Category-theoretic Fraïssé theory: an overview

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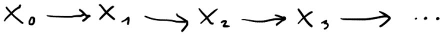
Overview

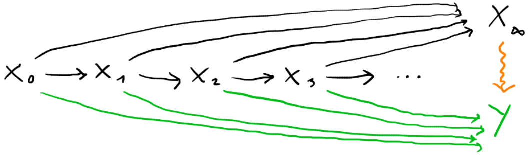
- Fraïssé-theoretic notions in the language of category theory
- “Common core” setup for countable discrete Fraïssé theory
  - synthesis of known results, mostly by Droste–Göbel and Kubiś
- Weak Fraïssé theory
  - KPT correspondence for weak Fraïssé categories (B., Bice, Dasilva Barbosa, Kubiś)
- Approximate Fraïssé theory
  - the pseudo-arc and pseudo-solenoids as metric Fraïssé limits (B., Kubiś)

The language of category theory

- Categories will be denoted by $\mathcal{K}, \mathcal{L}, \mathcal{C}, \ldots$
- Objects will be denoted by $x, y, z, X, Y, Z, \ldots$
- Morphisms will be denoted by $f \colon x \to y$, $g \colon y \to z$, $g \circ f \colon x \to z$, id_x, ...
- We shall often consider a pair $\langle \mathcal{K}, \mathcal{L} \rangle$ where $\mathcal{K} \subseteq \mathcal{L}$ is a subcategory.
- A sequence $\vec{x}$ in a category $\mathcal{K}$ consists of a sequence $\mathcal{K}$-objects $\langle x_n \rangle_{n \in \omega}$ and a coherent sequence of $\mathcal{K}$-maps $\langle x_n^m \colon x_n \to x_m \rangle_{n \leq m \in \omega}$.



- A colimit of the sequence $\vec{x}$ is an object x_∞ together with an initial cone $\vec{x}^\infty = \langle x_n^\infty \colon x_n \to x_\infty \rangle$.



The language of category theory

The main example to keep in mind

Let L be a first-order language. Let $\mathcal{L}$ be the category whose objects are all L-structures and whose morphisms are all embeddings.

- A sequence in $\mathcal{L}$ is without loss of generality an ω-chain

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

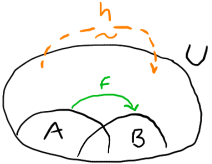
- Its colimit is the union $A_\infty = \bigcup_{n\in\omega} A_n$.

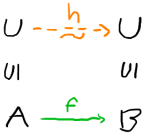
The language of category theory is flexible enough to cover:

- first-order structures and left-invertible embeddings,
- topological first-order structures and quotient maps,
- embedding-projection pairs,
- structures with relations as morphisms,
- a monoid as a category with a single object...

(Ultra)homogeneity

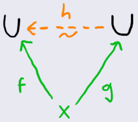
Recall that a countable relational structure U is *ultrahomogeneous* if every isomorphism $f\colon A \to B$ between finite substructures $A, B \subseteq U$ can be extended to an automorphism $h\colon U \to U$.





Definition

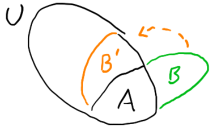
For a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ we say that an $\mathcal{L}$-object U is *homogeneous* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object x and every $\mathcal{L}$-maps $f, g\colon x \to U$ there is an $\mathcal{L}$-automorphism $h\colon U \to U$ such that $h \circ g = f$.

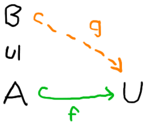


So a structure U is ultrahomogeneous if and only if it is homogeneous in $\langle \mathrm{Age}(U), \mathcal{L} \rangle$.

Extension property / injectivity

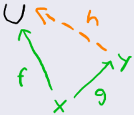
Recall that a countable relational structure U is *injective* or has the *extension property* if for every structures $A \subseteq B \in \mathrm{Age}(U)$ every embedding $f\colon A \to U$ can be extended to an embedding $g\colon B \to U$.





