An introduction to abstract Fraïssé theory

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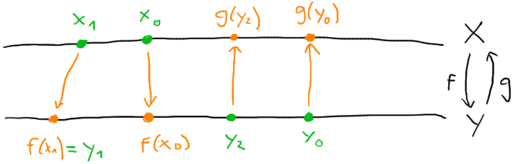
The rationals

Theorem (Cantor)

$\langle \mathbb{Q}, \leq \rangle$ is the unique countable dense linear order without endpoints.

Proof: back and forth construction.



Moreover:

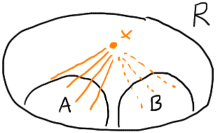
- Every isomorphism between finite $A, B \subseteq \mathbb{Q}$ extends to an automorphism (ultrahomogeneity).
- Every countable linear order embeds into $\mathbb{Q}$ (universality).

Random/Rado graph

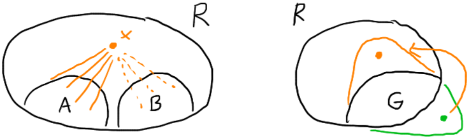
Theorem (Erdős–Rényi)

For $i \neq j \in \omega$ let us put an edge between the vertices i and j with probability $\frac{1}{2}$. Then almost surely we obtain an isomorphic copy of a particular graph R.

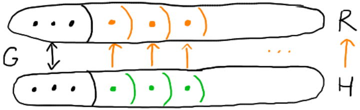
- R is characterized by the following property: For every disjoint finite $A, B \subseteq R$ there is a vertex $x \in R \setminus (A \cup B)$ such that $E(a, x)$ for every $a \in A$ and $\neg E(b, x)$ for every $b \in B$ (one-point extension property).



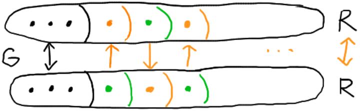
Random/Rado graph



- The extension property for finite graphs (i.e. for every finite graphs $G \subseteq H$ and an embedding $f\colon G \to R$ there is an embedding $g\colon H \to R$ such that $g{\restriction}_G = f$) and universality for countable graphs follow.



- Ultrahomogeneity and uniqueness follow as well.



Urysohn space

Theorem (Urysohn)

There is a unique separable metric space $\mathbb{U}$ such that

- for every isometry $f\colon A \to B$ between finite $A, B \subseteq \mathbb{U}$ there is an isometry $F\colon \mathbb{U} \to \mathbb{U}$ with $F{\restriction_A} = f$ (ultrahomogeneity),
- $\mathbb{U}$ contains every finite metric space (small universality).

- $\mathbb{U}$ contains even every separable metric space.
- There is a unique countable ultrahomogeneous rational metric space $\mathbb{U}_{\mathbb{Q}}$ that contains every finite rational metric space.
- $\mathbb{U}$ is the metric completion of $\mathbb{U}_{\mathbb{Q}}$.

The language of category theory

A category (denoted by $\mathcal{K}, \mathcal{L}, \mathcal{C}, \ldots$) consists of

- objects (denoted by $x, y, z, X, Y, Z, \ldots$) and
- morphisms that can be composed and include the identities (denoted by $f : x \to y$, $g : y \to z$, $g \circ f : x \to z$, id_x, ...).

Examples include the categories

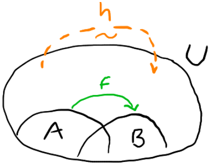
- **Set** of sets and functions,
- **Grp** of groups and group homomorphisms,
- **Top** of topological spaces and continuous maps.

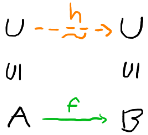
We will mostly consider the category $\mathbf{Emb}_L$ of all L-structures and embeddings for a first-order language L.

We shall often consider a pair $\langle \mathcal{K}, \mathcal{L} \rangle$ of "small" and "large" objects, where $\mathcal{K} \subseteq \mathcal{L}$ is a subcategory, e.g. finite and countable linear orders, respectively, with embeddings.

(Ultra)homogeneity

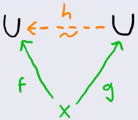
Recall that a countable relational structure U is ultrahomogeneous if every isomorphism $f\colon A \to B$ between finite substructures $A, B \subseteq U$ can be extended to an automorphism $h\colon U \to U$.





