Constructing compacta from relations between finite graphs

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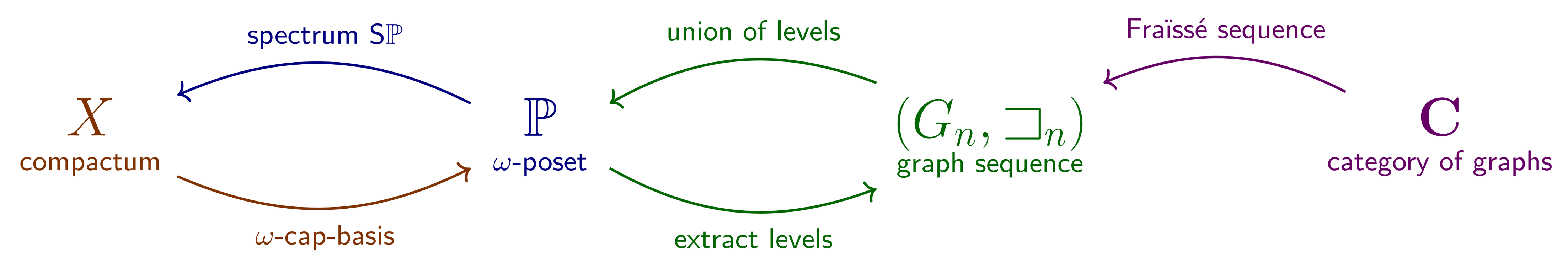


Every second-countable T_1 compactum admits a nice combinatorial basis

Every ω-poset encodes a basis and basic covers of a second-countable T_1 compactum

Graded ω-posets are sequences of graphs and relational morphisms

Relational categories of graphs admit Fraïssé sequences

ω-Posets, levels, caps

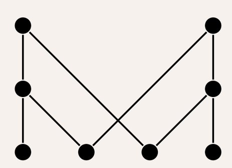
An *ω-poset* is a poset $\mathbb{P}$ such that

- every element $p \in \mathbb{P}$ has finite *rank*, i.e. $\mathsf{r}(p) < \omega$ where
$$\mathsf{r}(p) = \sup\{\mathsf{r}(q) + 1 : q > p\},$$
- every set $\mathbb{P}^n = \{p \in \mathbb{P} : \mathsf{r}(p) \leq n\}$ is finite.

We define the following special subsets.

- The n^{th} *level* $\mathbb{P}_n$ consists of minimal elements of $\mathbb{P}^n$.
- $C \subseteq \mathbb{P}$ is a *cap* ("abstract cover") if it is refined by some level:
$$\exists n\ \mathbb{P}_n \leq C, \text{ meaning } \forall p \in \mathbb{P}_n\ \exists c \in C\ p \leq c.$$

An ω-poset $\mathbb{P}$ is *graded* if for every $p < q$ and $n \in [\mathsf{r}(p), \mathsf{r}(q)]$ there is $r \in [p, q]$ with $\mathsf{r}(r) = n$.



ω-Cap-bases

An *ω-cap-basis* of a T_1 compactum X is a basis $\mathbb{P}$ such that

- $(\mathbb{P}, \subseteq)$ is an ω-poset,
- $\mathbb{P}$-covers of X are exactly $\mathbb{P}$-caps,
  or equivalently, every $\mathbb{P}$-cover is refined by a level $\mathbb{P}_n$.

Existence of ω-cap-bases:

- A countable basis $\{p_n : n \in \omega\}$ of non-empty sets of a metric space X is an ω-cap-basis if and only if $\operatorname{diam}(p_n) \to 0$.
- Every second-countable T_1 compactum X has an ω-cap-basis $\mathbb{P}$. Moreover, we can arrange any of the following (but not any two simultaneously).
  1. $\mathbb{P}$ is *weakly graded* and the levels $\mathbb{P}_n$ are members of a given co-initial family of minimal open covers.
  2. $\mathbb{P}$ is *predetermined* and its elements are members of a given countable basis.
  3. $\mathbb{P}$ is predetermined and graded.

Reconstruction of spaces

For every ω-cap-basis $\mathbb{P}$ of a T_1 compactum X the map
$$x \in X \mapsto x^{\in} = \{p \in \mathbb{P} : x \in p\} \in \mathsf{S}\mathbb{P}$$
is a homeomorphism inducing an order isomorphism $\mathbb{P} \to (p^{\in})_{p \in \mathbb{P}}$.

The spectrum $\mathsf{S}\mathbb{P}$

Given an ω-poset $\mathbb{P}$, we define its *spectrum* $\mathsf{S}\mathbb{P}$.

- A *selector* is a subset $S \subseteq \mathbb{P}$ intersecting every cap.
- Points of $\mathsf{S}\mathbb{P}$ are minimal selectors, or equivalently minimal filters intersecting every level.
- Basic open sets are $p^{\in} = \{S \in \mathsf{S}\mathbb{P} : p \in S\}$, $p \in \mathbb{P}$.