Definition

For a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ we say that an $\mathcal{L}$-object U is *injective* / has the *extension property* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-map $f\colon x \to U$ and $\mathcal{K}$-map $g\colon x \to y$ there is an $\mathcal{L}$-map $h\colon y \to U$ such that $h \circ g = f$.



Universality / cofinality

Recall that a structure U is *universal* for a class of structures $\mathcal{F}$ if every $X \in \mathcal{F}$ can be embedded to U.

Definition

For a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ we say that an $\mathcal{L}$-object U is *cofinal* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object x there is an $\mathcal{L}$-map $f : x \to U$.

What is the Fraïssé limit anyway?

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories, let U be an $\mathcal{L}$-object. We consider the properties:

1. U is homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is injective / has the extension property in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is cofinal in $\langle \mathcal{K}, \mathcal{L} \rangle$.

- Always, if U is cofinal and homogeneous, then U is injective.
- Sometimes U is cofinal homogeneous iff U is cofinal injective.
- Sometimes such U is unique.
- Sometimes such U is cofinal for the whole $\mathcal{L}$.

If it is the case, then it makes sense to call U the *Fraïssé limit*.

Free completion

A pair $\langle \mathcal{K}, \mathcal{L} \rangle$ is called a *free sequential cocompletion* or just a "*free completion*" if $\mathcal{L}$ arises from $\mathcal{K}$ by freely adding colimits of $\mathcal{K}$-sequences.

- We will give a precise definition later.
- Free completion establishes a correspondence

$$\mathcal{K}\text{-sequences} \quad \leftrightarrow \quad \mathcal{L}\text{-objects.}$$

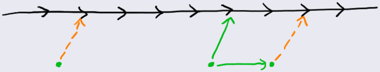
- This is the case in the classical setup when $\mathcal{K}$ is a class of finite structures and $\mathcal{L}$ is the class of their countable unions.

Fraïssé sequence

Definition

A $\mathcal{K}$-sequence $\vec{u}$ is *Fraïssé* if it is

- *cofinal*, i.e. for every $\mathcal{K}$-object x there is a $\mathcal{K}$-map $f: x \to u_n$ for some $n \in \omega$,
- *injective*, i.e. for every $\mathcal{K}$-maps $f: x \to u_n$ and $g: x \to y$ there is a $\mathcal{K}$-map $h: y \to u_m$ for some $m \geq n$ such that $h \circ g = u_n^m \circ f$.



Note that the definition is analogous to the definition of cofinal and injective object in $\langle \mathcal{K}, \mathcal{L} \rangle$.

Characterization of the Fraïssé limit

Theorem

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-colimit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$.

It follows that such U exists if and only if a Fraïssé sequence exists in $\mathcal{K}$.

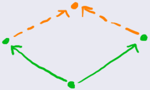
Existence

Theorem

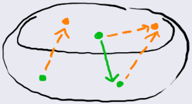
Let $\mathcal{K} \neq \emptyset$ be a category. There is a Fraïssé sequence in $\mathcal{K}$ if and only if

1. $\mathcal{K}$ is *directed* (JEP), i.e. for every $\mathcal{K}$-objects x, y there is a $\mathcal{K}$-object z and $\mathcal{K}$-maps $f\colon x \to z$, $g\colon y \to z$,

2. $\mathcal{K}$ has the *amalgamation property* (AP), i.e. for every $\mathcal{K}$-maps $f\colon x \to y$, $g\colon x \to z$ there are $\mathcal{K}$-maps $f'\colon y \to w$, $g'\colon z \to w$ such that $f' \circ f = g' \circ g$,



3. $\mathcal{K}$ has a countable *dominating subcategory*.



- Often $\mathcal{K}$ has an initial object and AP realized by one-point extensions.
- Adding the extra structure of *origin* and *transitions* leads to the notion of *abstract evolution scheme*, studied by Kubiś and Radecka.

