Projective Fraïssé theory and generalized limit constructions

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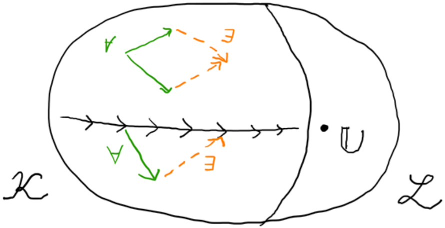
Fraïssé theory

Fraïssé theory: constructing and studying "large" symmetric structures through their "small" approximations.

- Classically, large means countable and small means finite.
- In projective Fraïssé theory, large means compact metrizable and small means finite or "simple" such as an arc or a circle.
- "Symmetric", i.e. (ultra)homogeneous means that all copies of a fixed small substructure / quotient can be moved to each other by an automorphism.
- Allows to study the automorphism group (e.g. the homeomorphism group of a continuum).

Fraïssé theory – the setup

- A category $\mathcal{K}$ of small structures is "freely completed" to a category $\mathcal{L}$ of large structures, which are limits of increasing sequences $\mathcal{K}$-objects.
- A (unique) universal/cofinal homogeneous $\mathcal{L}$-object $\mathbb{U}$ corresponds to a (unique) Fraïssé sequence in $\mathcal{K}$, which is constructed by generically amalgamating small structures.



- One can start with $\mathcal{K}$ and "compute" $\mathcal{L}$ and $\mathbb{U}$, or one can start with U and find a suitable pair $(\mathcal{K}, \mathcal{L})$.

Fraïssé theory – core theorems

Theorem (characterization of the Fraïssé limit)

Let $(\mathcal{K}, \mathcal{L})$ be a free completion and let $\mathbb{U}$ be an $\mathcal{L}$-object. Then the following are equivalent.

1. $\mathbb{U}$ is cofinal and homogeneous in $(\mathcal{K}, \mathcal{L})$,
2. $\mathbb{U}$ is cofinal and has the extension property in $(\mathcal{K}, \mathcal{L})$,
3. $\mathbb{U}$ is the $\mathcal{L}$-limit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such $\mathbb{U}$ is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-limit $\mathbb{U}$ is Fraïssé in $\mathcal{K}$.

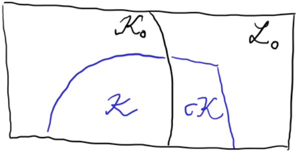
Theorem (existence of a Fraïssé sequence)

Let $\mathcal{K} \neq \emptyset$ be a category. $\mathcal{K}$ has a Fraïssé sequence if and only if

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
3. $\mathcal{K}$ has a countable dominating subcategory.

Fraïssé theory – the setup

- Often, we start with an ambient pair $(\mathcal{K}_0, \mathcal{L}_0)$ and we put $\mathcal{L} = \sigma\mathcal{K}$ for a suitable $\mathcal{K} \subseteq \mathcal{K}_0$, where $\sigma\mathcal{K}$ denotes the closure of $\mathcal{K}$ under $\mathcal{L}_0$-limits of countable sequences from $\mathcal{K}$.



- In "ordinary" (injective) Fraïssé theory, $\mathcal{L}_0$ consists of all countably generated structures in a fixed first-order language (e.g. one binary relation), with all embeddings, and $\mathcal{K}_0$ consists of the finitely generated ones.
- Example: $\mathcal{K} \subseteq \mathcal{K}_0$ consisting of all finite linear orders or finite graphs, leading to $\mathbb{U}$ being $(\mathbb{Q}, \leq)$ or the random graph.
- If $\mathcal{K} \subseteq \mathcal{K}_0$ restricts also the allowed morphisms, some conditions need to be checked.

Projective Fraïssé theory

- In injective Fraïssé theory, structures are growing forwards via embeddings.

$$X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots \quad X_\infty = \bigcup_n^{\uparrow} X_n$$

- In projective Fraïssé theory, arrows are reversed, structures are growing backwards via quotient mappings, and we take (variants of) inverse limits.

$$\begin{tikzcd}
 & & & & X_\infty \arrow[llll, bend right, "f^0_\infty"'] \arrow[lll, bend right, "f^1_\infty"'] \arrow[ll, bend right, "f^2_\infty"'] \\
X_0 & X_1 \arrow[l, "f_0"] & X_2 \arrow[l, "f_1"] & \dots \arrow[l, "f_2"] &
\end{tikzcd}$$

- A sequence (X_n, f^m_n) consists of structures X_n and bonding maps $f^m_n \colon X_m \leftarrow X_n$ for $m \leq n$, with $f^i_j \circ f^j_k = f^i_k$ for $i \leq j \leq k$.
- The limit is the space of threads

$$X_\infty = \{(x_n) \in \prod_n X_n : x_m = f^m_n(x_n) \text{ for } m \leq n\}$$

with the projections $f^n_\infty \colon X_n \leftarrow X_\infty$.

