Fraïssé-like constructions of compacta

Adam Bartoš
bartos@math.cas.cz

Institute of Mathematics, Czech Academy of Sciences

Hejnice, January 27 – February 3, 2024

Based on joint work with Wiesław Kubiś,
and on joint work with Tristan Bice and Alessandro Vignati

Abstract Fraïssé theory overview

[Fraïssé; Droste–Göbel; Kubiś; B.]

Theorem (characterization of the Fraïssé limit)

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and has the extension property in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-limit of a Fraïssé sequence in $\mathcal{K}$.

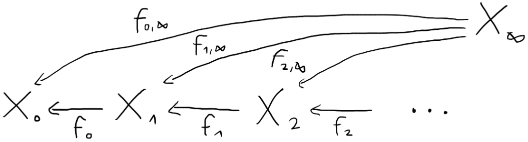
Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-limit U is Fraïssé in $\mathcal{K}$.

Theorem (existence of a Fraïssé sequence)

Let $\mathcal{K} \neq \emptyset$ be a category. $\mathcal{K}$ has a Fraïssé sequence if and only if

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
3. $\mathcal{K}$ has a countable dominating subcategory.

Inverse limits



Inverse sequence $\langle X_*, f_* \rangle$:

- $X_* = \langle X_n \rangle_{n \in \omega}$ sequence of structures,
- $f_* = \langle f_{n,m} \colon X_n \leftarrow X_m \rangle_{n \leq m \in \omega}$ structure-preserving maps such that $f_{n,m} \circ f_{m,k} = f_{n,k}$ and $f_{n,n} = \mathrm{id}_{X_n}$ for $n \leq m \leq k$.
- We write f_n for $f_{n,n+1}$.

The limit cone $\langle X_\infty, f_{*,\infty} \rangle$:

- $X_\infty = \{x_* \in \prod_{n \in \omega} X_n : x_n = f_{n,m}(x_m) \text{ for every } n \leq m\}$
- $f_{n,\infty} \colon X_n \leftarrow X_\infty$ is the restriction of the projection.

Inverse limits of compacta

- Take $\langle X_*, f_* \rangle$ such that every X_n is a finite discrete space and every $f_{n,m}$ is a (continuous) surjection.
- Then X_∞ is a zero-dimensional metrizable compactum.
- Every zero-dimensional metrizable compactum can be obtained this way.
- Let $\mathcal{K}$ be the category of all nonempty finite discrete spaces and all surjections and let $\mathcal{L}$ be the category of all zero-dimensional metrizable compacta and all continuous surjections.
- Then $\mathcal{K}$ is a Fraïssé category and the Cantor space is the Fraïssé limit.

How to obtain non-zero-dimensional spaces?

Approach 1: Classical projective Fraïssé theory

- [Irwin–Solecki]
- We endow zero-dimensional metrizable compacta with a closed symmetric relation E and consider E-quotient maps.
- We obtain the category $\mathcal{L}$ of topological graphs.
- $\mathcal{K}$ is then the subcategoty of all finite graphs.
- A topological graph $\langle X, E \rangle$ is a pre-space if a E is also transitive.
- Then the quotient space X/E is a metrizable compactum.
- Every metrizable compactum can be obtained this way.

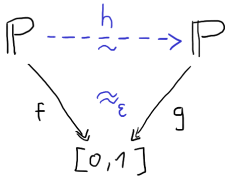
Approach 1: Classical projective Fraïssé theory

- We consider a subcategory $\mathcal{F} \subseteq \mathcal{K}$ that is Fraïssé and the subcategory $\sigma\mathcal{F} \subseteq \mathcal{L}$ of limits of $\mathcal{F}$-sequences
- If $\mathcal{F}$ consists of all linear graphs, then the Fraïssé limit is a pre-space of the pseudo-arc [Irwin–Solecki].
- Another example: if $\mathcal{F}$ consists of all connected graphs and monotone quotient maps, then the Fraïssé limit is a pre-space of the Menger curve [Panagiotopoulos–Solecki].
- Fraïssé limit is the pre-space, not the space. Properties of the space have to be transferred through the quotient map.

Can we obtain the desired compactum directly as a Fraïssé limit?

Approach 2: Approximate Fraïssé theory for compacta

- [B.–Kubiś]
- We work directly with metrizable compacta and continuous surjections.
- Fraïssé-theoretic properties involve approximate commutativity of diagrams.
- For example, homogeneity of the pseudo-arc $\mathbb{P}$: for every continuous surjections $f, g\colon \mathbb{P} \to [0,1]$ and every $\varepsilon > 0$ there is a homeomorphism $h\colon \mathbb{P} \to \mathbb{P}$ such that $f \approx_\varepsilon g \circ h$.



