Categorical approaches to projective Fraïssé theory

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TL;DR

- **Fraïssé theory** (= study of homogeneous structures) has a categorical formulation.
- There are several approaches to construction of compact spaces as projective Fraïssé limits.
- We would like to combine their strengths. In particular, we would like to build compacta **directly** as generalized **categorical limits** of **finite graphs** and relational morphisms.

	finite small objects	categorical limits	no quotient needed
classical projective FT	✓	✓	✗
approximate projective FT	✗	✓	✓
spectra of ω-posets	✓	✗	✓

Abstract framework for Fraïssé theory

(based on work of Droste and Göbel [4] and Kubiś [8])

Fraïssé theory

- Intuition: we are amalgamating finite building blocks to obtain an infinite generic structure.
- Originally in model theory [5] [6]: embeddings of finite structures leading to a countable ultrahomogeneous structure (every isomorphism between finite substructures extends to an automorphism, e.g. the rational order, the random graph).
- Abstractly we work with a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ representing "small" and "large" objects.
- If $\mathcal{K}$ is nice, it is possible to build a generic ω-sequence that "absorbs" all possible behavior in $\mathcal{K}$. Genericity of the sequence is then transferred to its $\mathcal{L}$-limit, which is a canonical object. Back and forth argument assures uniqueness and homogeneity.
- We use the projective convention, i.e. the domain of a morphism represents a more complicated object, ω-sequences grow backwards, we take (inverse) limits.

Characterization of the Fraïssé limit

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and **homogeneous** in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal with the **extension property** in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-limit of a **Fraïssé sequence** in $\mathcal{K}$.

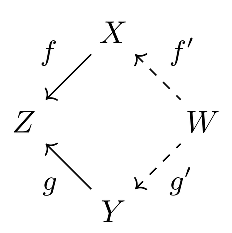
Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-limit U is Fraïssé in $\mathcal{K}$. Such U is called the **Fraïssé limit** of $\mathcal{K}$ in $\mathcal{L}$.

Existence of a Fraïssé sequence

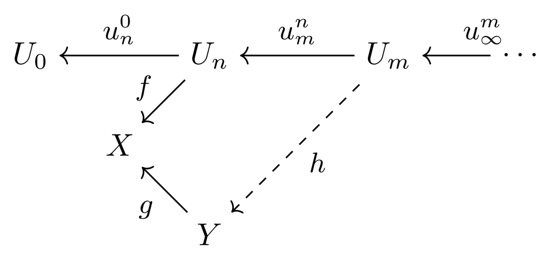
Let $\mathcal{K} \neq \emptyset$ be a category. Then $\mathcal{K}$ has a Fraïssé sequence if and only if

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the **amalgamation property**,
3. $\mathcal{K}$ has a countable dominating subcategory.

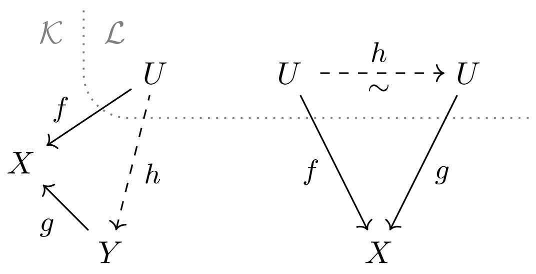
Such $\mathcal{K}$ is called a **Fraïssé category**.



amalgamation property



extension property



homogeneity

Free completion

- $\langle \mathcal{K}, \mathcal{L} \rangle$ is a **free completion** if $\mathcal{L} \supseteq \mathcal{K}$ arose from $\mathcal{K}$ by freely adding limits of sequences, i.e. we have
  - (L1) every $\mathcal{K}$-sequence has an $\mathcal{L}$-limit,
  - (L2) every $\mathcal{L}$-object is an $\mathcal{L}$-limit of a $\mathcal{K}$-sequence,

