Category-theoretic Fraïssé theory: an overview

Adam Bartoš
bartos@math.cas.cz

Institute of Mathematics, Czech Academy of Sciences

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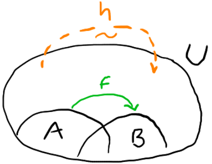
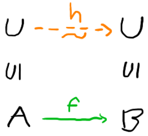
Abstract Convergence Schemes And Their Complexities

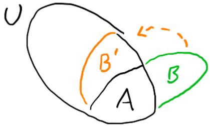
(Ultra)homogeneity and injectivity

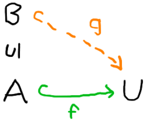
Recall that a countable relational structure U is *ultrahomogeneous* if every isomorphism $f\colon A \to B$ between finite substructures $A, B \subseteq U$ can be extended to an automorphism $h\colon U \to U$.





Recall that a countable relational structure U is *injective* or has the *extension property* if for every structures $A \subseteq B \in \mathrm{Age}(U)$ every embedding $f\colon A \to U$ can be extended to an embedding $g\colon B \to U$.





Classical Fraïssé theory

Theorem (Fraïssé)

Let L be a relational language.

1. For every countable homogeneous L-structure U, $\mathrm{Age}(U)$ is a hereditary, essentially countable class of finite L-structures satisfying JEP and AP – a *Fraïssé class*.
2. For every Fraïssé class $\mathcal{F}$ there is a unique (up to iso) countable homogeneous L-structure U with $\mathrm{Age}(U) = \mathcal{F}$ – the *Fraïssé limit*.

- Intuition: *amalgamating* the finite building blocks to obtain a chain $A_0 \subseteq A_1 \subseteq A_1 \subseteq \cdots$ and its colimit $U = \bigcup_{n \in \omega} A_n$.
- Examples: the linear order of rationals, the random graph, and the rational Urysohn metric space.

Abstract Fraïssé theory

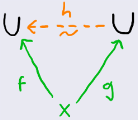
- We shall formulate the core theory in the language of category theory.
- This gives a clear presentation of the phenomena involved, and makes the theory suitable in many situations:
  - uniform treatment of classical and projective Fraïssé theory, embedding-projection pairs, comma categories, ...
- History:
  - Droste and Göbel (1993) – a semialgebroidal category $\mathcal{L}$ of "large objects";
  - Kubiś (2014) – a Fraïssé sequence in a category $\mathcal{K}$ of "small objects", category of sequences as the completion;
  - Caramello (2014) – translating the category of sequences to a category of large objects.
- My contribution: polishing the presentation a bit; stressing the role of the free completion.
- The countable core theory can be extended in various ways:
  - uncountable sequences, weakening AP, metric-enriched setting.

Abstract Fraïssé theory

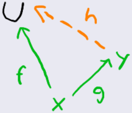
Definition

For a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ we say that an $\mathcal{L}$-object U is

- *homogeneous* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object x and every $\mathcal{L}$-maps $f, g \colon x \to U$ there is an $\mathcal{L}$-automorphism $h \colon U \to U$ such that $h \circ g = f$.

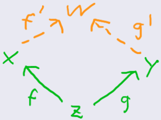


- *injective* / has the *extension property* in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-map $f \colon x \to U$ and $\mathcal{K}$-map $g \colon x \to y$ there is an $\mathcal{L}$-map $h \colon y \to U$ such that $h \circ g = f$.



Definition

A category $\mathcal{K}$ has the *amalgamation property* if for every $\mathcal{K}$-maps $f \colon z \to x$ and $g \colon z \to y$ there are $\mathcal{K}$-maps $f' \colon x \to w$ and $g' \colon y \to w$ such that $f' \circ f = g' \circ g$.

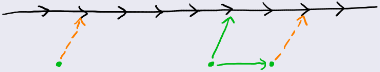


Abstract Fraïssé theory

Definition

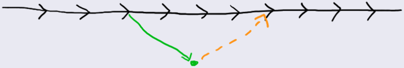
A $\mathcal{K}$-sequence $\vec{u}$ is *Fraïssé* if it is

- *cofinal*, i.e. for every $\mathcal{K}$-object x there is a $\mathcal{K}$-map $f : x \to u_n$ for some $n \in \omega$,
- *injective*, i.e. for every $\mathcal{K}$-maps $f : x \to u_n$ and $g : x \to y$ there is a $\mathcal{K}$-map $h : y \to u_m$ for some $m \geq n$ such that $h \circ g = u_n^m \circ f$.



