Compactifiable classes: a question

Adam Bartoš
drekin@gmail.com

October 26, 2019

Two classes of topological spaces $\mathcal{C}$ and $\mathcal{D}$ are called *equivalent* if every member of $\mathcal{C}$ has a homeomorphic copy in $\mathcal{D}$ and vice versa, i.e. the classes are essentially the same – up to different homeomorphic copies and their multiplicities. We write $\mathcal{C} \cong \mathcal{D}$. In [1] we have introduced the notion of a *compactifiable class*. A class $\mathcal{C}$ of metrizable compacta is *compactifiable* if there exists a continuous map $q\colon A \to B$ between metrizable compacta such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$. For continua, this is equivalent to the existence of a metrizable compactum whose set of components is equivalent to $\mathcal{C}$ – the idea is to disjointly pack the given class of spaces in a small (second countable) compact space.

We have found several related notions. The theorem below we summarizes various characterizations of these notions. We use the following notation. $\mathcal{K}(X)$ and $\mathcal{C}(X)$ denote the hyperspaces of all compacta and continua in X, endowed with the Vietoris topology τ_V. The upper and lower Vietoris topologies are denoted by τ_V^+ and τ_V^-. For a class of spaces $\mathcal{C}$, the class of all homeomorphic copies of members of $\mathcal{C}$ is denoted by $\mathcal{C}^{\cong}$. For a subset of a product of topological spaces $F \subseteq A \times B$ and $b \in B$, F^b denotes the section $\{a \in A : (a, b) \in F\}$.

Theorem 1. Let $\mathcal{C}$ be a nonempty class of nonempty metrizable compacta. We have the following lists of equivalent conditions (explanation for the abbreviations: [M]ap, [R]ectangle, [H]yperspace; [D]efinition, [S]trong, [W]eak, [N]ecessary, [C]ompact or [C]losed).

Strongly compactifiable

($\mathrm{SC_M}$) There are metrizable compacta A and B, and an open (and necessarily closed) continuous map $q\colon A \to B$ such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$.

($\mathrm{SC_{MS}}$) There is a metrizable compactum A and an open and closed continuous map $q\colon A \to 2^\omega$ such that $\{q^{-1}(b) : b \in 2^\omega\} \cong \mathcal{C}$.

($\mathrm{SC_{HS}}$) There is a closed zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

($\mathrm{SC_{HW}}$) There is a metrizable compactum X and an F_σ family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.

Compactifiable

($\mathrm{C_M}$) There are metrizable compacta A and B, and a continuous (and necessarily closed) map $q\colon A \to B$ such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$.

($\mathrm{C_{MS}}$) There is a metrizable compactum A and a closed continuous map $q\colon A \to 2^\omega$ such that $\{q^{-1}(b) : b \in 2^\omega\} \cong \mathcal{C}$.

($\mathrm{C_{RW}}$) There is a metrizable compactum A, a metrizable σ-compact space B, and a closed set $F \subseteq A \times B$ such that $\{F^b\colon b \in B\} \cong \mathcal{C}$.

(C$_{\mathrm{RS}}$) There is a closed set $F \subseteq [0,1]^\omega \times 2^\omega$ such that $\{F^b : b \in 2^\omega\} \cong \mathcal{C}$.

(C$_{\mathrm{HS}}$) There is a G_δ disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$ and $(\mathcal{F}, \tau_V^+)$ is a zero-dimensional metrizable compactum.

(C$_{\mathrm{HW}}$) There is a metrizable compactum X and a family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$ and $(\mathcal{F}, \tau)$ is a metrizable compactum for a topology $\tau \supseteq \tau_V^+$.

Strongly Polishable

(SP$_{\mathrm{M}}$) There are Polish spaces A and B, and an open and closed continuous map $q\colon A \to B$ such that such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$.

(SP$_{\mathrm{MS}}$) There is a Polish space A and an open and closed continuous map $q\colon A \to \omega^\omega$ such that such that $\{q^{-1}(b) : b \in \omega^\omega\} \cong \mathcal{C}$.