We obtain a second-countable T_1 compactum. Moreover,

- The map $p \mapsto p^{\in}$ is a monotone surjection of $\mathbb{P}$ onto a basis of $\mathsf{S}\mathbb{P}$ such that $\{p^{\in} : p \in \mathbb{P}\} \setminus \{\emptyset\}$ is an ω-cap-basis.
- For $C \subseteq \mathbb{P}$, the set $\{p^{\in} : p \in C\}$ is a cover of $\mathsf{S}\mathbb{P}$ if and only if C is a cap.

Regularity and metrizability

Given an ω-poset $\mathbb{P}$ we define

- the *compatibility* relation $p \wedge q \Leftrightarrow \exists r \leq p, q$,
- the *star* $Cp = \{q \in C : q \wedge p\}$ for a cap C,
- the *star-below* relation $p \lhd q \Leftrightarrow Cp \leq q$ for some cap/level C.

An ω-poset $\mathbb{P}$ is

- *prime* if for every $p \in \mathbb{P}$, $p^{\in} \neq \emptyset$, equivalently there is a cap C such that $C \setminus \{p\}$ is not a cap; then we have
$$p \wedge q \quad \Leftrightarrow \quad p^{\in} \cap q^{\in} \neq \emptyset, \qquad\qquad p \lhd q \quad \Leftrightarrow \quad \operatorname{cl}(p^{\in}) \subseteq q^{\in};$$
- *regular* if every cap/level is $\lhd$-refined by a cap/level.

For a prime ω-poset $\mathbb{P}$
$$\mathbb{P} \text{ is regular} \quad \Leftrightarrow \quad \mathsf{S}\mathbb{P} \text{ is Hausdorff/metrizable.}$$

Refiners and functoriality

A *refiner* $\mathbb{P} \to \mathbb{Q}$ between two ω-posets is a relation $\sqsupset \subseteq \mathbb{Q} \times \mathbb{P}$ such that every $\mathbb{Q}$-level/cap is $\sqsubset$-refined by a $\mathbb{P}$-level/cap.

- If $\sqsupset \subseteq \mathbb{Q} \times \mathbb{P}$ and $\sqsupset' \subseteq \mathbb{P} \times \mathbb{Q}$ are refiners such that $\sqsupset' \circ \sqsupset \subseteq \geq_{\mathbb{P}}$ and $\sqsupset \circ \sqsupset' \subseteq \geq_{\mathbb{Q}}$, then $\mathsf{S}\mathbb{P} \cong \mathsf{S}\mathbb{Q}$.
- Hence, if $\mathbb{Q} \subseteq \mathbb{P}$ consists of cofinally many levels, $\mathsf{S}\mathbb{P} \cong \mathsf{S}\mathbb{Q}$.

Let $\mathbf{P}$ denote the category of prime regular ω-posets and $\wedge$-preserving refiners; let $\mathbf{K}$ denote the category of metrizable compacta and continuous maps.

- By putting $\mathsf{S}(\sqsupset) \colon S \in \mathsf{S}\mathbb{P} \mapsto S^{\sqsubset\lhd} \in \mathsf{S}\mathbb{Q}$ we obtain a full essentially surjective functor $\mathsf{S} \colon \mathbf{P} \to \mathbf{K}$.

Graph categories

A *graph* is a nonempty finite set G endowed with a symmetric reflexive edge-relation $\sqcap$. We consider a category $\mathbf{G}$ of graphs and relational morphisms. A $\mathbf{G}$-morphism $G \to H$ is a relation $\sqsupset \subseteq H \times G$ that is

- *edge-preserving*: $\forall g \sqsubset h\ \forall g' \sqsubset h'\ g \sqcap g' \Rightarrow h \sqcap h'$,
- *edge-surjective*: $\forall h \sqcap h'\ \exists g \sqsubset h\ \exists g' \sqsubset h'\ g \sqcap g'$,
- *co-surjective*: $\forall g \in G\ \exists h \in H\ g \sqsubset h$,
- *co-injective*: $\forall h \in H\ \exists g \in G\ g^{\sqsubset} = \{h\}$.

We often consider following properties forming ideals:

- *anti-injective*: $\forall h \in H\ |h^{\sqsupset}| \geq 2$,
- *edge-witnessing*: $\forall h \sqcap h' \in H\ \exists g \in G\ g \sqsubset h, h'$,
- *star-refining*: $\forall g \in G\ \exists h \in H\ g^{\sqcap} \sqsubset h$.