If $\mathcal{K}$ is directed and has AP, the above reduces to being

- *absorbing*: for every $\mathcal{K}$-map $g : u_n \to x$ there is a $\mathcal{K}$-map $h : x \to u_m$ for some $m \geq n$ such that $h \circ g = u_n^m$.



Abstract Fraïssé theory

Theorem (characterization of the Fraïssé limit)

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-colimit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$.

Theorem (existence of a Fraïssé sequence)

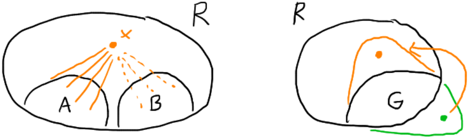
Let $\mathcal{K} \neq \emptyset$ be a category. $\mathcal{K}$ has a Fraïssé sequence if and only if

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
3. $\mathcal{K}$ has a countable dominating subcategory.

A simple example

Let $\mathcal{L}$ be the category of countable graphs and embeddings and let $\mathcal{K} \subseteq \mathcal{L}$ be the full subcategory of finite graphs.

- $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion – easy to see; mostly follows from a general result.
- $\mathcal{K}$ is locally finite and has countably many isomorphism types.
- $\mathcal{K}$ is directed – take the disjoint union; also follows from AP and initial object.
- $\mathcal{K}$ has the free amalgamation property.
- R is injective iff it is injective for one-point extensions iff
  - $(*)$ for every disjoint finite $A, B \subseteq U$ there is $x \in U \setminus (A \cup B)$ connected by an edge to every $a \in A$ and no $b \in B$.



- Every injective R is cofinal since $\mathcal{K}$ is directed.
- Hence, there is a unique countable graph R satisfying $(*)$, and it is cofinal and homogeneous – the *random graph*.

Free completion: $\mathcal{K}$-sequences $\leftrightarrow$ $\mathcal{L}$-objects

Definition

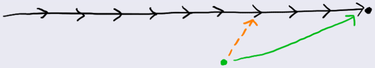
$\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion if

(L1) every $\mathcal{K}$-sequence has an $\mathcal{L}$-colimit,

(L2) every $\mathcal{L}$-object is an $\mathcal{L}$-colimit of a $\mathcal{K}$-sequence,

for every $\mathcal{K}$-sequence $\vec{x}$ and its $\mathcal{L}$-colimit $\langle X_\infty, \vec{x}^\infty \rangle$ we have that

(F1) for every $\mathcal{L}$-map from a $\mathcal{K}$-object $f\colon z \to X_\infty$ there is a $\mathcal{K}$-map $g\colon z \to x_n$ for some n such that $f = x_n^\infty \circ g$,



(F2) for every $\mathcal{K}$-maps $f, g\colon z \to x_n$ such that $x_n^\infty \circ f = x_n^\infty \circ g$ there is $m \geq n$ such that $x_n^m \circ f = x_n^m \circ g$.

- (F2) is trivial if $\mathcal{L}$ consists of monomorphisms.
- Given $\mathcal{K}$, $\mathcal{L}$ always exists and is essentially unique.
- Such $\mathcal{L}$ has colimits of all $\mathcal{L}$-sequences and has $\mathcal{K}$ as a full subcategory consisting of a rich family of *finitely presentable objects*.

How to get a free completion?

- Let L be a first-order language and let $\mathcal{K}$ and $\mathcal{L}$ be the categories of all finitely and countably generated L-structures, respectively, with all embeddings as morphisms. Then $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion.
- Similarly, let $\mathcal{K}$ be the category of all finite L-structures with all quotient homomorphisms, and let $\mathcal{L}$ be the category of all topological L-structures that are inverse limits of $\mathcal{K}$-sequences, with continuous quotient homomorphisms. Then $\langle \mathcal{K}^{\mathrm{op}}, \mathcal{L}^{\mathrm{op}} \rangle$ is a free completion.