Free completion

Definition

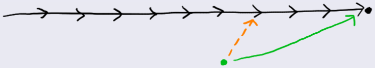
$\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion if

(L1) every $\mathcal{K}$-sequence has an $\mathcal{L}$-colimit,

(L2) every $\mathcal{L}$-object is an $\mathcal{L}$-colimit of a $\mathcal{K}$-sequence,

for every $\mathcal{K}$-sequence $\vec{x}$ and its $\mathcal{L}$-colimit $\langle X_\infty, \vec{x}^\infty \rangle$ we have that

(F1) for every $\mathcal{L}$-map from a $\mathcal{K}$-object $f \colon z \to X_\infty$ there is a $\mathcal{K}$-map $g \colon z \to x_n$ for some n such that $f = x_n^\infty \circ g$,



(F2) for every $\mathcal{K}$-maps $f, g \colon z \to x_n$ such that $x_n^\infty \circ f = x_n^\infty \circ g$ there is $m \geq n$ such that $x_n^m \circ f = x_n^m \circ g$.

- (F2) is trivial if $\mathcal{L}$ consists of monomorphisms.
- Given $\mathcal{K}$, $\mathcal{L}$ always exists and is essentially unique.
- Such $\mathcal{L}$ has all colimits of sequences and has $\mathcal{K}$ as a full subcategory consisting of a rich family of *finitely presentable objects*.

How to get a free completion?

- Let L be a first-order language and let $\mathcal{K}$ and $\mathcal{L}$ be the categories of all finitely and countably generated L-structures, respectively, with all embeddings are morphisms. Then $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion.
- Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion. If $\mathcal{F} \subseteq \mathcal{K}$ is a *full* subcategory and $\sigma\mathcal{F} \subseteq \mathcal{L}$ is the full subcategory of all $\mathcal{L}$-colimits of $\mathcal{F}$-sequences, then $\langle \mathcal{F}, \sigma\mathcal{F} \rangle$ is a free-completion.
- For a fixed $\mathcal{L}$-structure X we may take $\mathcal{F} = \mathrm{Age}(X)$. Then X is cofinal in $\langle \mathrm{Age}(X), \sigma\mathrm{Age}(X) \rangle$, so X is homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$ if and only if it is the Fraïssé limit of its age.

Projective Fraïssé theory

- Let $\mathcal{K}^{\mathsf{op}}$ consists of nonempty finite sets and surjections.
- Then $\mathcal{K}^{\mathsf{op}}$ is essentially countable, directed, and has AP.
- A $\mathcal{K}$-sequence is Fraïssé if and only if every point eventually splits.

Where to take the limit?

- For $\mathcal{L}^{\mathsf{op}}$ being all profinite sets and surjections, $\langle \mathcal{K}, \mathcal{L} \rangle$ is not a free completion and there is no cofinal object with the extension property.
- For $\mathcal{L}^{\mathsf{op}}$ being all profinite spaces (i.e. metrizable compact zero-dimensional) and continuous surjections, $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion, and 2^{ω} is the Fraïssé limit.

Projective Fraïssé theory (Irwin, Solecki)

- For L a relational first-order language, let $\mathcal{L}^{\mathsf{op}}$ be the category of all *topological L-structures* (profinite spaces with a closed interpretation of every relation) and *quotient maps*, and let $\mathcal{K}^{\mathsf{op}}$ be the full subcategory of finite L-structures. Then $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion.

Common core setup

Theorem (characterization of the Fraïssé limit)

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-colimit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$.

Theorem (existence of a Fraïssé sequence)

Let $\mathcal{K} \neq \emptyset$ be a category. $\mathcal{K}$ has a Fraïssé sequence if and only if

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
3. $\mathcal{K}$ has a countable dominating subcategory.

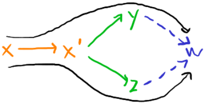
Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals
embeddings	finite graphs	countable graphs	Rado/random graph
embeddings	finite groups	locally finite countable groups	Hall's universal group
embeddings	finite rational metric spaces	countable rational metric spaces	rational Urysohn space
quotients	finite discrete spaces	zero-dimensional metrizable compacta	Cantor space
quotients	finite discrete linear graphs	zero-dimensional metrizable compacta with a special closed symmetric relation	pseudo-arc prespace

Weak Fraïssé theory

...sometimes we just don't have the full amalgamation property, but the theory still works.