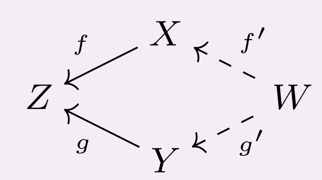
Sequences of graphs

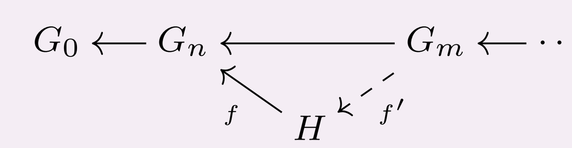
A *sequence* $(G_n, \sqsupset_n)$ in the category $\mathbf{G}$ is
$$G_0 \xleftarrow{\sqsupset_0} G_1 \xleftarrow{\sqsupset_1} G_2 \xleftarrow{\sqsupset_2} G_3 \longleftarrow \cdots$$
$(G_n, \sqsupset_n)$ yields an atomless predetermined graded ω-poset
$$\mathbb{P} = \textstyle\bigcup_n G_n, \qquad \leq\, = \textstyle\bigcup_{m \leq n} \sqsubset_n^m .$$
Every such ω-poset $\mathbb{P}$ yields a $\mathbf{G}$-sequence
$$(G_n, \sqcap) = (\mathbb{P}_n, \wedge\restriction_{\mathbb{P}_n}), \qquad \sqsupset_n\, = \,\geq\restriction_{\mathbb{P}_n \times \mathbb{P}_{n+1}}.$$

- $(G_n, \sqsupset_n)$ has an edge-witnessing subsequence $\Leftrightarrow$ $\sqcap = \wedge$ on G_ns.
- Then every G_n faithfully represents a basic minimal cover of $\mathsf{S}\mathbb{P}$.
- $(G_n, \sqsupset_n)$ has a star-refining subsequence $\Leftrightarrow$ $\mathbb{P}$ is regular.

Fraïssé theory

Every essentially countable directed category with the *amalgamation property* has a *Fraïssé sequence*:





Its limit in a *free completion* is *cofinal* and *homogeneous*.

different approaches	small category setup	Fraïssé limit
Irwin–Solecki	discrete	the pre-space
B.–Kubiś	continuous	the space
our goal	discrete	the space

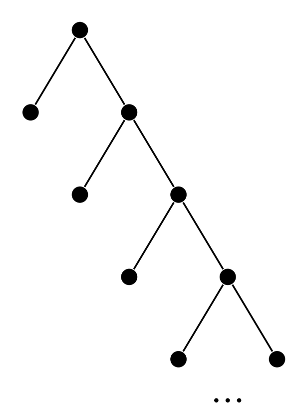
Applications

We represent spaces of interest as spectra of Fraïssé sequences in corresponding graphs categories.

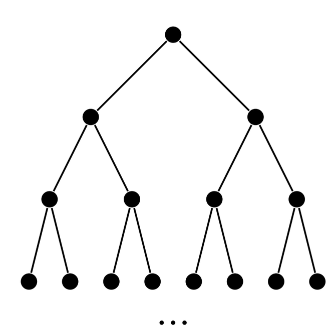
graphs	relational morphisms	$\mathsf{S}\mathbb{P}$
discrete	all ($\Leftrightarrow$ surjective functions)	Cantor space
paths	monotone	arc
paths	all	pseudo-arc
fans	root-monotone end-preserving	Cantor fan (?)
fans	root-monotone	Lelek fan (?)
connected	monotone	Menger curve (??)

More goals:

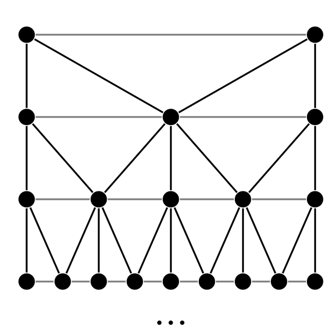
- Represent more spaces, find new ones.
- Characterize the corresponding Fraïssé sequences.
- Use the combinatorial description to investigate automorphism groups (point homogeneity, generic homeomorphisms, . . .).



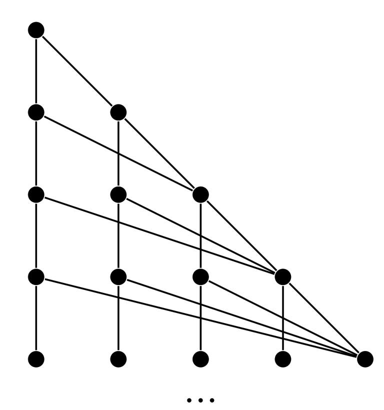
convergent sequence



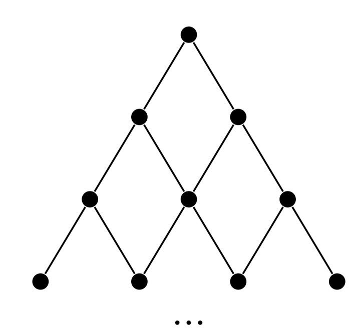
Cantor space 2^{ω}



unit interval $[0, 1]$



cofinite topology on ω



discrete space $\{0, 1\}$