  for every $\mathcal{K}$-sequence $\langle X_*, u_* \rangle$ and its $\mathcal{L}$-limit $\langle X_\infty, u_\infty^* \rangle$
  - (F1) for every $\mathcal{L}$-map to a $\mathcal{K}$-object $f\colon Z \leftarrow X_\infty$ there is a $\mathcal{K}$-map $f'\colon Z \leftarrow X_n$ for some n with $f' \circ u_\infty^n = f$,
  - (F2) for every $\mathcal{K}$-maps $f, g\colon Z \leftarrow X_n$ with $f \circ u_\infty^n = g \circ u_\infty^n$ there is $m \geq n$ such that $f \circ u_m^n = g \circ u_m^n$.
- For every $\mathcal{F} \subseteq \mathcal{K}$ we define its σ**-closure** $\sigma\mathcal{F} \subseteq \mathcal{L}$ as the closure under limits of sequences.
- $\mathcal{F} \subseteq \mathcal{K}$ is **iso-consistent** if every isomorphism between $\mathcal{L}$-limits of $\mathcal{F}$-sequences is witnessed by a back and forth sequence in $\mathcal{F}$.
- If $\mathcal{F} \subseteq \mathcal{K}$ is iso-consistent, then $\langle \mathcal{F}, \sigma\mathcal{F} \rangle$ is a free completion. If $\mathcal{F} \subseteq \mathcal{K}$ is full, then it is iso-consistent, and $\sigma\mathcal{F} \subseteq \mathcal{L}$ is full.

Classical projective Fraïssé theory

(introduced by Irwin and Solecki [7])

Topological L-structures

- For a relational language L let $\mathcal{L}$ denote the category of all **topological L-structures**, i.e. L-structures endowed with a metrizable compact zero-dimensional topology making the relations closed. Morphisms are quotient maps (both L-quotient and topologically quotient).
- Let $\mathcal{K} \subseteq \mathcal{L}$ be the full subcategory of all finite (= discrete) structures. Then $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion, and so the abstract Fraïssé theory applies.
- If L has a distinguished binary relation R, and $\mathcal{F} \subseteq \mathcal{K}$ is a subcategory such that R is reflexive and symmetric on $\mathcal{F}$-objects, then R is reflexive and symmetric also on $\sigma\mathcal{F}$-objects. Example: $L = \{R\}$ and $\mathcal{F}$ consisting of all finite graphs.
- A $\sigma\mathcal{F}$-object X is a **prespace** if R is also transitive on X, i.e. if it is an equivalence relation. Then the **topological realization** of X is the quotient space X/R, which is a metrizable compactum.

Applications

- Let $\mathcal{I}_\Delta$ be the category of all finite linear graphs. Then $\mathcal{I}_\Delta$ is a Fraïssé category, and the Fraïssé limit of $\langle \mathcal{I}_\Delta, \sigma\mathcal{I}_\Delta \rangle$ is a prespace of the **pseudo-arc**, the unique hereditarily indecomposable arc-like continuum.
- This way Irwin and Solecki obtained a new characterization of the pseudo-arc: it is the unique arc-like continuum $\mathbb{P}$ that is **approximately projectively homogeneous**, i.e. for every continuous surjections $f, g\colon \mathbb{P} \to Y$ onto an arc-like continuum Y and for every $\varepsilon > 0$ there is a homeomorphism $h\colon \mathbb{P} \to \mathbb{P}$ such that $f \approx_\varepsilon g \circ h$.
- More spaces were obtained as topological realizations of Fraïssé limits:

$\mathcal{F}$-objects	$\mathcal{F}$-maps	Fraïssé limit quotient
discrete	all	the Cantor space
linear	all	the pseudo-arc
ordered trees	all	the Lelek fan
trees of degree ≤ 3	monotone	the Ważewski dendrite D_3
connected	monotone	the Menger curve
connected	confluent	a new continuum

Spectra as unital lax adjoint limits

ULA limits

- Categories like Rel do not have limits of sequences.
- In a Pos-enriched category, lax notions are defined by relaxing strict commutativity "$f \circ g = h$" to "$f \circ g \leq h$".
- A **unital lax adjoint limit** of $\langle X_*, f_* \rangle$ is a lax cone $\langle X_\infty, f_\infty^* \rangle$ such that every lax cone $\langle Y, g^* \rangle$ admits the largest map $h\colon X_\infty \leftarrow Y$ with $f_\infty^* \circ h \leq g^*$ (see [9]) and such that id_{X_∞} is the only lax-factorizing map for $\langle X_\infty, f_\infty^* \rangle$.

Results

- For an ω-poset $\mathbb{P}$ corresponding to a lax sequence $\langle P_*, \sqsupset_* \rangle$ in FinRel, we may put $P_\infty = \mathsf{S}\mathbb{P}$ and $p \sqsupset_\infty^n S$ iff $p \in S$. Then $\langle P_\infty, \sqsupset_\infty^* \rangle$ is a ULA limit of $\sqsupset_*$ in Rel.
- The topology of $\mathsf{S}\mathbb{P}$ is the initial LSC topology (making every $U^{\sqsupset_\infty^n}$ open), however it does not make $\mathsf{S}\mathbb{P}$ an ULA limit in $\mathrm{Rel}_{\mathrm{LSC}}$.