- A $\mathcal{K}$-map $e\colon x \to x'$ is called *amalgamable* if for every $\mathcal{K}$-maps $f\colon x' \to y$, $g\colon x' \to z$ there are $\mathcal{K}$-maps $f'\colon y \to w$ and $g'\colon z \to w$ such that $f' \circ f \circ e = g' \circ g \circ e$.



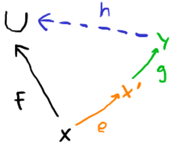
- A $\mathcal{K}$-object x is *amalgamable* if id_x is amalgambable.
- $\mathcal{K}$ has the *cofinal amalgamation property* (CAP) if for every $\mathcal{K}$-object x there is a $\mathcal{K}$-map $e\colon x \to x'$ such that x' is amalgamable.
- $\mathcal{K}$ has the *weak amalgamation property* (WAP) if for every $\mathcal{K}$-object x there is an amalgamable $\mathcal{K}$-map $e\colon x \to x'$.

(WAP) was introduced by Iwanow and later independently by Kechris and Rosendal.

For examples of herereditary classes with (WAP) and not (CAP) see Krawczyk–Kruckman–Kubiś–Panagiotopoulos.

Weak Fraïssé theory

- Throughout the theory we add "guardian arrows", e.g. U is *weakly injective* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-map from a $\mathcal{K}$-object $f\colon x \to U$ there is a $\mathcal{K}$-map $e\colon x \to x'$ such that for every $\mathcal{K}$-map $g\colon x' \to y$ there is an $\mathcal{L}$-map $h\colon y \to U$ such that $h \circ g \circ e = f$.



- Then the theory covers more examples and is more stable under constructions.

Connections with genericity

- The generic automorphism of the Fraïssé limit of $\mathcal{K}$ is exactly the weak Fraïssé limit of the induced category of partial automorphisms $\mathcal{K}_{\mathrm{p}}$ (Kechris–Rosendal).
- The weak Fraïssé limit of $\langle \mathcal{K}, \mathcal{L} \rangle$ can be characterized by existence of the winning strategy in the *abstract Banach–Mazur game* played in $\mathcal{K}$ (Kubiś). Hence *generic limit*.

Weak Fraïssé theory

Theorem (characterization of the weak Fraïssé limit)

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and weakly homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and weakly injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-colimit of a weak Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal for $\mathcal{L}$-colimits of $\mathcal{K}$-sequences of amalgamamable maps, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is weak Fraïssé in $\mathcal{K}$.

Theorem (existence of a weak Fraïssé sequence)

Let $\mathcal{K} \neq \emptyset$ be a category. $\mathcal{K}$ has a weak Fraïssé sequence iff

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the weak amalgamation property,
3. $\mathcal{K}$ has a countable weakly dominating subcategory.

Induced topology and uniform structure

- A free completion $\langle \mathcal{K}, \mathcal{L} \rangle$ induces a uniform structure on every homset $\mathcal{L}(X, Y)$.
- For every $\mathcal{K}$-object z and $\mathcal{L}$-map $u \colon z \to X$ we put

$$f \approx_u g \iff f \circ u = g \circ u$$

  for every $f, g \colon X \to Y$. Clasically this means that the maps agree on a given finite substructure.
- This defines a basis of a complete uniformity metrized by the complete ultrametric

$$d(f, g) < 1/n \iff f \circ x_n^\infty = g \circ x_n^\infty$$

  for any fixed $\mathcal{K}$-sequence $\vec{x}$ with $\mathcal{L}$-colimit $\langle X, \vec{x}^\infty \rangle$.
- This induces the topology of pointwise / uniform convergence in the classical / projective setup.
- $\mathrm{Aut}(X) \subseteq \mathcal{L}(X, X)$ becomes a non-archimedean completely metrizable topological group.
- $\mathcal{L}(X, Y)$ and $\mathrm{Aut}(X)$ are Polish if $\mathcal{K}$ is *locally countable*.

