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**DOCTORAL THESIS**

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# Families of connected spaces

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In Prague, May 30 Adam Bartoš

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Abstract: We deal with two completely different kinds of connected spaces – maximal connected spaces and metrizable continua. A topological space is maximal connected if it is connected, but every strictly finer topology on the same base set is disconnected. Here, the name "Families of connected spaces" refers to the collection of all connected topologies on a given set, which is ordered by inclusion, and maximal connected topologies are its maximal elements. We study the construction of tree sums of topological spaces, and how this construction preserves maximal connectedness. We also characterize finitely generated maximal connected spaces as $T_{\frac{1}{2}}$-compatible tree sums of copies of the Sierpiński space. On the other hand, we are interested in a general question when for a given class of continua there exists a metrizable compactum whose set of components is equivalent to the given class. (Two classes are equivalent if they contain the same spaces up to homeomorphic copies.) We introduce compactifiable, Polishable, strongly compactifiable, and strongly Polishable classes of compacta, and we investigate their properties. This is related to the descriptive complexity of equivalent realizations of the given class in the hyperspace of all compacta. We prove that in the hyperspace every analytic family is equivalent to a $G_\delta$ family, every $F_\sigma$ family is equivalent to a closed family, and every open family is equivalent to one of countably many saturated open families.

Keywords: maximal connectedness, tree sum, continuum, compactifiable class

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# I. Introduction

Connectedness is one of the most fundamental properties of topological spaces. A topological space is *connected* if it contains no nontrivial clopen subset, or equivalently, if it cannot be continuously mapped onto the two-point discrete space. So connected spaces are "indecomposable" in the sense that they cannot be represented as nontrivial topological sums.

In the present thesis we study two completely different kinds of connected spaces – *maximal connected spaces* and *metrizable continua*. A topological space is maximal connected if it is connected, but every strictly finer topology on the same base set is disconnected. Here, the name "Families of connected spaces" refers to the collection of all connected topologies on a given set, which is ordered by inclusion, and maximal connected topologies are its maximal elements. These are quite pathological, e.g. it is difficult to obtain even a Hausdorff maximal connected space (see below). On the other hand, when we combine connectedness with Hausdorff compactness and with existence of a countable base, we obtain the class of metrizable continua – a very well studied class of well-behaving spaces. Here we are interested in various "structural" properties of families of metrizable continua like existence of universal elements or common models, and Borel complexity of realizations of the families as subsets in a hyperspace.

The thesis is organized as follows. The rest of the introductory chapter gives some context and motivation for studying maximal connected spaces and classes of metrizable continua, and summarizes our results. The other chapters correspond to individual papers on the topic(s), namely,

- Adam Bartoš, *Tree sums of maximal connected spaces*, Topology. Appl., 252 (2019), pp. 50–71.
- A. Bartoš, J. Bobok, J. van Mill, P. Pyrih, B. Vejnar, *Compactifiable classes of compacta*, submitted to Topology Appl., arXiv:1801.01826.
- Adam Bartoš, *Borel complexity up to the equivalence*, submitted to Fund. Math., arXiv:1812.00484.

## 1 The lattice of topologies and extremal properties

The categorical constructions in the category of topological spaces are quite simple – one first perform the corresponding construction in the category of sets and then endows the resulting set with the initial or final topology. For example, the coproduct of a family of topological spaces is just the disjoint union of their underlying sets endowed with the finest topology such that the inclusions of the summands are continuous – a subset of the sum is open if and only if all its intersections with the summands are open in the the respective summands.

The reason for the possibility of splitting a topological construction into a set-theoretical one followed by endowing the resulting set with a suitable topology lies in the fact that for every set $X$ the family $\mathcal{T}(X)$ of all topologies on $X$ is a complete lattice. (We order $\mathcal{T}(X)$ by the inclusion, so coarser topologies are smaller and finer topologies are larger. Also, topologies finer than $\tau \in \mathcal{T}(X)$ are called its *expansions*.) Since formally a topology is just a family of sets closed under finite intersections and arbitrary unions, any intersection of topologies is a topology, so we have arbitrary meets in $\mathcal{T}(X)$. This gives arbitrary joins as well – the join is just the smallest topology containing the union of the given topologies.

It makes sense to study topological properties from the point of view of $\mathcal{T}(X)$. Many fundamental topological properties like compactness (not necessarily Hausdorff) and connectedness are stable under taking coarser topologies – they form a lower subset of $\mathcal{T}(X)$ for every set $X$. These are called *contractive properties*. It is natural to ask about the maximal elements of this lower set – for a contractive property $\mathcal{P}$, a topology is called *maximal* $\mathcal{P}$ if it is a maximal element of all $\mathcal{P}$ topologies in $\mathcal{T}(X)$ for some set $X$. A maximal $\mathcal{P}$ topology has $\mathcal{P}$, but no strictly finer topology has $\mathcal{P}$. There are also topological properties stable under taking finer topologies – *expansive properties* – e.g. point separating axioms: $T_0$, $T_1$, $T_2$, Urysohn, functionally Hausdorff, totally separated. So dually, for such properties $\mathcal{P}$, the notion of being *minimal* $\mathcal{P}$ is considered.

We say that a pair consisting of a contractive property $\mathcal{P}$ and an expansive property $\mathcal{Q}$ is a pair of *opposing properties* if every topology that is $\mathcal{P}$ and $\mathcal{Q}$ is both maximal $\mathcal{P}$ and minimal $\mathcal{Q}$ – this happens if there is no pair of topologies $\tau < \tau^*$ such that $\tau$ is $\mathcal{Q}$ and $\tau^*$ is $\mathcal{P}$. The classical example is compactness and $T_2$ – every continuous map from a compact space to a Hausdorff space is continuous, so a compact Hausdorff space is both maximal compact and minimal Hausdorff. Sometimes the opposing pair $\mathcal{P}, \mathcal{Q}$ may be in a stronger opposition: every maximal $\mathcal{P}$ space may already be $\mathcal{Q}$ and/or every minimal $\mathcal{Q}$ space may already be $\mathcal{P}$. (Larson [32] calls $\mathcal{P}$ and $\mathcal{Q}$ *complementary* if being maximal $\mathcal{P}$ is equivalent to being minimal $\mathcal{Q}$.) This is not the case for compactness and $T_2$ – there is even a maximal compact topology strictly coarser than a minimal Hausdorff topology [48].

However, there is a better property opposing to compactness. A topological space is a *KC-space* if every compact subset is closed. This is an expansive property between $T_2$ and $T_1$. It is easy to see that every compact KC-space is both maximal compact and minimal KC, and that every maximal compact space is KC. This was observed long time ago [43]. On the other hand, the remaining implication that every minimal KC-space is compact was proved quite recently [8].

There is a vast study of minimal and maximal properties, see [11], [14], [15]. The last paragraphs were just an illustration.

We are interested in maximal connected spaces, i.e., the contractive property $\mathcal{P}$ considered is connectedness. These were introduced by Thomas [50], who characterized finitely generated maximal connected spaces. We provide details on finitely generated maximal connected spaces in Chapter II, Section 3.