Definition

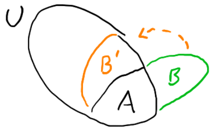
For a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ we say that an $\mathcal{L}$-object U is homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object x and every $\mathcal{L}$-maps $f, g\colon x \to U$ there is an $\mathcal{L}$-automorphism $h\colon U \to U$ such that $h \circ g = f$.

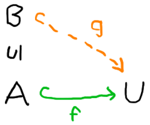


So a structure U is ultrahomogeneous if and only if it is homogeneous in $\langle \mathrm{Age}(U), \mathcal{L} \rangle$.

Extension property / injectivity

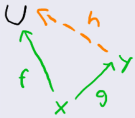
Recall that a countable relational structure U is injective or has the extension property if for every structures $A \subseteq B \in \mathrm{Age}(U)$ every embedding $f\colon A \to U$ can be extended to an embedding $g\colon B \to U$.





Definition

For a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ we say that an $\mathcal{L}$-object U is injective / has the extension property in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{L}$-map $f\colon x \to U$ and $\mathcal{K}$-map $g\colon x \to y$ there is an $\mathcal{L}$-map $h\colon y \to U$ such that $h \circ g = f$.



Universality / cofinality

Recall that a structure U is universal for a class of structures $\mathcal{F}$ if every $X \in \mathcal{F}$ can be embedded to U.

Definition

For a pair of categories $\mathcal{K} \subseteq \mathcal{L}$ we say that an $\mathcal{L}$-object U is cofinal in $\langle \mathcal{K}, \mathcal{L} \rangle$ if for every $\mathcal{K}$-object x there is an $\mathcal{L}$-map $f : x \to U$.

What is the Fraïssé limit anyway?

Let $\mathcal{K} \subseteq \mathcal{L}$ be categories, let U be an $\mathcal{L}$-object. We consider the properties:

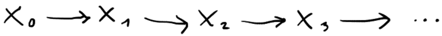
1. U is homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is injective / has the extension property in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is cofinal in $\langle \mathcal{K}, \mathcal{L} \rangle$.

- Always, if U is cofinal and homogeneous, then U is injective.
- Sometimes U is cofinal homogeneous iff U is cofinal injective.
- Sometimes such U is unique.
- Sometimes such U is cofinal for the whole $\mathcal{L}$.

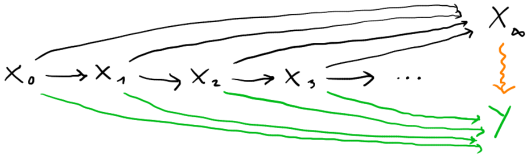
If it is the case, then it makes sense to call U the Fraïssé limit.

Sequences and colimits

- A sequence $\vec{x}$ in a category $\mathcal{K}$ consists of a sequence $\mathcal{K}$-objects $\langle x_n \rangle_{n\in\omega}$ and a coherent sequence of $\mathcal{K}$-maps $\langle x_n^m \colon x_n \to x_m \rangle_{n\le m\in\omega}$.



- A colimit of the sequence $\vec{x}$ is an object x_∞ together with an initial cone $\vec{x}^\infty = \langle x_n^\infty \colon x_n \to x_\infty \rangle$.



- A sequence in $\mathbf{Emb}_L$ is without loss of generality an ω-chain

$$A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$$

and its colimit is the union $A_\infty = \bigcup_{n\in\omega} A_n$.

Free completion

A pair $\langle \mathcal{K}, \mathcal{L} \rangle$ is called a free sequential cocompletion or just a "free completion" if $\mathcal{L}$ arises from $\mathcal{K}$ by freely adding colimits of $\mathcal{K}$-sequences.

- We will give a precise definition later.
- Free completion establishes a correspondence

$$\mathcal{K}\text{-sequences} \quad \leftrightarrow \quad \mathcal{L}\text{-objects.}$$

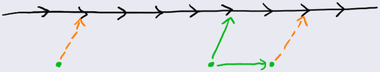
- This is the case in the classical setup when $\mathcal{K}$ is a class of finite structures and $\mathcal{L}$ is the class of their countable unions.