Kechris–Pestov–Todorčević correspondence

- A topological group G is *extremely amenable* if every continuous action $G \curvearrowright X$ on a compact space has a fixed point (i.e. the UMF is trivial).
- A locally finite category $\mathcal{C}$ has the *Ramsey property* if for every $\mathcal{C}$-objects a, b and every $k \in \omega$ there is a $\mathcal{C}$-object c such that for every coloring $\varphi\colon \mathcal{C}(a, c) \to k$ there is a $\mathcal{K}$-map $e\colon b \to c$, such that φ is monochromatic on $e \circ \mathcal{C}(a, b)$.

Theorem (B., Bice, Dasilva Barbosa, Kubiś)

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion, $\mathcal{K}$ a weak Fraïssé category, U the limit. Then the following are equivalent.

1. $\mathrm{Aut}(U)$ is extremely amenable.
2. $\mathcal{K}$ has the weak Ramsey property.

Approximate Fraïssé theory

Consider the category $\mathcal{K}$ of metric compact spaces and continuous maps.

- Every $\mathcal{K}(X, Y)$ can be endowed with the uniform distance

$$d(f, g) = \sup_{x \in X} d_Y(f(x), g(x)).$$

- Let us write $f \approx_\varepsilon g$ if $d(f, g) < \varepsilon$.

1. For every $\mathcal{K}$-map $h\colon Z \to X$ we have

$$d(f \circ h, g \circ h) \le d(f, g).$$

2. For every $\mathcal{K}$-map $h\colon Y \to Z$ and $\varepsilon > 0$ there is $\delta > 0$ such that for every $\mathcal{K}$-object X and every $\mathcal{K}$-maps $f, g\colon X \to Y$ we have

$$h \circ f \approx_\varepsilon h \circ g \quad \text{if} \quad f \approx_\delta g.$$

3. If h is *non-expansive*, then also

$$d(h \circ f, h \circ g) \le d(f, g).$$

Approximate Fraïssé theory

Definition

An *MU-category* is a category $\mathcal{K}$ such that every homset $\mathcal{K}(X, Y)$ is a metric space satisfying the following.

1. For every $h\colon X \to Y$ and $f, g\colon Y \to Z$ we have
$$d(f \circ h, g \circ h) \leq d(f, g).$$
2. For every $h\colon Y \to Z$ and $\varepsilon > 0$ there is $\delta > 0$ such that h is *$\langle \varepsilon, \delta \rangle$-continuous*, i.e. for every $\mathcal{K}$-object X and $\mathcal{K}$-maps $f, g\colon X \to Y$ we have
$$h \circ f \approx_\varepsilon h \circ g \quad \text{if} \quad f \approx_\delta g.$$

$\mathcal{K}$ is called *metric-enriched* if additionally every $h\colon Y \to Z$ is

3. *non-expansive*, i.e. for every $f, g\colon X \to Y$ we have
$$d(h \circ f, h \circ g) \leq d(f, g).$$

Every category $\mathcal{K}$ can be viewed as a *discrete MU-category* when endowed with the 0-1 metric.

Approximate Fraïssé theory

- Now throughout the theory we add epsilons (and we switch to the projective convention for convenience).
- The amalgamation property now means: for every $\mathcal{K}$-maps $f\colon Z \leftarrow X$, $g\colon Z \leftarrow Y$ and every $\varepsilon > 0$ there are $\mathcal{K}$-maps $f'\colon X \leftarrow W$ and $g'\colon Y \leftarrow W$ such that $f \circ f' \approx_\varepsilon g \circ g'$.
- U is homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object X, $\mathcal{L}$-maps $f, g\colon X \leftarrow U$, and $\varepsilon > 0$ there is an automorphism $h\colon U \leftarrow U$ such that $f \approx_\varepsilon g \circ h$.
- For a discrete MU-category, the definitions reduce to the basic ones.
- Our motivation: Irwin and Solecki characterized the *pseudo-arc* by a condition that becomes the actual homogeneity in our setup.

