Categorical approaches to projective Fraïssé theory

A. Bartoš

Adam Bartoš (bartos@math.cas.cz)
Institute of Mathematics, Czech Academy of Sciences

Abstract.

Fraïssé theory, understood as the study of ultrahomogeneous structures, is classical in model theory [5]. Recall that a first-order structure U is *ultrahomogeneous* if every isomorphism $f\colon A \to B$ between finitely generated substructures $A, B \subseteq U$ can be extended to an automorphism $\tilde{f}\colon U \to U$. Most classical countable ultrahomogeneous structures include the *linear order of rationals*, the *random graph*, and the *rational Urysohn metric space.*

It is natural to formulate Fraïssé theory in the language of category theory. This allows for clear and general definitions and proofs capturing the essence of the constructions involved. For example, given a pair of categories $\mathcal{K} \subseteq \mathcal{L}$, we call an $\mathcal{L}$-object U *homogeneous* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every pair of $\mathcal{L}$-maps from a $\mathcal{K}$-object $f, g\colon x \to U$ there is an automorphism $h\colon U \to U$ such that $h \circ g = f$. Based on work of Droste and Göbel [4] and Kubiś [7], the core of Fraïssé theory can be summarized in the following two theorems. We say that $\langle \mathcal{K}, \mathcal{L} \rangle$ is a *free completion* (or more precisely, a free sequential cocompletion) if $\mathcal{L}$ essentially arises from $\mathcal{K}$ by freely adding colimits of $\mathcal{K}$-sequences.

Theorem (Characterization of the Fraïssé limit)**.** Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

(1) U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(2) U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,

(3) U is the $\mathcal{L}$-colimit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$. Such U is called the *Fraïssé limit* of $\mathcal{K}$ in $\mathcal{L}$.

Theorem (Existence of a Fraïssé sequence)**.** Let $\mathcal{K} \neq \emptyset$ be a category. Then $\mathcal{K}$ has a Fraïssé sequence if and only if

(1) $\mathcal{K}$ is directed,

(2) $\mathcal{K}$ has the amalgamation property,

(3) $\mathcal{K}$ has a countable dominating subcategory.

Such $\mathcal{K}$ is called a *Fraïssé category.*

In 2006, Irwin and Solecki [6] introduced *projective Fraïssé theory*, where instead of embeddings of first-order structures, quotients of topological graphs are considered. The (projectively) homogeneous structure is obtained as a limit of an inverse sequence of quotient maps, instead of taking the union of

an increasing chain. The particular limit obtained by Irwin and Solecki was the Cantor space endowed with a special closed equivalence relation with the quotient space being the *pseudo-arc*, a well-known continuum. Projective Fraïssé theory fits the categorical framework presented above (with taking opposite categories everywhere), but the Fraïssé limit obtained is a pre-space (profinite space with a closed equivalence relation), not the actual space (the induced quotient).

Recently, we have considered alternative approaches to projective Fraïssé theory. With W. Kubiś we have developed Fraïssé theory of *MU-categories* [3], a specific generalization of metric-enriched categories tailored to work with categories of metrizable compacta and continuous surjections. Here the key notions are approximate in the sense that defining diagrams are not commuting exactly, but up to ε for arbitrary $\varepsilon > 0$. In this setup we have obtained the pseudo-arc as well as P-adic pseudo-solenoids as Fraïssé limits directly. Here the category of small objects $\mathcal{K}$ consists of all continuous surjections of the unit interval and all continous surjections of the unit circle whose degree uses primes only from P, respectively, so the small objects are not finite and discrete any more.

Another approach, which is a joint work in progress with T. Bice and A. Vignati [1, 2], we start with a category $\mathcal{K}$ of finite graphs, as in the classical projective Fraïssé theory, but we allow relations instead of functions as morphisms, and instead of taking the limit of an inverse sequence, we turn the sequence into a *graded ω-poset* and take its *spectrum*. This way we can construct compact spaces directly from finite graphs, but categorical limit is replaced by an ad hoc construction. Recently we found out that the spectrum can be to some extent viewed as a limit, by utilizing the notion of *lax-adjoint limits* [8] in poset-enriched categories.

In the talk I will give an overview of the three approaches and discuss new ideas about viewing the spectrum as a *unital* lax-adjoint limit.

References

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