Fraïssé sequence

Definition

A $\mathcal{K}$-sequence $\vec{u}$ is Fraïssé if it is

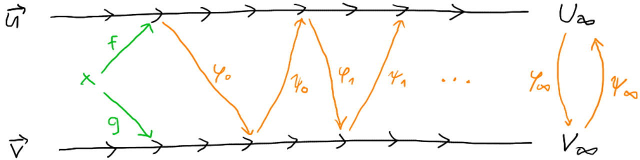
- cofinal, i.e. for every $\mathcal{K}$-object x there is a $\mathcal{K}$-map $f: x \to u_n$ for some $n \in \omega$,
- injective, i.e. for every $\mathcal{K}$-maps $f: x \to u_n$ and $g: x \to y$ there is a $\mathcal{K}$-map $h: y \to u_m$ for some $m \geq n$ such that $h \circ g = u_n^m \circ f$.



Note that the definition is analogous to the definition of cofinal and injective object in $\langle \mathcal{K}, \mathcal{L} \rangle$.

Abstract back and forth

- Let $\vec{u}, \vec{v}$ be Fraïssé sequences in a category $\mathcal{K}$ and let $f: x \to u_{m_0}$, $g: y \to v_n$ be $\mathcal{K}$-maps.
- Then there are $\mathcal{K}$-maps $\varphi_k: u_{m_k} \to v_{n_k}$ and $\psi_k: v_{m_k} \to u_{n_{k+1}}$ such that the following diagram commutes.



- Hence, there are mutually inverse isomorphisms $\varphi_\infty: U_\infty \to V_\infty$ and $\psi_\infty: V_\infty \to U_\infty$ in a free completion $\mathcal{L}$ such that $\varphi_\infty \circ u_{m_0}^\infty \circ f = v_n^\infty \circ g$.
- This gives uniqueness and homogeneity of the Fraïssé limit.

Characterization of the Fraïssé limit

Theorem

Let $\langle \mathcal{K}, \mathcal{L} \rangle$ be a free completion and let U be an $\mathcal{L}$-object. Then the following are equivalent.

1. U is cofinal and homogeneous in $\langle \mathcal{K}, \mathcal{L} \rangle$,
2. U is cofinal and injective in $\langle \mathcal{K}, \mathcal{L} \rangle$,
3. U is the $\mathcal{L}$-colimit of a Fraïssé sequence in $\mathcal{K}$.

Moreover, such U is unique and cofinal in $\mathcal{L}$, and every $\mathcal{K}$-sequence with $\mathcal{L}$-colimit U is Fraïssé in $\mathcal{K}$.

It follows that such U exists if and only if a Fraïssé sequence exists in $\mathcal{K}$.

Existence

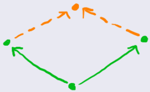
The Fraïssé limit U exists iff a Fraïssé sequence exists in $\mathcal{K}$.

Theorem

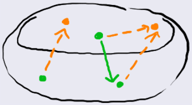
Let $\mathcal{K} \neq \emptyset$ be a category. There is a Fraïssé sequence in $\mathcal{K}$ if and only if $\mathcal{K}$ is a Fraïssé category, i.e.

1. $\mathcal{K}$ is directed (JEP), i.e. for every $\mathcal{K}$-objects x, y there is a $\mathcal{K}$-object z and $\mathcal{K}$-maps $f : x \to z$, $g : y \to z$,

2. $\mathcal{K}$ has the amalgamation property (AP), i.e. for every $\mathcal{K}$-maps $f : x \to y$, $g : x \to z$ there are $\mathcal{K}$-maps $f' : y \to w$, $g' : z \to w$ such that $f' \circ f = g' \circ g$,



3. $\mathcal{K}$ has a countable dominating subcategory.



Often $\mathcal{K}$ is locally countable (or even locally finite) and has countably many isomorphism types, which gives 3.

Free completion

Definition

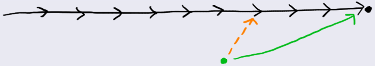
$\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion if

(L1) every $\mathcal{K}$-sequence has an $\mathcal{L}$-colimit,

(L2) every $\mathcal{L}$-object is an $\mathcal{L}$-colimit of a $\mathcal{K}$-sequence,

for every $\mathcal{K}$-sequence $\vec{x}$ and its $\mathcal{L}$-colimit $\langle X_\infty, \vec{x}^\infty \rangle$ we have that

(F1) for every $\mathcal{L}$-map from a $\mathcal{K}$-object $f \colon z \to X_\infty$ there is a $\mathcal{K}$-map $g \colon z \to x_n$ for some n such that $f = x_n^\infty \circ g$,



(F2) for every $\mathcal{K}$-maps $f, g \colon z \to x_n$ such that $x_n^\infty \circ f = x_n^\infty \circ g$ there is $m \geq n$ such that $x_n^m \circ f = x_n^m \circ g$.