Free MU-completion

Definition

A pair of MU-categories $\langle \mathcal{K}, \mathcal{L} \rangle$ is a *free MU-completion* if

(L1) every $\mathcal{K}$-sequence has an $\mathcal{L}$-limit, and every $\mathcal{L}$-homset is a complete metric space,

(L2) every $\mathcal{L}$-object is an $\mathcal{L}$-limit of a $\mathcal{K}$-sequence,

(F1) for every $\mathcal{K}$-object z, $\mathcal{K}$-sequence $\vec{x}$ with $\mathcal{L}$-limit $\langle X_\infty, \vec{x}_\infty \rangle$, $\mathcal{L}$-map $f \colon z \leftarrow X_\infty$ and $\varepsilon > 0$ there is a $\mathcal{K}$-map $g \colon z \leftarrow z_n$ for some n such that $g \circ x_\infty^n \approx_\varepsilon f$,

(F2) for every $\mathcal{K}$-object z and $\varepsilon > 0$ there is $\delta > 0$ such that for every $\mathcal{K}$-sequence $\vec{x}$ with $\mathcal{L}$-limit $\langle X_\infty, \vec{x}_\infty \rangle$ and $\mathcal{K}$-maps $f, g \colon z \leftarrow x_n$ such that $f \circ x_\infty^n \approx_\delta g \circ x_\infty^n$ there is $m \geq n$ with $f \circ x_m^n \approx_\varepsilon g \circ x_m^n$,

(C) for every $\mathcal{K}$-sequence $\vec{x}$ with $\mathcal{L}$-limit $\langle X_\infty, \vec{x}_\infty \rangle$ and $\varepsilon > 0$ there is $n \in \omega$ and $\delta > 0$ such that for every $\mathcal{L}$-maps $f, g \colon X_\infty \leftarrow Y$ with $x_\infty^n \circ f \approx_\delta x_\infty^n \circ g$ we have $f \approx_\varepsilon g$.

If $\mathcal{K}$ is a discrete MU-category, then $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free MU-completion iff $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion and $\mathcal{L}$ is endowed with the induced uniformity.

Approximate Fraïssé theory

Theorem (characterization of the Fraïssé limit)

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free MU-completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-limit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal and homogeneous in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$.

Theorem (existence of a Fraïssé sequence)

Let $\mathcal{K} \neq \emptyset$ be an MU-category. $\mathcal{K}$ has a Fraïssé sequence if and only if

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
3. $\mathcal{K}$ has a countable dominating subcategory.

Pseudo-arc and pseudo-solenoids

- Let $\mathcal{I}$ / $\mathcal{S}$ be the category of all continuous surjections of the unit interval / unit circle and let $\sigma\mathcal{I}$ / $\sigma\mathcal{S}$ be the category of all *arc-like* / *circle-like* continua and all continuous surjections.
- For a set of primes P let $\mathcal{S}_P \subseteq \mathcal{S}$ be the subcategory of all maps whose *degree* uses only primes from P, and let $\sigma\mathcal{S}_P$ be its *σ-closure* in $\sigma\mathcal{S}$.

Theorem (B., Kubiś)

1. $\langle \mathcal{I}, \sigma\mathcal{I} \rangle$ is a free MU-completion, $\mathcal{I}$ is a Fraïssé MU-category, and the *pseudo-arc* is the Fraïssé limit.
2. $\langle \mathcal{S}_P, \sigma\mathcal{S}_P \rangle$ is a free MU-completion, $\mathcal{S}_P$ is a Fraïssé MU-category, and the *P-adic pseudo-solenoid* is the Fraïssé limit.

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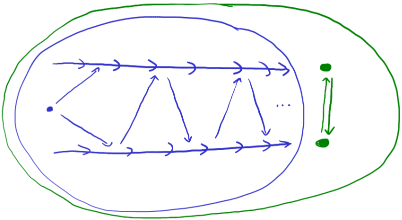

Thank you!