One opposing property $\mathcal{Q}$ to connectedness is being a *door space*, i.e. a space where every subset is open or closed. It is easy to see that no connected topology can be strictly finer than a door topology, and so the properties are indeed opposing. This was observed by McCartan [34], who also classified door spaces – a door space $\langle X, \tau\rangle$ is either discrete, or has exactly one non-isolated point, or $\tau$ is an expansion of an *ultrafilter topology* by a set of new isolated points (i.e. there is an ultrafilter $\mathcal{U} \subseteq \mathcal{P}(X)$ and a set $A \subseteq X$ such that $U \in \tau$ if and only if $U \in \mathcal{U}$ or $U \subseteq A$). It follows that every minimal door topology is connected – every such topology is an ultrafilter topology (i.e. $\tau \setminus \{\emptyset\}$ is an ultrafilter) or an *excluded point topology* (i.e. there is a point $x \in X$ such that $U \in \tau$ if and only if $x \notin U$ or $U = X$). Of course, not every maximal connected space is door. There are counterexamples even among finitely generated spaces.

Connected door spaces and finitely generated maximal connected spaces provide quite simple examples of maximal connected spaces. They are easy to describe, but they are not $T_2$ and with the exception of free ultrafilter spaces not even $T_1$. It is much harder to construct a Hausdorff maximal connected space. This was first done by Simon [47] and Guthrie–Stone–Wage [23] as an expansion of the real line. For every infinite cardinal $\kappa$, El'kin [20] constructed a Hausdorff maximal connected space each nonempty open subset of which has cardinality $\kappa$.

It is useful to view expansions of topologies from the point of view of their *regular open* sets. For every topology $\tau \in \mathcal{T}(X)$ let $\mathsf{RO}(\tau)$ denote the family of all regular open sets in $\tau$. Recall that $\tau$ is called *semiregular* if $\mathsf{RO}(\tau)$ is its open base, and that the topology $\tau_{\mathrm{s}}$ generated by $\mathsf{RO}(\tau)$ is called the *semiregularization* of $\tau$. Since $\mathsf{RO}(\tau_{\mathrm{s}}) = \mathsf{RO}(\tau)$, we have that $\tau_{\mathrm{s}}$ is the finest semiregular topology coarser than $\tau$. Let us call two topologies on the same set $\mathsf{RO}$-*equivalent* if they have the same regular open sets. It turns up [13, §8, Exercise 20 c)] that two topologies $\tau \leq \tau^*$ are $\mathsf{RO}$-equivalent if and only if $\tau^*$ is the expansion of $\tau$ by a filterbase of $\tau$-dense sets. Namely, the collection $\mathsf{DO}(\tau^*)$ of all $\tau^*$-open $\tau^*$-dense sets works as the filterbase since every topology $\tau$ is generated by $\mathsf{RO}(\tau) \cup \mathsf{DO}(\tau)$. Let $\mathcal{A} \subseteq \mathcal{T}(X)$ be an $\mathsf{RO}$-equivalence class. $\mathcal{A}$ has the coarsest element – the semiregularization of any member of $\mathcal{A}$. It follows that semiregular topologies are exactly these coarsest elements of $\mathsf{RO}$-equivalence classes. Every $\tau \in \mathcal{A}$ is below a (not necessarily unique) maximal element of $\mathcal{A}$ – it is enough to expand $\tau$ by a maximal filter of $\tau$-dense sets (which will necessarily contain $\mathsf{DO}(\tau)$). The resulting expansion $\tau^*$ will have the property that every dense set is open. Such topologies are called *submaximal* (see [4] for more characterizations), and we see that they are exactly the maximal elements of $\mathsf{RO}$-equivalence classes. $\mathcal{A}$ is also convex in the sense that for every $\tau \leq \tau' \leq \tau^*$ such that $\tau, \tau^* \in \mathcal{A}$ we have $\tau' \in \mathcal{A}$. This is because two topologies $\tau \leq \tau^*$ are $\mathsf{RO}$-equivalent if and only if for every two disjoint $\tau^*$-open sets $U$ and $V$ there are disjoint $\tau$-open sets $U' \supseteq U$ and $V' \supseteq V$. Finally, a topological property $\mathcal{P}$ is called *semiregular* if for every topology having $\mathcal{P}$ also every $\mathsf{RO}$-equivalent topology has $\mathcal{P}$. In other words, whole $\mathsf{RO}$-equivalence classes either have or do not have $\mathcal{P}$. It is easy to see that being Hausdorff or Urysohn are semiregular properties. Connectedness is also a semiregular property since $\mathsf{RO}$-equivalent topologies have the same clopen sets. It follows that every maximal connected space is submaximal.

Clark and Schneider [19] proved that a topological space $X$ is maximal connected if and only if it is both submaximal and *nearly maximal connected*, which means connected and having the following property: for every regular open set

$V$ and $x \in \partial V$ the space $X$ can be decomposed as $A \cup \{x\} \cup B$ for some open sets $A, B$ such that $x \in \overline{A} \cap \overline{B}$ (and so they are regular open) so that only one of the regular open sets $V \cap A$ and $V \cap B$ has $x$ in its closure – this holds if and only if expanding the topology with $V \cup \{x\}$ would make it disconnected. As we can see, near maximal connectedness depends only on regular open sets, and so it is a semiregular property. Hence, construction of a Hausdorff maximal connected expansion of a given topology $\tau$ may be decomposed into two steps – the first and essential step is to find a nearly maximal connected expansion $\tau^*$, so we end up in an $\mathsf{RO}$-equivalence class of nearly maximal connected spaces; the second step is to take any expansion of $\tau^*$ by a maximal filter of $\tau^*$-dense sets, so we end up with a maximal element of the class. Near maximal connectedness may be also characterized in the language of *singular sets*, and finding a nearly maximal expansion is possible via *singular expansions* [30], [40].

Probably the main open problem on maximal connected spaces is whether there exists a regular nondegenerate maximal connected space [41, Question 4]. Note that finding a regular maximal connected topology is much harder than finding a Hausdorff one since regularity is not an expansive property. Also, splitting the problem into the two steps sketched above does not help since a regular maximal connected topology or even a semiregular submaximal topology forms a singleton $\mathsf{RO}$-equivalence class. Recently, it was shown by Kalapodi and Tzannes [28] that there is no regular maximal connected expansion of the real line, which settles [41, Question 5]. Under $\mathsf{V} = \mathsf{L}$, every submaximal space $X$ is a countable union of discrete subsets [1]. These are necessarily closed if $X$ has no isolated points. Consequently, $X$ is zero-dimensional if it is $T_1$ normal. Therefore, consistently, there are no nondegenerate $T_1$ normal maximal connected spaces. Let us briefly mention some other related results. It is not known whether there exists a nondegenerate Tychonoff maximal connected space. Such space has to have $\pi$-weight $> \mathfrak{c}$, every subset of cardinality $< \mathfrak{c}$ has to be closed (and discrete), and the space cannot be locally connected, locally separable, or have a point of countable character (see [4], [46], [36] where even stronger results are proved). On the other hand, it is also not known whether there exists a nondegenerate regular submaximal connected space [41, Question 3]. In such space every countable subset has to be closed (and discrete), and so the space cannot be separable [4].