(SP$_{\mathrm{MW}}$) There is a Polish space A, a (necessarily analytic) space B, and a closed continuous map $q\colon A \to B$ such that such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$.

(SP$_{\mathrm{RW}}$) There is a metrizable compactum A, an analytic space B, and a closed set $F \subseteq A \times B$ such that $\{F^b : b \in B\} \cong \mathcal{C}$.

(SP$_{\mathrm{RS}}$) There is a closed set $F \subseteq [0,1]^\omega \times \omega^\omega$ such that $\{F^b : b \in B\} \cong \mathcal{C}$.

(SP$_{\mathrm{RC}}$) There is a closed set $F \subseteq [0,1]^\omega \times 2^\omega$ and a G_δ set $G \subseteq 2^\omega$ such that $\{F^b : b \in G\} \cong \mathcal{C}$ and $\{F^b : b \in 2^\omega\} = \overline{\{F^b : b \in G\}}$ in $\mathcal{K}([0,1]^\omega)$.

(SP$_{\mathrm{HS}}$) There is a G_δ zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

(SP$_{\mathrm{HC}}$) There is a closed zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}((0,1)^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

(SP$_{\mathrm{HW}}$) There is a Polish space X and an analytic family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.

(SP$_{\mathrm{HN}}$) For every Polish space X we have that $\mathcal{C}^{\cong} \cap \mathcal{K}(X)$ is analytic.

Polishable

(P$_{\mathrm{M}}$) There are Polish spaces A and B, and a continuous map $q\colon A \to B$ such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$.

(P$_{\mathrm{MS}}$) There is a Polish space A and a continuous map $q\colon A \to \omega^\omega$ such that such that $\{q^{-1}(b) : b \in \omega^\omega\} \cong \mathcal{C}$.

(P$_{\mathrm{MW}}$) There is a Polish space A, an analytic space B, and a continuous map $q\colon A \to B$ such that such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$.

(P$_{\mathrm{RW}}$) There is a Polish space A, an analytic space B, and a G_δ set $F \subseteq A \times B$ such that $\{F^b\colon b \in B\} \cong \mathcal{C}$.

(P$_{\mathrm{RS}}$) There is a G_δ set $F \subseteq [0,1]^\omega \times \omega^\omega$ such that $\{F^b : b \in \omega^\omega\} \cong \mathcal{C}$.

(P$_{\mathrm{RC}}$) There is a closed set $F \subseteq (0,1)^\omega \times \omega^\omega$ such that $\{F^b : b \in \omega^\omega\} \cong \mathcal{C}$.

Clearly, we have the implications

$$\begin{matrix}\text{strongly}\\\text{compactifiable}\end{matrix} \implies \text{compactifiable} \implies \begin{matrix}\text{strongly}\\\text{Polishable}\end{matrix} \implies \text{Polishable},$$

but the following question remains open.

Question 2. Are the four classes indeed different, or do some of them collapse?

The problem is the lack tools for showing that a class is *not* (strongly) compactifiable/Polishable. Only the condition ($\mathrm{SP_{HN}}$) tells us that the saturated family $\mathcal{C}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ has to be analytic for a strongly Polishable class $\mathcal{C}$ of compacta. So the class of all countable compacta, which is known to be coanalytically complete [4, Theorem 27.5], is not strongly Polishable. This does not happen for other complexities. The class of all uncountable compacta $\mathcal{C}$ is strongly compactifiable even though $\mathcal{C}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ is not Borel. In fact, in [2] we have proved that for every open family $\mathcal{U} \subseteq \mathcal{K}([0,1]^\omega)$ the class $\mathcal{U}^{\cong}$ is one of countably many explicitly described classes, so it is almost never the case that $\mathcal{C}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ is closed.

This is connected to the topic of the complexity in hyperspaces up to the equivalence. It follows from the characterizations above that every analytic family in $\mathcal{K}([0,1]^\omega)$ is equivalent to a G_δ family. There are four clopen families in $\mathcal{K}([0,1]^\omega)$, and as we mentioned, countably many open families up to the equivalence. In [2] we prove that every F_σ family in $\mathcal{K}([0,1]^\omega)$ is equivalent to a closed family. What remains open is the question whether there exits a G_δ (or equivalently analytic) family that is not equivalent to a closed (or equivalently F_σ) family, i.e. whether there exists a strongly Polishable class that is not strongly compactifiable, so this is a part of Question 2.