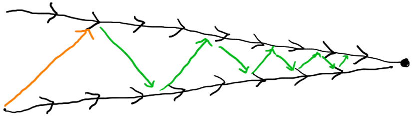
- For $\mathcal{F} \subseteq \mathcal{K}$ we define its *σ-closure* $\sigma\mathcal{F} \subseteq \mathcal{L}$ as the smallest subcategory that agrees with $\mathcal{L}$ on colimits of $\mathcal{F}$-sequences.
- Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion. If $\mathcal{F} \subseteq \mathcal{K}$ is a *full* subcategory, then $\sigma\mathcal{F} \subseteq \mathcal{L}$ is the full subcategory of all $\mathcal{L}$-colimits of $\mathcal{F}$-sequences, and $\langle \mathcal{F}, \sigma\mathcal{F} \rangle$ is a free-completion.

Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals
	finite graphs	countable graphs	Rado/random graph
	finite groups	locally finite countable groups	Hall's universal group
	finite rational metric spaces	countable rational metric spaces	rational Urysohn space
quotients	finite discrete spaces	zero-dimensional metrizable compacta	Cantor space
	finite discrete linear graphs	zero-dimensional metrizable compacta with a special closed symmetric relation	pseudo-arc prespace

How to get a free completion?

- More generally, we say that $\mathcal{F} \subseteq \mathcal{K}$ is *iso-consistent* if every isomorphism between $\mathcal{L}$-colimits of $\mathcal{F}$-sequences is witnessed by a back and forth sequence in $\mathcal{F}$, equivalently, every colimit cone map factorizes through any other colimit cone with the same apex.



- If $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion and $\mathcal{F} \subseteq \mathcal{K}$ is iso-consistent, then $\langle \mathcal{F}, \sigma\mathcal{F} \rangle$ is a free completion.
- If $\mathcal{F} \subseteq \mathcal{K}$ is full, then $\mathcal{F}$ is iso-consistent and $\sigma\mathcal{F} \subseteq \mathcal{K}$ is full.
- Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion, let $\mathcal{W} \subseteq \mathcal{L}$ be an iso-full wide subcategory, and let $\mathcal{F} \subseteq \mathcal{K} \cap \mathcal{W}$ be full. If
  1. $g \circ f \in \mathcal{W}$ and $g \in \mathcal{W}$ implies $f \in \mathcal{W}$,
  2. every $\mathcal{L}$-colimit cone f_*^∞ of an $\mathcal{F}$-sequence lie in $\mathcal{W}$,
  3. every $g \circ f_*^\infty \in \mathcal{W}$ implies $g \in \mathcal{W}$,

  then $\mathcal{F}$ is iso-consistent and $\sigma\mathcal{F} \subseteq \mathcal{W}$ is full.

Projective examples with special morphisms

$\mathcal{F}$-objects	$\mathcal{F}$-maps	Fraïssé limit quotient
discrete	all	the Cantor space
linear	all	the pseudo-arc
ordered trees	all	the Lelek fan
trees of degree ≤ 3	monotone	the Ważewski dendrite D_3
connected	monotone	the Menger curve
connected	confluent	a new continuum

Approximate Fraïssé theory

- Joint work with Wiesław Kubiś.
- We have extended the theory from ordinary (discrete) categories to *MU-categories* – a generalized version of metric-enriched categories, abstracting from the category of metric spaces and uniformly continuous maps.

Definition

An *MU-category* is a category $\mathcal{K}$ endowed with distance maps $d\colon \mathcal{K}(X, Y)^2 \to [0, \infty]$ such that for every $\mathcal{K}$-map $f\colon X \to Y$ we have

1. $d(g \circ f, h \circ f) \leq d(g, h)$ for every $\mathcal{K}$-maps $g, h\colon Y \to Z$,
2. for every $\varepsilon > 0$ there is $\delta > 0$ such that f is $\langle\varepsilon, \delta\rangle$*-continuous*, i.e. for every $\mathcal{K}$-maps $g, h\colon W \to X$ such that $g \approx_\delta h$ we have $f \circ g \approx_\varepsilon f \circ h$.