Recall that a point in a connected space $x \in X$ is a *cut point* if $X \setminus \{x\}$ is disconnected, and that it is a *dispersion point* if $X \setminus \{x\}$ is hereditarily disconnected. In a nearly maximal connected space every regular open set has a cut point in its boundary. Hence, in a regular maximal connected space the set of all cut points is dense, and in a nondegenerate Hausdorff maximal connected space the set of all cut points is infinite. On the other hand, a dispersion point is the unique cut point, and hence no Hausdorff topology with a dispersion point has a maximal connected expansion. This was already observed by Guthrie and Stone [22]. Baggs [6] studied a particular example of a countable Hausdorff topology without a maximal connected expansion. Examples of Hausdorff connected spaces with a dispersion point include the *Roy's countable space* and the *Cantor's leaky tent* (also called the *Knaster–Kuratowski fan*) [49]. Examples of Hausdorff maximal connected spaces where the set of all cut points is not dense were constructed in

[29] and [27].

We think a way to get more insight on maximal connected spaces is to study general topological constructions preserving maximal connectedness. In Chapter II we consider such a construction – so called *tree sum*. It is a quotient of an ordinary sum of topological spaces such that the "gluing structure" corresponds to a tree graph, so all the summands become retracts of the tree sum (Proposition 2.7). In Section 2 we first systematically study general properties of tree sums (for example we reformulate the standard separation axioms in terms of existence of continuous maps onto special topological spaces, so we may prove preservation of these separation axioms under tree sums in a uniform way – Proposition 2.23). Then we prove the main result concerning preservation of maximal connectedness and related properties under tree sums (Theorem 2.44, Theorem 2.53). We also give a simple proof of the fact that maximal connectedness and related properties are preserved by connected subspaces (Proposition 1.12). In Section 3 we reformulate the characterization of finitely generated maximal connected spaces given by Thomas [50] in the language of specialization preorder and graphs (they correspond to tree graphs with fixed bipartition – Corollary 3.12), and in the language of tree sums (they are exactly $T_{\frac{1}{2}}$-*compatible* tree sums of copies of the *Sierpiński space* – Corollary 3.14). We also suggest a natural way how to visualize these spaces (Notation 3.15).

# 2 Families of continua and their complexity

A *continuum* is a compact connected space. Here we will be interested in metrizable or equivalently second countable continua, so let this be a part of the definition in this section. As before, we are interested in the overall structure, but instead of considering connected topologies on a fixed set, we shall consider the structure of continuous maps on a fixed class of continua and the descriptive complexity of the class.

To unify various problems we are interested in, let us introduce the following framework, based on [17]. Let $\mathcal{C}$ be a class of continua and let $\mathcal{F}$ be a class of continuous maps between continua from $\mathcal{C}$. For $X, Y \in \mathcal{C}$ we may write $X \to_{\mathcal{F}} Y$ if there is a map $f\colon X \to Y$ in $\mathcal{F}$. Natural examples of the class $\mathcal{F}$ include the class of all embeddings (in this case we write $X \hookrightarrow Y$ instead of $X \to_{\mathcal{F}} Y$) and the class of all continuous surjections (in this case we write $X \twoheadrightarrow Y$ instead of $X \to_{\mathcal{F}} Y$). If $\mathcal{F}$ is closed under compositions and contains the identity maps on all continua from $\mathcal{C}$ (in other words, if $\langle \mathcal{C}, \mathcal{F} \rangle$ forms a category – this is the case for all classes $\mathcal{F}$ considered), the relation $\to_{\mathcal{F}}$ is a preorder. We however interpret the "orientation" of the preorder in different ways. If $\mathcal{F}$ consists of embeddings, we interpret $X \to_{\mathcal{F}} Y$ as $X \leq_{\mathcal{F}} Y$, while if $\mathcal{F}$ consists of quotients, we interpret $X \to_{\mathcal{F}} Y$ as $X \geq_{\mathcal{F}} Y$. The terms "smaller", "larger", "minimal", "maximal", etc. are interpreted according to $\leq_{\mathcal{F}}$ rather than $\to_{\mathcal{F}}$. Of course, the same definitions may be considered for any topological spaces, but here we are interested only in metrizable continua and sometimes in metrizable compacta. The point of this framework is that many interesting problems and properties of classes of continua may be stated in the language of the preorder $\leq_{\mathcal{F}}$. In [17], greatest, least, maximal, and minimal elements, chains and antichains for the class of all *dendrites* and for families of all *monotone*, *open*, *confluent*, *weakly confluent*, and *retractive* maps are studied.

Let $\mathcal{C}$ be a class of continua and let $\mathcal{F}$ be a family of (some) embeddings or quotients on $\mathcal{C}$. An $\leq_{\mathcal{F}}$-greatest element $U \in \mathcal{C}$ is called a *$\mathcal{F}$-universal continuum* (in $\mathcal{C}$). If $\mathcal{F}$ is the family of all embeddings (i.e. $U \in \mathcal{C}$ such that $X \hookrightarrow U$ for every $X \in \mathcal{C}$), $U$ is called just a *universal continuum.* If $\mathcal{F}$ is the family of all quotients, an $\leq_{\mathcal{F}}$-upper bound of $\mathcal{C}$ in the class of all continua (i.e. a continuum $U$ such that $U \twoheadrightarrow X$ for every $X \in \mathcal{C}$) is called a *common model* for $\mathcal{C}$. If additionally $U \in \mathcal{C}$, we call it *co-universal.* Let us mention some classical examples of (co-) universal continua. The *Hilbert cube* $[0,1]^\omega$ is universal for all continua (as well as for all separable metrizable spaces). By the Hahn–Mazurkiewicz theorem [39, 8.14], the unit interval $[0,1]$ is co-universal for all Peano continua (and it follows that every nondegenerate Peano continuum co-universal). The *Menger universal spaces* $M_n^m$ [21, p. 121], $1 \leq n \leq m$, are universal for all $n$-dimensional continua embeddable into $\mathbb{R}^m$, and so for all $n$-dimensional continua if $m \geq 2n+1$. $M_1^2$ is the *Sierpiński universal curve* or the *Sierpiński carpet*; $M_1^3$ is the *Menger universal curve* or the *Menger sponge.* $M_0^m$ for $0 < m$ is the *Cantor space*, which is universal for all zero-dimensional spaces and also a common model for all compacta. The *Ważewski dendrite* $D_\omega$ [39, 10.37] is universal for all dendrites. The *pseudo-arc* is co-universal in the class of all chainable continua (as well as all weakly chainable continua). This classical result (together with more properties of the pseudo-arc) was recently proved using *projective Fraïssé theory* [25], [31]. For more examples see [16, §9].

There are also results on non-existence of $\mathcal{F}$-universal continua, sometimes related to the existence of a large *$\mathcal{F}$-incomparable family.* A family of continua $\mathcal{P} \subseteq \mathcal{C}$ is *$\mathcal{F}$-incomparable* if for every $X \neq Y \in \mathcal{C}$ we have neither $X \to_{\mathcal{F}} Y$ nor $Y \to_{\mathcal{F}} X$. Again, we say just "*incomparable*" if $\mathcal{F}$ is the family of all quotients. There is a classical result of Waraszkiewicz [53], [54] who constructed an incomparable family of size $\mathfrak{c}$ with no common model consisting of spirals over the circle. (By a *spiral over a continuum* $X$ we mean a compactification of the ray $[0,\infty)$ with $X$ as the remainder.) It follows that the class of all continua has no co-universal element. A shorter proof of the incomparability is given in [42]. Russo [45] constructed a $\mathfrak{c}$-sized family with no common model consisting of spirals over the simple triod. Bellamy [9] constructed a $\mathfrak{c}$-sized incomparable family of chainable continua. The chainable continua were sequences of attached "double-spirals over double-spirals" and had infinitely many path-components. Later, Awartani [5] found a $\mathfrak{c}$-sized incomparable family of spirals over the arc. Note that spirals over the arc are chainable, and so have a common model. In [7] we have generalized the construction from [42] and obtained a $\mathfrak{c}$-sized incomparable family of spirals over $X$ for any fixed nondegenerate Peano continuum other than the arc, so with the Awartani's result all nondegenerate Peano continua are covered. As for arcwise connected continua, Minc [37] and Islas [26] have constructed $\mathfrak{c}$-sized incomparable families of fans. The family of Minc consists of retracts of a single fan, while the family of Islas consists of planar fans based on the Awartani's construction.