Fraïssé theory of MU-categories overview

Theorem (characterization of the Fraïssé limit)

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free MU-completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-limit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal and homogeneous in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$.

Theorem (existence of a Fraïssé sequence)

Let $\mathcal{K} \neq \emptyset$ be an MU-category. $\mathcal{K}$ has a Fraïssé sequence if and only if

1. $\mathcal{K}$ is directed,
2. $\mathcal{K}$ has the amalgamation property,
3. $\mathcal{K}$ has a countable dominating subcategory.

Approach 2: Approximate Fraïssé theory for compacta

- Let $\mathcal{I}$ be the category consisting of the unit interval and all continuous surjections.
- Then $\mathcal{I}$ is Fraïssé, $\sigma\mathcal{I}$ consists of all arc-like continua, and the Fraïssé limit is the pseudo-arc.
- Let $\mathcal{S}_P$ be the category consisting of the unit circle and all continuous surjections whose degree uses only primes from a set P.
- Then $\mathcal{S}_P$ is Fraïssé, $\sigma\mathcal{S}_P$ consists of circle-like continua of "type $\leq P^\infty$", and the Fraïssé limit is the P-adic pseudo-solenoid.

However, our small objects are not finite any more.

Approach 3: Spectra of ω-posets

- [B.–Bice–Vignati]
- Small objects are still finite graphs, but morphisms are relations.
- A morphism $R\colon Y \leftarrow X$ is $R \subseteq Y \times X$ that is
  - co-surjective: $\forall x \; \exists y \; yRx$,
  - co-injective: $\forall y \; \exists x \; R(x) = \{y\}$,
  - edge-preserving: $yRx \wedge y'Rx' \wedge xEx' \Rightarrow yEy'$.
- Let $\mathcal{K}$ denote the corresponding category.

Approach 3: Spectra of ω-posets

- A sequence $\langle X_*, R_* \rangle$ in $\mathcal{K}$ induces an ω-poset $\mathbb{P} = \bigsqcup_{n \in \omega} X_n$ where $\langle n, y \rangle \geq \langle m, x \rangle$ if $n \leq m$ and $y R_{n,m} x$.
- $\mathbb{P}$ induces the spectrum $\mathsf{S}\mathbb{P} = \{S \subseteq \mathbb{P} : S \text{ minimal selector}\}$, where $S \subseteq \mathbb{P}$ is a selector if S is upwards closed and $S \cap X_n \neq \emptyset$ for every $n \in \omega$.
- $\mathsf{S}\mathbb{P}$ is endowed with the topology generated by the sets $p^{\in} = \{S \in \mathsf{S}\mathbb{P} : p \in S\}$ for $p \in \mathbb{P}$.
- Then $\mathsf{S}\mathbb{P}$ is a second-countable T_1 compactum.
- Every second-countable T_1-compactum can be obtained this way.

Approach 3: Spectra of ω-posets

- If $\mathcal{F} \subseteq \mathcal{K}$ is the category of all linear graphs and monotone morphisms, then $\mathcal{F}$ is Fraïssé and the unit interval is the "limit".
- If $\mathcal{F} \subseteq \mathcal{K}$ is the category of all linear graphs and all morphisms, then $\mathcal{F}$ is a Fraïssé and the pseudo-arc is the "limit".
- If $\mathcal{F} \subseteq \mathcal{K}$ is the category of all fan graphs and spokewise monotone morphisms, then $\mathcal{F}$ is Fraïssé and the Lelek fan is the "limit".

But instead of taking the limit, we introduce an ad hoc construction.

Summary

	finite small objects	proper limits	no quotient needed
classical projective FT	✓	✓	✗
approximate projective FT	✗	✓	✓
spectra of ω-posets	✓	✗*	✓

Spectra as limits

- Maybe the spectrum can be viewed as a limit after all.
- Let us put $X_\infty = \mathrm{S}\mathbb{P}$ and $yR_{n,\infty}x$ iff $y \in x$.
- Then $\langle X_*, R_{*,\infty} \rangle$ is the unital lax adjoint limit (as a set) endowed with the initial topology with respect to lower semicontinuity.
- ... this is work in progess.

References

T. Irwin, S. Solecki.
Projective Fraïssé limits and the pseudo-arc.
Trans. Amer. Math. Soc., 358 (2006)

A. Bartoš, W. Kubiś.
Hereditarily indecomposable continua as generic mathematical structures.
arXiv:2208.06886

A. Bartoš, T. Bice, A. Vignati
Constructing compacta from posets.
arXiv:2307.01143