Projective Fraïssé theory – three approaches

- Projective Fraïssé theory allows us to study metrizable compacta and their homeomorphism groups.
- Besides the "classical" projective Fraïssé theory (Irwin–Solecki, 2006), we have developed two complementary approaches: "approximate" (Bartoš–Kubiś, 2022) and "spectral" (Bartoš–Bice–Vignati, 2023).

	small objects are finite	proper limits	no quotient is needed
classical	✓	✓	✗
approximate	✗	✓	✓
spectral	✓	✗	✓

Classical projective Fraïssé theory – the setup

- $\mathcal{L}_0$ consists of all topological graphs $(G, \sqcap)$: zero-dimensional metrizable compact spaces with a reflexive symmetric closed binary edge relation, and possibly with more structure.
- $\mathcal{L}_0$-maps are "epimorphisms" / quotient maps: surjective continuous homomorphisms such that every edge is an image of an edge.
- $\mathcal{K}_0$ consists of all finite graphs (with the discrete topology).
- The inverse limit G_∞ of a $\mathcal{K}_0$-sequence (G_n, f_n) comes with the limit topology and the limit edge relation $\sqcap_\infty$.
- If $\sqcap_\infty$ is transitive, we call G_∞ a prespace, and the quotient $G_\infty/\sqcap_\infty$ is a metrizable compactum.
- The quotient also induces a continuous homomorphism $\mathrm{Aut}(G_\infty) \to \mathrm{Homeo}(G_\infty/\sqcap_\infty)$.
- A continuum X can be studied by realizing it as $G_\infty/\sqcap_\infty$ for G_∞ the Fraïssé limit of a suitable category $\mathcal{K} \subseteq \mathcal{K}_0$.

Classical projective Fraïssé theory – some results

- Irwin–Solecki (2006): Introduced projective Fraïssé theory; $\mathcal{K} = \{\text{paths}\}$, leading to the (prespace of) the pseudo-arc and a new characterization of the pseudo-arc.
- Bartošová–Kwiatkowska (2015): $\mathcal{K} = \{$fan graphs + spoke-monotone epimorphisms$\}$ leading to the Lelek fan.
- Panagiotopoulos (2016): Every metrizable compactum K can be realized with $\mathrm{Aut}(G_\infty) \to \mathrm{Homeo}(K)$ dense embedding if dual predicates are allowed.
- Panagiotopulos–Solecki (2018): $\mathcal{K} = \{$connected graphs + monotone epimorphisms$\}$ leading to the Menger curve.
- Basso–Camerlo (2021): $\mathcal{K} = \{$finite sums of paths and singletons + weakly component-surjective monotone epimorphisms$\}$ leading to the so-called Fraïssé fence.
- Charatonik–Kwiatkowska–Roe (2025): $\mathcal{K} = \{$connected graphs + confluent epimorphisms$\}$ leading to a new generic continuum.

Approximate projective Fraïssé theory

- $\mathcal{L}_0$ consists of all metrizable continua and continuous surjections; $\mathcal{K}_0$ consists of all connected polyhedra.
- The continuum of interest is obtained directly as an inverse limit, avoiding the quotient.
- The framework is formally developed for MU-categories, where every abstract homset $\mathcal{L}(X, Y)$ is a metric space.
- For $\mathcal{K} = \{\text{arc}\}$ we get $\sigma\mathcal{K} = \{\text{arc-like continua}\}$ and $\mathbb{U} =$ the pseudo-arc.
- For a set of primes P and $\mathcal{K} = \{\text{circle + maps of allowed degree}\}$ we get $\sigma\mathcal{K} = \{\text{circle-like continua of type} \leq P^\infty + \text{maps of allowed degree}\}$ and $\mathbb{U} =$ the P-adic pseudo-solenoid.
- Cantier–Vilalta (2024): approximate Fraïssé theory for Cuntz semigroups in the study of C^*-algebras.