Note that every incomparable family consists of pairwise non-homeomorphic continua. A *non-homeomorphic family* is an $\mathcal{F}$-incomparable family for $\mathcal{F}$ being the family of all homeomorphisms (note that the preorder $\to_{\mathcal{F}}$ is an equivalence in this case). Recall that the pseudo-arc is the unique *hereditarily indecomposable* chainable continuum [12]. We have already mentioned that it is a co-universal chainable continuum. By the result of Bellamy [10], every nondegenerate hereditarily indecomposable continuum maps onto the pseudo-arc (i.e. the pseudo-arc is the $\leq_{\mathcal{F}}$-least nondegenerate hereditarily indecomposable continuum for the family $\mathcal{F}$ of all quotients). Moreover, by [24] for every spiral over a hereditarily indecomposable continuum $X$ and every spiral over a chainable continuum $Y$, every continuous surjection $X \twoheadrightarrow Y$ can be extended to a continuous surjection between the spirals. Hence, an incomparable family of spirals over the pseudo-arc is impossible. On the other hand, Martínez-de-la-Vega [51] constructed a $\mathfrak{c}$-sized non-homeomorphic family of spirals over the pseudo-arc. This was generalized by Martínez-de-la-Vega and Minc [52], who obtained an uncountable non-homeomorphic family of spirals over any fixed nondegenerate continuum $X$, and later by Minc [38] who obtained such a family of size $\mathfrak{c}$.

Minc in fact constructed a (metrizable) compactum whose set of components is the desired non-homeomorphic family of spirals, and the corresponding quotient space is homeomorphic to the Cantor space. He also asked about the dual situation – whether there is a compactum whose components are, up to homeomorphism and allowing multiple copies, all spirals over a fixed Peano continuum. In Chapter III we are interested in the general form of this question. We say that a class of continua $\mathcal{C}$ is *compactifiable* if there is a compactum whose set of components is equivalent to $\mathcal{C}$. (We call two classes of spaces *equivalent* if every member of one class has a homeomorphic copy in the other class and vice versa.) It turns out (Chapter III, Observation 2.12) that a class of continua $\mathcal{C}$ is compactifiable if and only if there is a continuous map $q\colon A \to B$ between some compacta $A$ and $B$ such that the family of fibers $\{q^{-1}(b) : b \in B\}$ is equivalent to $\mathcal{C}$. This condition may be easily generalized, so we define compactifiable classes of compacta in the obvious way. We also define *Polishable* classes by a weaker condition – it is enough if the witnessing spaces $A$ and $B$ are Polish. Moreover, we define *strongly compactifiable* and *strongly Polishable* classes by the extra requirement that the map $q$ is closed and open. The motivation for these modified notions is their close connection to *hyperspaces*.

For a metrizable space $X$, the *hyperspace* $\mathcal{C}(X)$ is the family of all subcontinua of $X$ endowed with the *Hausdorff metric* and the *Vietoris topology*. The hyperspace $\mathcal{K}(X)$ of all compacta in $X$ is defined analogously. More details on the definition and some properties of hyperspaces are summarized in Chapter III, Section 3. Since the Hilbert cube $[0,1]^\omega$ is universal for metrizable compacta, every class of compacta may be realized by an equivalent family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$. Having such a realization, we may talk about its topological properties and about its complexity with respect to the Borel hierarchy. We prove (see Theorem 3.13 and Theorem 3.14) that a class of compacta $\mathcal{C}$ is strongly compactifiable if and only if it can be realized by a closed family, and that $\mathcal{C}$ is strongly Polishable if and only if it can be realized by a $G_\delta$ family or equivalently by an analytic family. Note that a fixed class $\mathcal{C}$ may be realized by families of different complexities. In the results above, we are interested how low may the complexity be. It is not hard to show that compactifiability, strong Polishability, and Polishability is preserved by countable unions (Observation 2.14, Proposition 4.1). On the other hand, the corresponding result for strongly compactifiable classes, i.e. the fact that every $F_\sigma$ subset of $\mathcal{K}([0,1]^\omega)$ is equivalent to a closed set, led to a separate paper – Chapter IV.

Note that hyperspaces fit to the framework from the beginning of this section by considering $\mathcal{C}(X)$ as the class of continua $\mathcal{C}$ and the family of all inclusions as the family $\mathcal{F}$. Furthermore, the resulting structure is endowed with the Vietoris topology. The structure $\langle \mathcal{C}, \mathcal{F} \rangle$ for a class of continuous surjections $\mathcal{F}$ may be endowed with yet another topology. A continuous surjection $f\colon X \to Y$ is called an *$\varepsilon$-map* for some $\varepsilon > 0$ if all the fibers $f^{-1}(y)$ for $y \in Y$ have diameter $< \varepsilon$. The closure operator $\mathrm{cl}_\mathcal{F}$ is defined by $X \in \mathrm{cl}_\mathcal{F}(\mathcal{P})$ for $X \in \mathcal{C}$ and $\mathcal{P} \subseteq \mathcal{C}$ if for every $\varepsilon > 0$ there is an $\varepsilon$-map $f\colon X \to Y$ for some $Y \in \mathcal{P}$ and $f \in \mathcal{F}$, i.e. $\mathrm{cl}_\mathcal{F}(\mathcal{P})$ consists of all *$\mathcal{P}$-like* spaces (with respect to $\mathcal{F}$). Note that being an $\varepsilon$-map depends on the choice of metric on the domain space $X$, but the closure $\mathrm{cl}_\mathcal{F}$ does not since all metrics on a compact space are uniformly equivalent. The operator $\mathrm{cl}_\mathcal{F}$ is indeed a topological closure, and if $\mathcal{C}$ is a set containing exactly one homeomorphic copy of each continuum, the resulting topological space is called the *representation space for $\mathcal{F}$*. Representation spaces were introduced in [2], and the families of all, all confluent, and all monotone continuous surjections were considered. Clearly, with more maps in the family $\mathcal{F}$, the topology of the corresponding representation space becomes coarser. Even for the coarsest topology (the one corresponding to the family of all continuous surjections) the degenerate spaces are clopen points. That is why the representation space $\mathcal{N}$ consisting of all nondegenerate continua is often considered. It is proved in [18] that with respect to all continuous surjections, $\mathcal{N}$ is connected and contains a dense point. The representation space for confluent maps is further studied in [3]. Let us mention some more results regarding the representation spaces. The representation space (for any $\mathcal{F}$) has countable character but weight $\mathfrak{c}$. Every incomparable family is discrete, and so the Waraszkiewicz spirals form a discrete set of size $\mathfrak{c}$. The representation space is not $T_0$ even for monotone maps. For all surjections, the family of all chainable continua correspond to a closed nowhere dense subset, and the arc is its dense point. On the other hand, for confluent maps, the arc is an isolated point, and its closure is the family of all *Knaster type continua*. For monotone maps, the arc is a clopen point. The Peano continua form an open dense set for all surjections; the closure for confluent maps is strictly bellow the family of all *Kelley continua*.

