xi)\,d\xi\right)^2 \le |s-t| \int_a^b v^2(\xi)\,d\xi \quad \text{for} \quad t,s\in[a,b], \quad v\in C([a,b];\mathbb{R}).$$

Hence, it is clear that

$$\|\widetilde{\ell}(v)\|_{L_2}^2 = \int_a^b \left(\int_a^b [h(t,s)]_+ \left(\int_t^s v(\xi)\,d\xi\right)^2 ds\right) dt \le$$

$$\le \|v\|_{L_2}^2 \int_a^b \left(\int_a^b |s-t|\,[h(t,s)]_+\,ds\right) dt \quad \text{for} \quad v\in C([a,b];\mathbb{R}).$$

Consequently,

$$\|\widetilde{\ell}\|^2 \le \int_a^b \left(\int_a^b |s-t|\,[h(t,s)]_+\,ds\right) dt.$$

The last inequality, together with (1.10), yields (1.2). On the other hand, it is clear that (1.1) holds as well. □

Acknowledgement

The research was supported by the Ministry of Education of the Czech Republic under the project MSM0021622409 and by the Academy of Sciences of the Czech Republic, Institutional Research Plan No. AV0Z10190503.

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(Received 17.10.2006)

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