Approximate Fraïssé theory

- The notions like free completion, amalgamation, and homogeneity have their corresponding generalizations.
  - AP now means (in projective the convention): for every $\mathcal{K}$-maps $f\colon Z \leftarrow X$, $g\colon Z \leftarrow Y$ and every $\varepsilon > 0$ there are $\mathcal{K}$-maps $f'\colon X \leftarrow W$ and $g'\colon Y \leftarrow W$ such that $f \circ f' \approx_\varepsilon g \circ g'$.
  - U is homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object X, $\mathcal{L}$-maps $f, g\colon X \leftarrow U$, and $\varepsilon > 0$ there is an automorphism $h\colon U \leftarrow U$ such that $f \approx_\varepsilon g \circ h$.
- This forms a conservative extension of the original theory in the sense that for a discrete MU-category, the definitions reduce to the basic ones.
- Our motivation: Irwin and Solecki (2006) characterized the *pseudo-arc* by a condition that becomes the actual homogeneity in our setup.

Pseudo-arc and pseudo-solenoids

- Let $\mathcal{I}$ / $\mathcal{S}$ be the MU-category of all continuous surjections of the unit interval / unit circle.
- Then $\sigma\mathcal{I}$ / $\sigma\mathcal{S}$ is the MU-category of all *arc-like* / *circle-like* continua and all continuous surjections.
- For a set of primes P let $\mathcal{S}_P \subseteq \mathcal{S}$ be the MU-subcategory of all maps whose *degree* uses only primes from P.

Theorem (B., Kubiś)

1. $\langle \mathcal{I}, \sigma\mathcal{I} \rangle$ is a free MU-completion, $\mathcal{I}$ is a Fraïssé MU-category, and the *pseudo-arc* is the Fraïssé limit.
2. $\langle \mathcal{S}_P, \sigma\mathcal{S}_P \rangle$ is a free MU-completion, $\mathcal{S}_P$ is a Fraïssé MU-category, and the *P-adic pseudo-solenoid* is the Fraïssé limit.

References

Adam Bartoš and Wiesław Kubiś.
Hereditarily indecomposable continua as generic mathematical structures, 2022.
Preprint [arXiv:2208.06886].

Dana Bartošová and Aleksandra Kwiatkowska.
Lelek fan from a projective Fraïssé limit.
Fund. Math., 231(1):57–79, 2015.

Olivia Caramello.
Fraïssé's construction from a topos-theoretic perspective.
Log. Univers., 8(2):261–281, 2014.

Manfred Droste and Rüdiger Göbel.
Universal domains and the amalgamation property.
Math. Structures Comput. Sci., 3(2):137–159, 1993.

Roland Fraïssé.
Sur l'extension aux relations de quelques propriétés des ordres.
Ann. Sci. École Norm. Sup. (3), 71:363–388, 1954.

Wilfrid Hodges.
Model theory, volume 42 of *Encyclopedia of Mathematics and its Applications*.
Cambridge University Press, Cambridge, 1993.

References

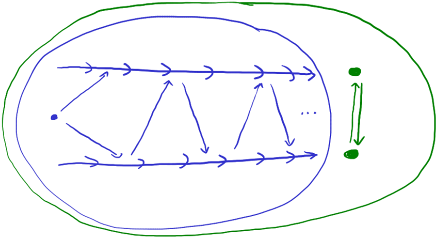
Trevor Irwin and Sławomir Solecki.
Projective Fraïssé limits and the pseudo-arc.
Trans. Amer. Math. Soc., 358(7):3077–3096, 2006.

Wiesław Kubiś.
Metric-enriched categories and approximate Fraïssé limits, 2013.
Preprint [arXiv:1210.6506].

Wiesław Kubiś.
Fraïssé sequences: category-theoretic approach to universal homogeneous structures.
Ann. Pure Appl. Logic, 165(11):1755–1811, 2014.

Wiesław Kubiś.
Weak Fraïssé categories.
Theory Appl. Categ., 38:27–63, 2022.

Aristotelis Panagiotopoulos and Sławomir Solecki.
A combinatorial model for the Menger curve.
J. Topol. Anal., 14(1):203–229, 2022.



Thank you!