The property that a continuum $X$ is $\mathcal{P}$-like (with respect to $\mathcal{F}$) is closely related to the condition that there is an inverse sequence of spaces from $\mathcal{P}$ with bonding maps from $\mathcal{F}$ such that $X$ is its limit – let us call this other condition "*sequentially $\mathcal{P}$-like*". If for every inverse sequence of maps from $\mathcal{F}$ the corresponding limit maps are also from $\mathcal{F}$ (which happens for all families considered here), every sequentially $\mathcal{P}$-like space is $\mathcal{P}$-like. The two conditions are equivalent if $\mathcal{P}$ is a class of *polyhedra* and $\mathcal{F}$ is the class of all continuous surjections [33]. Sometimes the term "$\mathcal{P}$-like" is even defined as sequentially $\mathcal{P}$-like. A continuum is called *arc-like* if it is {arc}-like, and this is equivalent to being chainable; *circle-like* continua are defined analogously, and the condition is equivalent to being *circularly chainable*. We have already mentioned that the pseudo-arc is a co-universal arc-like continuum. There is also a co-universal circle-like continuum [44], sometimes called *the pseudo-solenoid*, and since the pseudo-arc is a continuous image of a circle-like continuum, the pseudo-solenoid is also co-universal for {arc, circle}-like continua. Russo [45] proved that for any other family $\mathcal{P}$ of nondegenerate connected polyhedra, the family of all $\mathcal{P}$-like continua has no common model. Besides the co-universal arc-like continuum, there is also a universal arc-like continuum [39, Theorem 12.22]. We adapt its construction and show that for every countable class of compacta $\mathcal{P}$, the class of all sequentially $\mathcal{P}$-like compacta is compactifiable. In particular, the class of all circle-like continua is compactifiable, even though there is no universal circle-like continuum (Chapter III, Remark 5.9). With a grain of salt, we may say that the closure of a countable class in the representation space is a compactifiable class.

To summarize our results, in Chapter III we define the main notions – (strongly) compactifiable and (strongly) Polishable classes and their witnessing objects, called *compositions* – and we systematically study their properties. We establish several characterizations of the properties – in the language of *rectangular compositions* (Theorem 2.10, Theorem 2.11, Theorem 3.18), and in terms of existence of a suitable family in a hyperspace (Theorem 3.13, Theorem 3.14, Theorem 3.22). We also prove that every compactifiable class is strongly Polishable (Corollary 3.17), and that sometimes a strongly Polishable class is compactifiable (Corollary 3.21). In Section 4 we study preservation of the properties under various constructions, and we demonstrate the obtained results on several examples. Among other results, we prove the following. All four properties are preserved under countable unions (Proposition 4.1). Every hereditary class of compacta with a universal element is strongly compactifiable (Corollary 4.10). Every class of continua closed under continuous images with a co-universal element is compactifiable (Corollary 4.25). For every strongly Polishable class of compacta, the family of all homeomorphic copies of its members in $\mathcal{K}(X)$ is analytic for any Polish space $X$ (Theorem 4.26) – this gives a necessary condition. In Section 5 we see how compactifiability is related to inverse limits and prove the already mentioned result that the class of all sequentially $\mathcal{P}$-like spaces for a countable family $\mathcal{P}$ of compacta is compactifiable (Theorem 5.7).

Chapter IV is devoted to the Borel complexity in the universal hyperspace $\mathcal{K}([0,1]^\omega)$ up to the equivalence of families. We have observed that every analytic family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ is equivalent to a $G_\delta$ family. In Section 2, we show that there are only countably many open saturated families in $\mathcal{K}([0,1]^\omega)$ and that every open family is equivalent to (exactly) one of them (Theorem 2.18). To every compactum $X$ we associate a its type $t(X)$ depending on the number of components and the number of nondegenerate components. The set of types $T \cup \{\infty\}$ is naturally ordered, and we use the theory of *Z-sets* [35, §5.1] to prove that for $X, Y \in \mathcal{K}([0,1]^\omega)$ a homeomorphic copy of $Y$ is in every neighborhood of $X$ if and only if $t(Y) \geq t(X)$ (Proposition 2.16). In Section 3 we extend our analysis of the hyperspace neighborhoods from single spaces to compact families, and we prove our main result of the chapter – every $F_\sigma$ family in $\mathcal{K}([0,1]^\omega)$ is equivalent to a closed family (Theorem 3.6). We end in Section 4 with several observations on *saturated* and *type-saturated* families. For example, the *saturation* (i.e. the closure under homeomorphic copies) of a clopen, open, or analytic family remains clopen, open, or analytic, respectively, but the saturation of a closed family is almost never closed, and sometimes is not even Borel (Corollary 4.4). The equivalence of complexity classes is summarized in Figure 1.

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# II. Tree sums of maximal connected spaces

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*Dedicated to the memory of Petr Simon,*
*who introduced me to the beautiful realm of general topology.*

**Abstract**

A topology $\tau$ on a set $X$ is called maximal connected if it is connected, but no strictly finer topology $\tau^* > \tau$ is connected. We consider a construction of so-called tree sums of topological spaces, and we show how this construction preserves maximal connectedness and also related properties of strong connectedness and essential connectedness.

We also recall the characterization of finitely generated maximal connected spaces and reformulate it in the language of specialization preorder and graphs, from which it is clear that finitely generated maximal connected spaces are precisely $T_{\frac{1}{2}}$-compatible tree sums of copies of the Sierpiński space.

*Classification:* 54A10, 54D05, 54B17, 54D10, 54G15.

*Keywords:* maximal connected, strongly connected, essentially connected, tree sum, I-subset, submaximal, nodec, specialization preorder.

## 1 Introduction

For every fixed set $X$ we may consider the collection of all topologies on $X$. These form a complete lattice $\mathcal{T}(X)$ when ordered by inclusion. For every topological property $\mathcal{P}$ we may consider the subcollection of $\mathcal{T}(X)$ consisting of all topologies having the property $\mathcal{P}$. Then we may consider maximal and minimal elements of this collection. A topology $\tau \in \mathcal{T}(X)$ is called *maximal* $\mathcal{P}$ if it satisfies $\mathcal{P}$ but no strictly finer topology in $\mathcal{T}(X)$ satisfies $\mathcal{P}$. The property of being *minimal* $\mathcal{P}$ is defined dually. Often, the maximality is considered when $\mathcal{P}$ is stable under coarser topologies, and minimality is considered when $\mathcal{P}$ is stable under finer topologies.

Probably the most classical result in this context is the fact that Hausdorff compact spaces are both minimal Hausdorff and maximal compact (but not every minimal Hausdorff space is compact and not every maximal compact space is Hausdorff). References, many other properties, and a general treatment can be found in a paper by Cameron [3]. There are also *maximal* spaces where the implicit property $\mathcal{P}$ is "having no isolated points". Let us mention van Douwen's example of countable regular maximal space that can be found in [4].