- (F2) is trivial if $\mathcal{L}$ consists of monomorphisms.
- Given $\mathcal{K}$, $\mathcal{L}$ always exists and is essentially unique.
- Such $\mathcal{L}$ has all colimits of sequences and has $\mathcal{K}$ as a full subcategory consisting of a rich family of small objects.

How to get a free completion?

- Let L be a first-order language.
- Let $\mathcal{F}$ be a class of finitely generated L-structures with all embeddings.
- Let $\sigma\mathcal{F}$ be the class of all colimits of $\mathcal{F}$-sequences (which are necessarily countably generated) with all embeddings.
- Then $\langle \mathcal{F}, \sigma\mathcal{F} \rangle$ is a free completion, i.e. in the classical case the conditions are always satisfied.

Projective Fraïssé theory

- Let $\mathcal{K}^{\mathsf{op}}$ consist of nonempty finite sets and surjections.
- Then $\mathcal{K}^{\mathsf{op}}$ is essentially countable, directed, and has AP.
- A $\mathcal{K}$-sequence is Fraïssé if and only if every point eventually splits.

Where to take the limit?

- For $\mathcal{L}^{\mathsf{op}}$ being all profinite sets and surjections, $\langle \mathcal{K}, \mathcal{L} \rangle$ is not a free completion and there is no cofinal object with the extension property.
- For $\mathcal{L}^{\mathsf{op}}$ being all profinite spaces (i.e. metrizable compact zero-dimensional) and continuous surjections, $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion, and 2^ω is the Fraïssé limit.

Projective Fraïssé theory (Irwin, Solecki)

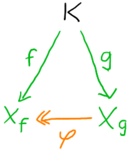
- For L a relational first-order language, let $\mathcal{L}^{\mathsf{op}}$ be the category of all topological L-structures (profinite spaces with a closed interpretation of every relation) and quotient maps, and let $\mathcal{K}^{\mathsf{op}}$ be the full subcategory of finite L-structures. Then $\langle \mathcal{K}, \mathcal{L} \rangle$ is a free completion.

Examples

	$\mathcal{K}$	$\mathcal{L}$	U
embeddings	finite linear orders	countable linear orders	the rationals
embeddings	finite graphs	countable graphs	Rado/random graph
embeddings	finite groups	locally finite countable groups	Hall's universal group
embeddings	finite rational metric spaces	countable rational metric spaces	rational Urysohn space
quotients	finite discrete spaces	zero-dimensional metrizable compacta	Cantor space
quotients	finite discrete linear graphs	zero-dimensional metrizable compacta with a special closed symmetric relation	pseudo-arc prespace

Knaster–Reichbach theorem Fraïssé theoretically

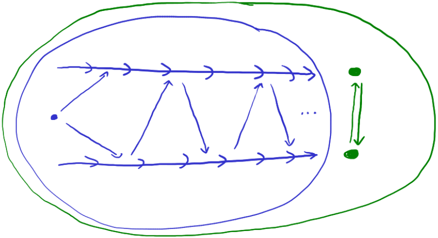
- Let K be a fixed zero-dimentional metrizable compactum.
- Let $\mathcal{K}_K$ be the following comma category.
  - A $\mathcal{K}_K$-object is a continuous map $f\colon K \to X_f$ to (not necessarily onto) a finite discrete space.
  - A $\mathcal{K}_K$-map $\varphi\colon f \to g$ is a continuous surjection $\varphi\colon X_f \leftarrow X_g$ such that $f = \varphi \circ g$.



- $\mathcal{K}_K$ is a Fraïssé category.
- Every embedding $f\colon K \to 2^\omega$ onto a nowhere dense subset is a Fraïssé limit.
- Hence, by the uniqueness of the Fraïssé limit, every homeomorphism of two closed nowhere dense subsets of 2^ω can be extended to a homeomorphism $2^\omega \to 2^\omega$.

Takeaway

1. There are these interesting properties
   - (ultra)homogeneity, extension property, universality
   - seen in the wild: rationals, random graph, Urysohn space, ...
2. There is this abstract theory about them
   - characterization of the Fraïssé limit
   - existence of a Fraïssé sequence
3. Using categories makes the theory flexible
   - projective Fraïssé theory, embedding-projection pairs, comma categories, categories of partial automorphisms, ...



Thank you!