Question

Is there a suitable category of topological graphs having metrizable compact spaces and continuous maps as a full subcategory such that spectra are ULA limits of finite graphs? Would it behave like a free completion?

Approximate Fraïssé theory of MU-categories

(joint work with Wiesław Kubiś [3])

MU-categories

- MU-categories generalize metric-enriched categories and abstract from the category of metric spaces and uniformly continuous maps.
- An **MU-category** is a category $\mathcal{K}$ endowed with distance maps $d\colon \mathcal{K}(X, Y)^2 \to [0, \infty]$ such that for every $\mathcal{K}$-map $f\colon X \to Y$ we have
  1. $d(g \circ f, h \circ f) \leq d(g, h)$ for every $\mathcal{K}$-maps $g, h\colon Y \to Z$,
  2. for every $\varepsilon > 0$ there is $\delta > 0$ such that f is $\langle \varepsilon, \delta \rangle$**-continuous**, i.e. for every $\mathcal{K}$-maps $g, h\colon W \to X$ such that $g \approx_\delta h$ we have $f \circ g \approx_\varepsilon f \circ h$.
- Every category can be viewed as a discrete MU-category.
- $\mathrm{Met}_{\mathrm{u}}$, the category of metric spaces and uniformly continuous maps with the supremum distance, is an MU-category.
- MCpt, the category of metrizable compacta and continuous maps, can be viewed as an MU-category in an essentially unique way.

Approximate Fraïssé theory

- The abstract framework for Fraïssé theory can be extended to MU-categories.
- There is an appropriate notion of the free completion and of the σ-closure for MU-categories. The free MU-completion of a discrete category is the discrete free completion with the induced MU-structure.
- Fraïssé-theoretic notions also have their analogues that reduce to the classical ones in the discrete case. Namely, homogeneity is the approximate projective homogeneity of Irwin and Solecki, and the amalgamation property means "for every f, g and $\varepsilon > 0$ there are f', g' with $f \circ f' \approx_\varepsilon g \circ g'$.
- The analogues of the characterization of the Fraïssé limit theorem and of the existence of a Fraïssé sequence theorem hold. The Fraïssé limit is homogeneous also with respect to large objects.

Applications

- Consider MU-categories of metrizable compacta and continuous surjections. For every class $\mathcal{P}$ of connected polyhedra, with all continuous surjections as morphisms, $\langle \mathcal{P}, \sigma\mathcal{P} \rangle$ is a free MU-completion, and $\sigma\mathcal{P} \subseteq \mathrm{MCpt}_{\mathrm{s}}$ is full.
- In particular, for $\mathcal{I} = \{[0, 1]\}$, $\sigma\mathcal{I}$ consists of all arc-like continua and the Fraïssé limit of $\langle \mathcal{I}, \sigma\mathcal{I} \rangle$ is the pseudo-arc.
- For $\mathcal{S} = \{\text{circle}\}$, $\sigma\mathcal{S}$ consists of all circle-like continua, but there is no Fraïssé limit since $\mathcal{S}$ does not have the amalgamation property.
- For every set of primes P, the wide subcategory $\mathcal{S}_P \subseteq \mathcal{S}$ consisting of maps whose degree uses only primes from P is iso-consistent and Fraïssé. The Fraïssé limit of $\langle \mathcal{S}_P, \sigma\mathcal{S}_P \rangle$ is the P-**adic pseudo-solenoid**.

Spectra of ω-posets from sequences of graphs

(joint work in progress with Tristan Bice and Alessandro Vignati [1] [2])

ω-Posets

- An ω**-poset** $\mathbb{P}$ is a poset consisting of countably many finite levels.
- More precisely, let $\langle P_*, \sqsupset_* \rangle$ be a lax sequence in FinRel, i.e. $\sqsupset_n^m\colon P_m \leftarrow P_n$ are total relations between finite sets such that $\sqsupset_m^l \circ \sqsupset_n^m \subseteq \sqsupset_m^l$ for every $l \leq m \leq n \in \omega$. Then by putting $\mathbb{P} = \bigsqcup_n P_n$ and $\geq = \bigsqcup_{m \leq n} \sqsupset_n^m$ we obtain an ω-poset, and every ω-poset is obtained this way.
- An ω-poset is **graded** if it comes from a sequence, i.e. $\sqsupset_m^l \circ \sqsupset_n^m = \sqsupset_m^l$.