We are interested in the situation where $\mathcal{P}$ means connectedness, i.e. in *maximal connected* spaces. These were first considered by Thomas in [14], where he

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proved among other results that every open connected subspace of a maximal connected space is maximal connected, and also characterized finitely generated maximal connected spaces. There are also related notions of *strongly connected* and *essentially connected* spaces. Following Cameron, for a topological property $\mathcal{P}$ when we consider being maximal $\mathcal{P}$, we say that a topological space is *strongly* $\mathcal{P}$ if it admits finer maximal $\mathcal{P}$ topology. Essentially connected are those connected spaces whose every connected expansion has the same connected subsets – these spaces were considered by Guthrie and Stone in [7].

In this paper we first recall the facts about maximal, strongly, and essentially connected spaces that we use later. Clearly the construction of topological sum does not preserve the properties since it does not preserve connectedness. In the second section we consider another sum-like construction – a *tree sum*. It is a certain quotient of a topological sum – such quotient that it preserves the original spaces as subspaces, glues them only at individual points, and the overall structure of gluing corresponds to a tree graph. First, we systematically treat the properties of tree sums of topological spaces, so we may next show how this construction preserves maximal, strong, and essential connectedness (Theorem 2.44 and 2.53).

In the third section we revise Thomas' characterization of finitely generated maximal connected spaces. We describe them in the language of specialization preorder and graphs. With this description their structure is crystal clear, they can be easily visualized, and it is evident that they are exactly $T_{\frac{1}{2}}$-compatible tree sums of copies of the Sierpiński space (Corollary 3.14).

**Definition 1.1.** Let $X$ be a set or a topological space. We say that $X$ is *degenerate* if $|X| \leq 1$. Otherwise, we say that $X$ is *nondegenerate*.

By a *decomposition* of $X$ we mean an indexed family $\langle A_i : i \in I \rangle$ of subsets of $X$ such that $\bigcup_{i \in I} A_i = X$ and $A_i \cap A_j = \emptyset$ for every $i \neq j \in I$. If additionally every set $A_i$ is nonempty, we say that the decomposition is *proper*. Hence, a topological space is connected if and only if it admits no clopen proper decomposition $\langle U, V \rangle$.

**Notation 1.2.** Let $X$ be a set. The order of the lattice $\mathcal{T}(X)$ of all topologies on $X$ is denoted simply by $\leq$. So $\tau \leq \tau^*$ means that $\tau$ is coarser and $\tau^*$ is finer. Also, $\tau < \tau^*$ means that $\tau^*$ is *strictly* finer than $\tau$. Additionally, when $\tau \leq \tau^*$ (or $\tau < \tau^*$), we say that the topology $\tau^*$ is an *expansion* of $\tau$ (or a *strict expansion* of $\tau$). In that case we also say that the space $\langle X, \tau^* \rangle$ is an expansion of $\langle X, \tau \rangle$.

The join operation on $\mathcal{T}(X)$ is denoted by $\vee$ and is extended to all subsystems of $\mathcal{P}(X)$, so $\mathcal{A} \vee \mathcal{B}$ denotes the topology generated by $\mathcal{A} \cup \mathcal{B}$ for any $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$. Hence, for $\tau$ a topology on $X$ and $\mathcal{A} \subseteq \mathcal{P}(X)$ the expansion of $\tau$ by $\mathcal{A}$ is denoted by $\tau \vee \mathcal{A}$. For $A \subseteq X$ the expansion $\tau \vee \{A\}$ is called a *simple expansion* of $\tau$.

For a topology $\tau \in \mathcal{T}(X)$ and a set $Y \subseteq X$ the induced subspace topology on $Y$ is denoted by $\tau{\restriction}_Y$.

**Definition 1.3.** Recall that a topological space $\langle X, \tau \rangle$ or its topology $\tau$ is called

- *maximal connected* if it is connected and has no connected strict expansion;
- *strongly connected* if it has a maximal connected expansion;

- *essentially connected* if it is connected and every connected expansion has the same connected subsets.

**Observation 1.4.** When testing maximal or essential connectedness, it is enough to consider only expansions by finite families. Let $\langle X, \tau \rangle$ be a connected topological space and let $\tau^*$ be a connected expansion of $\tau$.

(i) If $A \in \tau^* \setminus \tau$, then $\tau' := \tau \vee \{A\}$ is also a connected expansion of $\tau$. Hence, $\tau$ is maximal connected if and only if it has no connected strict simple expansion. Also note that every $\tau'$-open set is of the form $U \cup (A \cap V)$ and also of the form $(U' \cup A) \cap V'$ where $U$, $V$, $U'$, $V'$ are $\tau$-open.

(ii) If $C \subseteq X$ is not $\tau^*$-connected, then there are $\tau^*$-open sets $U, V \subseteq X$ such that $\langle C \cap U, C \cap V \rangle$ is a proper decomposition of $C$. Hence, for $\tau' := \tau \vee \{U, V\}$ we have that $C$ is $\tau'$-disconnected while $\tau'$ is connected. Therefore, it is enough to test essential connectedness on expansions by two of sets.

The following lemmata provide conditions to test whether a set open in an expansion is open in the original topology as well, and whether a subspace of an expansion is a subspace of the original space as well.

**Lemma 1.5.** Let $\langle X, \tau \rangle$ be a topological space, let $\tau^* = \tau \vee \mathcal{A}$ be an expansion of $\tau$ for some $\mathcal{A} \subseteq \mathcal{P}(X)$. If a set $U$ is $\tau^*$-open, then the following conditions are equivalent.

(i) $U$ is $\tau$-open.

(ii) For every $A \in \mathcal{A}$ there is a $\tau$-open set $V_A$ such that $U \cap A \subseteq V_A \subseteq U \cup A$.

*Proof.* For "$\Longrightarrow$" it is enough to put $V_A := U$. For "$\Longleftarrow$" note that the set $U$ is of the form $\bigcup_{i \in I}(W_i \cap \bigcap_{j \in J_i} A_{i,j})$ where the sets $W_i$ are $\tau$-open, the sets $A_{i,j}$ are members of $\mathcal{A}$, and the index sets $J_i$ are finite. Consider the function $f$ that maps every indexed family $\langle X_{i,j} : i \in I, j \in J_i \rangle$ of subsets of $X$ to the set $\bigcup_{i \in I}(W_i \cap \bigcap_{j \in J_i} X_{i,j})$. Clearly, $f$ is monotone in the sense that for every two families $\langle X_{i,j} \rangle$, $\langle Y_{i,j} \rangle$ such that $X_{i,j} \subseteq Y_{i,j}$ for every $i, j$ we have $f(\langle X_{i,j} \rangle) \subseteq f(\langle Y_{i,j} \rangle)$. Note that $f(\langle U \cap A_{i,j} \rangle) = f(\langle U \cup A_{i,j} \rangle) = U$. Therefore, the $\tau$-open set $f(\langle V_{A_{i,j}} \rangle)$ is also equal to $U$. □

**Corollary 1.6.** Let $\langle X, \tau \rangle$ be a topological space, let $\tau^* = \tau \vee \mathcal{A}$ be an expansion of $\tau$ for some $\mathcal{A} \subseteq \mathcal{P}(X)$. A $\tau^*$-open set $U$ is $\tau$-open if for every $A \in \mathcal{A}$ any of the following conditions holds.

(i) $U \cap A$ is $\tau$-open, in particular $U \cap A = \emptyset$.