Sometimes it is possible to show that a strongly Polishable class is in fact compactifiable. The condition ($\mathrm{SP_{RC}}$) says that we are quite close to being compactifiable. The problem are the extra fibers $\{F^b : b \in 2^\omega \setminus G\}$, which do not have to be in our class. If we were able to alter these fiber so they belong to our class while keeping the set F closed, we would prove the compactifiability.

Theorem 3. Let (X, d) be a metric compactum and for every $n \in \omega$ let $\mathcal{A}_n$ be a finite covering of X by closed sets of diameter $< 2^{-n}$. For every $F \in \mathcal{K}(X)$ let $\mathcal{A}_n(F)$ denote the space $\bigcup\{A \in \mathcal{A}_n : A \cap F \neq \emptyset\}$. Every G_δ family $\mathcal{F} \subseteq \mathcal{K}(X)$ containing a copy of every space from $\{\mathcal{A}_n(F) : F \in \overline{\mathcal{F}}, n \in \omega\}$ is compactifiable.

Hence, if we have a metrizable compactum X that is universal in the sense that $\mathcal{A}_n(F) \cong X$ for suitable coverings $\mathcal{A}_n$ and for *every* space $F \in \mathcal{K}(X)$ (or $\mathcal{C}(X)$), then every G_δ family $\mathcal{F} \subseteq \mathcal{K}(X)$ (or $\mathcal{C}(X)$) containing a copy of X is compactifiable. Moreover, as we said, every analytic family in $\mathcal{K}([0,1]^\omega)$ is equivalent to a G_δ family. This is true in $\mathcal{K}(X)$ for every Polish space X such that $X \times \omega^\omega$ embeds into X. So if our universal space satisfies also this condition, than we have can extend the compactifiability result from G_δ families to analytic families.

This is the case with $X = 2^\omega$, so every analytic family in $\mathcal{K}(2^\omega)$ containing a copy of 2^ω is compactifiable, but not with $X = D_\omega$ (the Ważewski's universal dendrite), so only every G_δ family in $\mathcal{C}(D_\omega)$ containing a copy of D_ω is compactifiable. The Hilbert cube $[0,1]^\omega$ satisfies $X \times \omega^\omega \hookrightarrow X$, but not the universality condition. But we may replace a universal space with a universal family of spaces, so we obtain the result that every analytic family in $\mathcal{C}([0,1]^\omega)$ containing a copy of every Peano continuum is compactifiable. In particular, the class of all Peano continua is compactifiable.

Note that the universality condition is a very strong self-similarity property. If for a continuum X there exist finite closed coverings $\mathcal{A}_n$ by sets of diameter $< 2^{-n}$ such that every $\mathcal{A}_n(F)$ is homeomorphic to X for every $F \in \mathcal{C}(X)$, then for every open set $U \subseteq X$ and every continuum $F \subseteq U$ there is a continuum $F \subseteq X' \subseteq U$ homeomorphic to X. This latter property could be called "*strongly continuumwise self-homeomorphic*" and it is a strengthening of the property of being *strongly pointwise self-homeomorphic* [3].

References

[1] A. Bartoš, J. Bobok, J. van Mill, P. Pyrih, B. Vejnar, *Compactifiable classes of compacta*, Topology Appl. 266 (2019), 106836.

[2] Adam Bartoš, *Borel complexity up to the equivalence*, submitted to Fund. Math., 2018, arXiv:1812.00484.

[3] Włodzimierz J. Charatonik, Anne Dilks, *On self-homeomorphic spaces*, Topology Appl. 55 (1994), no. 3, 215–238.

[4] Alexander S. Kechris, *Classical descriptive set theory*, Graduate Texts in Mathematics, 156, Springer-Verlag, New York, 1995.