Spectra

- An ω-poset $\mathbb{P}$ is meant to represent a basis of a compact space, with the levels P_n corresponding to basic open covers.
- A **selector** is an upset $S \subseteq \mathbb{P}$ intersecting every level P_n.
- The **spectrum** $\mathsf{S}\mathbb{P}$ is the set of all minimal selectors endowed with the topology generated by the basic open sets $p^\in = \{S \in \mathsf{S}\mathbb{P} : p \in S\}$ for $p \in P_n$ and $n \in \omega$. It is always a second-countable T_1 compactum.

Graph categories

- Let $\mathcal{G}$ denote the category of nonempty finite graphs $\langle G, \sqcap \rangle$ and quotient-like relational morphisms $\sqsupset\colon H \leftarrow G$ that are
  - **co-surjective/total**: $\forall g \in G\ \exists h \in H\ g \sqsubset h$,
  - **edge-preserving**: $\forall g \sqsubset h\ \forall g' \sqsubset h'\ g \sqcap g' \Rightarrow h \sqcap h'$,
  - **co-injective**: $\forall h \in H\ \exists g \in G\ g^{\sqsubset} = \{h\}$,
  - **edge-surjective**: $\forall h \sqcap h'\ \exists g \sqsubset h\ \exists g' \sqsubset h'\ g \sqcap g'$.
- We often consider the following properties forming ideals:
  - **edge-witnessing**: $\forall h \sqcap h' \in H\ \exists g \in G\ g \sqsubset h, h'$,
  - **star-refining**: $\forall g \in G\ \exists h \in H\ g^{\sqcap} \sqsubset h$.
- A sequence $\langle G_*, \sqsupset_* \rangle$ in $\mathcal{G}$ is **lax-Fraïssé** if it is cofinal and lax-absorbing, i.e. for every $\sqsupset\colon G_m \leftarrow H$ there is $\sqsupseteq\colon H \leftarrow G_n$ such that $\sqsupset \circ \sqsupseteq \subseteq \sqsupset$.

Results

- Every nonempty second-countable T_1 compactum X is the spectrum of an edge-witnessing $\mathcal{G}$-sequence.
- The spectrum of a $\mathcal{G}$-sequence $\sqsupset_*$ is Hausdorff/metrizable if and only if $\sqsupset_*$ has a star-refining subsequence.
- The spectrum construction yields a full essentially wide functor from a certain category of ω-posets to the category of metrizable compacta and continuous maps.
- For $\mathcal{F} \subseteq \mathcal{G}$ with wide ideals of edge-witnessing and star-refining morphisms, all lax-Fraïssé sequences have homeomorphic spectra.
- We have shown that the following categories are Fraïssé and we have characterized their (lax-)Fraïssé sequences and limits:

$\mathcal{F}$-objects	$\mathcal{F}$-maps	Fraïssé spectrum
discrete	all	the Cantor space
linear	all	the pseudo-arc
linear	monotone	the arc
fans	spokewise monotone end-preserving	the Cantor fan
fans	spokewise monotone	the Lelek fan

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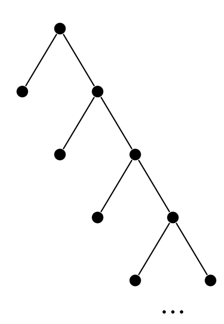
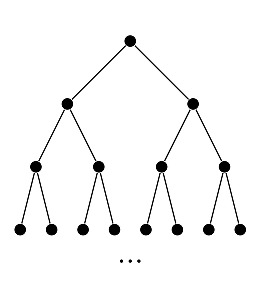
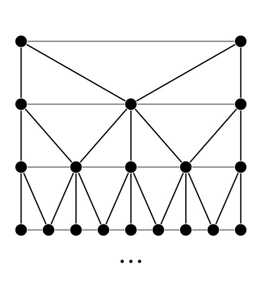


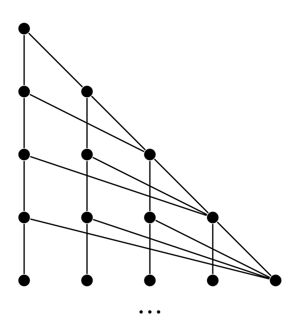
convergent sequence



Cantor space 2^ω



unit interval $[0, 1]$



cofinite topology on ω