(ii) $U \cup A$ is $\tau$-open, in particular $U \cup A = X$.

(iii) There is a $\tau$-open set $V$ such that $U \supseteq V \supseteq A$.

(iv) There is a $\tau$-open set $V$ such that $U \subseteq V \subseteq A$.

**Lemma 1.7.** Let $\langle X, \tau \rangle$ be a topological space, let $\tau^* = \tau \vee \mathcal{A}$ be an expansion of $\tau$ for some $\mathcal{A} \subseteq \mathcal{P}(X)$. If $Y \subseteq X$, then $\tau^* \restriction_Y = \tau \restriction_Y$ if and only if $Y \cap A$ is $\tau$-open in $Y$ for every $A \in \mathcal{A}$, in particular if for every $A \in \mathcal{A}$ we have $Y \cap A = \emptyset$ or $Y \subseteq A$.

*Proof.* "$\Longrightarrow$" is obvious since $Y \cap A$ is $\tau^*$-open in $Y$ for every $A$. For "$\Longleftarrow$" let $U \subseteq Y$ be $(\tau^* {\restriction}_Y)$-open, and for every $A \in \mathcal{A}$ let $V_A \subseteq X$ be a $\tau$-open set such that $V_A \cap Y = A \cap Y$. There are $\tau$-open sets $W_i$, finite sets $J_i$, and sets $A_{i,j} \in \mathcal{A}$ for $i \in I$, $j \in I_j$ such that $U = \bigcup_{i \in I}(Y \cap W_i \cap \bigcap_{j \in J_i} A_{i,j}) = \bigcup_{i \in I}(Y \cap W_i \cap \bigcap_{j \in J_i} V_{A_{i,j}})$, which is $(\tau {\restriction}_Y)$-open. □

**Definition 1.8.** Recall that a topological space $X$ is called

- *submaximal* if every dense subset is open, equivalently if every co-dense subset is closed (and so discrete), equivalently if $\overline{A} \setminus A$ is closed for every $A \subseteq X$;
- *nodec* if every nowhere dense subset is closed, equivalently if every nowhere dense subset is discrete, equivalently if $\overline{U} \setminus U$ is discrete for every open $U \subseteq X$;
- $T_{\frac{1}{2}}$ if the singleton $\{x\}$ is open or closed for every $x \in X$.

Note that co-dense sets are exactly sets of the form $\overline{A} \setminus A$ for $A \subseteq X$, and that closed nowhere dense sets are exactly sets of the form $\overline{U} \setminus U$ for open $U \subseteq X$.

Submaximal spaces without isolated points were introduced by Hewitt, who called them MI-spaces. The name submaximal is due to Bourbaki. Many references and an excellent overview can be found in [2]. Nodec spaces were considered by van Douwen in [4, 1.14]. $T_{\frac{1}{2}}$ spaces were introduced by McSherry in [12] under name $T_{ES}$. The name $T_{\frac{1}{2}}$ comes from Levine, who earlier introduced a different but equivalent condition in [11].

**Proposition 1.9.** All implications in Figure 1 hold.

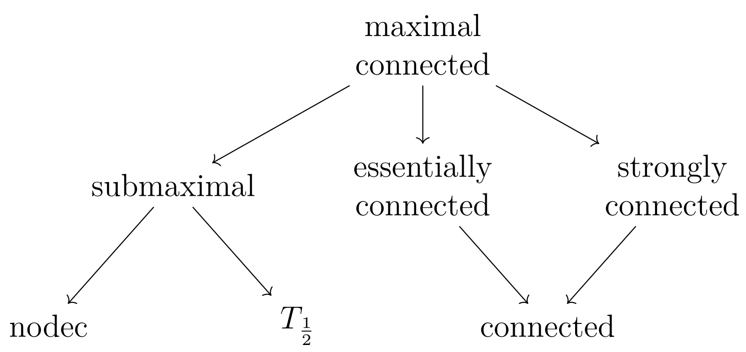



Figure 1: Implications between the properties considered.

*Proof.* Every maximal connected space is submaximal since making any dense subset open (or even a filter of dense subsets [1, Lemma 1]) preserves connectedness. Every submaximal space is $T_{\frac{1}{2}}$ since every singleton is either open or co-dense and hence closed. The other implications are clear from definitions. □

## 1.1 Preservation under subspaces

**Observation 1.10.** The properties of being submaximal, nodec, or $T_{\frac{1}{2}}$ are hereditary to all subspaces.

*Proof.* This was proved before, see for example [2, Proposition 2.1]. It is enough to observe that every co-dense, nowhere dense, or one-point subset of a subspace is co-dense, nowhere dense, or one-point in the original space as well, respectively. □

Clearly, the properties of maximal connectedness, strong connectedness, and essential connectedness can be preserved only by connected subspaces. Thomas proved in [14, Theorem 3] that maximal connectedness is hereditary with respect to open connected subspaces. Later, Guthrie, Reynolds, and Stone proved the same first for closed connected subspaces in [6, Lemma 2], and then using sub-maximality they observed that every connected subspace of a maximal connected space is open in its closure, and so is itself maximal connected [6, Theorem 7]. In [7, Theorem 1] Guthrie and Stone proved that essential connectedness is hereditary with respect to connected subspaces as well. The core argument of the proofs can be stated as follows.

**Lemma 1.11.** Let $\langle Y, \sigma \rangle$ be a subspace of a connected space $\langle X, \tau \rangle$. For every connected expansion $\sigma^* \geq \sigma$ there exists a connected expansion $\tau^* \geq \tau$ such that $\tau^* \restriction_Y = \sigma^*$.

*Proof.* We put $\tau^* := \tau \vee \mathcal{A}$ where $\mathcal{A} := \{S \cup (X \setminus \overline{Y}) : S \in \sigma^*\}$. Clearly, $\tau^*$ is an expansion of $\tau$ such that $\tau^* \restriction_Y = \sigma^*$. We need to show that it is connected. $\overline{Y}$ is $\tau^*$-connected since $Y$ is $\tau^*$-connected and $\overline{Y} = \mathrm{cl}_\tau(Y) = \mathrm{cl}_{\tau^*}(Y)$. Let $\langle U, V \rangle$ be a $\tau^*$-clopen decomposition of $X$. Without loss of generality $\overline{Y} \subseteq U$. $U$ is $\tau$-open by Corollary 1.6 since $U \cup A = X$ for every $A \in \mathcal{A}$. Let $W$ be the $\tau$-open set $X \setminus \overline{Y}$. $V$ is $\tau$-open by Corollary 1.6 since $V \subseteq W \subseteq A$ for every $A \in \mathcal{A}$. Hence, $\langle U, V \rangle$ is a $\tau$-clopen decomposition of $X$, so $U = \emptyset$ or $V = \emptyset$ since $\tau$ is connected. □

Now, the above-mentioned results on preservation under connected subspaces can be re-proved easily.

**Proposition 1.12.**

(i) Every connected subspace of a maximal connected space is maximal connected.

(ii) Every connected subspace of an essentially connected space is essentially connected.

(iii) Every connected subspace of a strongly connected and essentially connected space is both strongly connected and essentially connected.

*Proof.* Let $\langle X, \tau \rangle$ be a topological space and let $\langle Y, \sigma \rangle$ be its connected subspace.