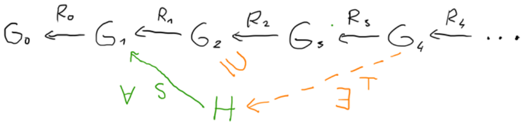
Spectral projective Fraïssé theory – reconstruction

- Let X be a metrizable compactum and let $(\mathcal{C}_n)$ be a fundamental sequence of covers:
  - every $\mathcal{C}_n$ is a finite open cover of X,
  - $\mathcal{C}_n$ refines $\mathcal{C}_m$ for $n \geq m$,
  - every open cover of X is refined by some $\mathcal{C}_n$, or equivalently $\max\{\mathrm{diam}(U) : U \in \mathcal{C}_n\} \to 0$.
- Let us view every $\mathcal{C}_n$ as a graph by putting $U \sqcap V$ if $U \cap V \neq \emptyset$.
- We consider the refinement relations $V \sqsupseteq_n^m U$ for $m \leq n$ defined by $\mathcal{C}_m \ni V \supseteq U \in \mathcal{C}_n$ and view them as multivalued maps $\sqsupseteq_n^m : \mathcal{C}_m \leftarrow \mathcal{C}_n$.
- Then X can be reconstructed from the system $(\mathcal{C}_n, \sqsupseteq_n^m)$.
- More precisely, for every $x \in X$ and n let $\Phi_n(x) := \{U \in \mathcal{C}_n : x \in C\}$ and $\Phi(x) := (\Phi_n(x))$, then $\Phi \colon X \to \mathrm{Spec}(\mathcal{C}_n, \sqsupseteq_n^m)$ is a homeomorphism.

Spectral projective Fraïssé theory – the spectrum

- We let $\mathcal{K}_0$ be the category of all finite graphs as in the classical approach, but as morphisms we allow relations $R\colon G \to H$ that are
  - co-surjective: every $R(g)$ is nonempty,
  - co-injective: for every $h \in H$ there is $g \in G$ with $R(g) = \{h\}$,
  - edge-preserving: for every $g \sqcap g'$ in G, and $h \in R(g)$, $h' \in R(g')$ we have $h \sqcap h'$,
  - edge-surjective: for every $h \sqcap h'$ in H there are $g \in R^{-1}(h)$ and $g' \in R^{-1}(h')$ with $g \sqcap g'$.
- Let (G_n, R_n^m) be a lax-sequence in $\mathcal{K}_0$, i.e. $R_j^i \circ R_k^j \subseteq R_k^i$ for $i \leq j \leq k$.
- A selector for (G_n, R_n^m) is a sequence $S = (S_n)$ with $\emptyset \neq S_n \subseteq G_n$ and $S_m \supseteq R_n^m[S_n]$ for every $m \leq n$.
- We define the spectrum $\operatorname{Spec}(G_n, R_n^m)$ as the set $\{S : S \text{ is a } \subseteq\text{-minimal selector}\}$ with the projections $R_\infty^n(S) := S_n$ and the topology generated by the sets $[g]_n = (R_\infty^n)^{-1}(g) = \{S : g \in S_n\}$.

- The spectrum is always a second-countable T_1 compactum.
- The spectrum of an edge-faithful lax-sequence is T_2 if and only if it is metrizable if and only if the lax-sequence has a star-refining sub-lax-sequence.
- $(\{[g]_n : g \in G_n\})$ is a fundamental sequence of covers for the spectrum.
- For R_n^m being functions, the spectrum reduces to the inverse limit, which is then zero-dimensional.
- A Fraïssé category $\mathcal{K} \subseteq \mathcal{K}_0$ has Fraïssé sequences, but typically also lax-Fraïssé sequences that are not Fraïssé.



Spectral projective Fraïssé theory – some results

- Bartoš–Bice–Vignati (2024): examples of Fraïssé $\mathcal{K} \subseteq \mathcal{K}_0$, their spectral Fraïssé limits, and characterizations of (lax-)Fraïssé sequences:
  - $\mathcal{K} = \{$paths + monotone morphisms$\}$ gives the arc,
  - $\mathcal{K} = \{$fan graphs + spokewise monotone end-preserving morphisms$\}$ gives the Cantor fan,
  - $\mathcal{K} = \{$fan graphs + spokewise monotone morphisms$\}$ gives the Lelek fan.
- Bice–Malicki (2024): characterization of existence of a dense / comeager conjugacy class of a homeomorphism group using the spectral approach; the pseudo-arc has a dense conjugacy class.
- Poór–Solecki (2025): the diagonal conjugacy action $\mathrm{Homeo}(\mathbb{P}) \curvearrowright \mathrm{Homeo}(\mathbb{P})^{\mathbb{N}}$ has a dense conjugacy class, using the classical approach.

- I believe the three approaches can be unified in a common framework.
- This is a part of an ongoing project...

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classical	✓	✓	✗
approximate	✗	✓	✓
spectral	✓	✗	✓

Thank you!

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