(i) Let $\sigma^*$ be a connected expansion of $\sigma$. By Lemma 1.11 there is a connected expansion $\tau^* \geq \tau$ such that $\tau^* \restriction_Y = \sigma^*$. Since $\tau$ is maximal connected, we have that $\tau^* = \tau$ and so $\sigma^* = \sigma$.

(ii) Let $C \subseteq Y$ be connected and let $\sigma^*$ be a connected expansion of $\sigma$. By Lemma 1.11 there is $\tau^*$ a connected expansion of $\tau$ such that $\tau^* \restriction_Y = \sigma^*$. $C$ is $\tau^*$-connected since $\tau$ is essentially connected, and hence $C$ is $\sigma^*$-connected.

(iii) Let $\tau^*$ be a maximal connected expansion of $\tau$. Since $\tau$ is essentially connected, $Y$ is $\tau^*$-connected, and hence $\tau^*\restriction_Y$ is a maximal connected expansion of $\sigma$ by (i). $\square$

**Example 1.13.** Not every connected subspace of a strongly connected space is strongly connected. By [7, Theorem 15] no Hausdorff connected space with a dispersion point is strongly connected. Cantor's leaky tent is such a space. Yet, it is a subspace of $\mathbb{R}^2$, which is strongly connected by [8, Corollary 5A] and also by Corollary 2.45.

**Observation 1.14.** The interval $[0, 1]$ is both strongly connected and essentially connected. Hence, the same holds for the real line $\mathbb{R}$ and the interval $[0, 1)$.

*Proof.* The fact that $[0, 1]$ is essentially connected was first proved in [9, Theorem 4.2] and also follows from [7, Theorem 10]. A maximal connected expansion of $[0, 1]$ was constructed independently in [13] and [8]. The equivalence of $\mathbb{R}$, $[0, 1]$, and $[0, 1)$ with respect to having the properties follows from Proposition 1.12 and from the fact that $\mathbb{R} \hookrightarrow [0, 1) \hookrightarrow [0, 1] \hookrightarrow \mathbb{R}$. $\square$

# 2 Tree sums of topological spaces

**Definition 2.1.** By a *gluing structure* $\mathcal{G}$ we mean an indexed family of topological spaces $\langle X_i : i \in I\rangle$ together with an equivalence $\sim$ on $\sum_{i\in I} X_i$. These are exactly the data needed to form a *glued sum* $X_\mathcal{G} := \sum_{i\in I} X_i/\sim$, which is a quotient of the topological sum. We denote the associated canonical maps $X_i \to X_\mathcal{G}$ by $e_{\mathcal{G},i}$ and the canonical quotient map $\sum_{i\in I} X_i \to X_\mathcal{G}$ by $q_\mathcal{G}$.

We define the *set of gluing points* by $S_\mathcal{G} := \{x \in X_\mathcal{G} : |q_\mathcal{G}^{-1}(x)| > 1\}$, and we define the *gluing graph* $G_\mathcal{G}$ as the quiver (a directed graph allowing multiple edges between a pair of vertices) such that the set of vertices is $I \sqcup S_\mathcal{G}$, and $\langle s, i, x\rangle$ is a directed edge from $s$ to $i$ if and only if $e_{\mathcal{G},i}(x) = s$. Even though the edges are directed in order to stress the bipartite nature of the graph, we consider graph notions like connectedness or paths in the corresponding undirected version of the graph if not stated otherwise.

We say that $\mathcal{G}$ *induces a tree sum* if the corresponding gluing graph $G_\mathcal{G}$ is a tree, i.e. for every pair of distinct vertices there is a unique path connecting them. In that case, $X_\mathcal{G}$ is called the *tree sum* of $\mathcal{G}$ and the spaces $X_i$ are called *summands*.

Often, when the gluing structure is implied, we just write "$X$ is a glued/tree sum of $\langle X_i : i \in I\rangle$", or "$X := \sum_{i\in I} X_i/\sim$ is a glued/tree sum" when we want to name the equivalence. In that case, we write $e_{X,i}$, $q_X$, $S_X$, $G_X$ or even $e_i$, $q$, $S$, $G$ (with a short reminder) instead of $e_{\mathcal{G},i}$, $q_\mathcal{G}$, $S_\mathcal{G}$, $G_\mathcal{G}$, respectively.

**Remark 2.2.** Despite the lengthy definition above, the notion of tree sum is quite natural. We just glue topological spaces in a way that the spaces are preserved, two spaces may be glued only at one point, and the global structure of connections forms a tree.

**Remark 2.3.** Because of the connectedness of the gluing graph, all the summands of a tree sum have to be nonempty unless the whole space is empty.

**Example 2.4.** A wedge sum, that is a space $\sum_{i \in I} X_i/{\sim}$ such that one point is chosen in each space $X_i$ and $\sim$ glues these points together, is an example of a tree sum.

**Example 2.5.** The Arens' space, which is the canonical example of a sequential space that is not Fréchet–Urysohn (see [5, Example 1.6.19]), is a certain tree sum of convergent sequences.

**Observation 2.6.** Let $X := \sum_{i \in I} X_i/{\sim}$ be a glued sum. All the maps $e_i \colon X_i \to X$ are injective if and only if there is at most one edge between any two vertices in $G_X$.

*Proof.* For $i \in I$ the map $e_i$ is injective if and only if there are no points $x \neq y \in X_i$ such that $e_i(x) = e_i(y) \in S_X$, that is if and only if there are no points $x \neq y \in X_i$ and $s \in S_X$ such that $\langle s, i, x \rangle$ and $\langle s, i, y \rangle$ are edges in $G_X$. □

**Proposition 2.7.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$. There exist unique maps $q_i \colon X \to X_i$ for $i \in I$ such that $q_i \circ e_j = \mathrm{id}_{X_{i=j}}$ if $i = j$ and $q_i \circ e_j$ is constant if $i \neq j$. Hence, all the maps $e_i$ are embeddings, all the maps $q_i$ are quotients (even retractions), and all the spaces $X_i$ are retracts of $X$.

*Proof.* By Observation 2.6 all the maps $e_i$ are injective, and hence we may assume $X_i \subseteq X$ (as sets) for every $i \in I$.

Let $i \in I$. The map $q_i$ has to be the identity on $X_i$, and for $j \neq i$ it has to be constant on $X_j$. There exists a unique path between $i$ and $j$ in $G_X$, which goes through vertices $i, s_0, i_0, s_1, \ldots, i_{n-1}, s_n, j$. Since $s_n \in X_j$, the constant value of $q_i$ on $X_j$ has to be $q_i(s_n)$, but since $s_n \in X_{i_{n-1}} \ni s_{n-1}$, the constant value of $q_i$ on $X_{i_{n-1}}$ has to be the same and also equal to $q_i(s_{n-1})$, and so on. Hence, the constant value on $X_j$ is $q_i(s_0) = s_0$. This is a consistent definition of $q_i$, and it is the only possible.

Since the topology on $X$ is inductively generated by the maps $e_i$, and $q_i \circ e_j$ is continuous for every $i, j \in I$, all the maps $q_i$ are continuous. And since $q_i \circ e_i = \mathrm{id}_{X_i}$, we have that $q_i$ is a quotient map and $e_i$ is an embedding. □

## 2.1 Internal characterization

We dedicate a few following paragraphs to an internal characterization of tree sums. Similarly to other sum-like constructions it makes sense to ask if a given topological space is an "inner" tree sum of some of its subspaces.

**Definition 2.8.** Let $f \colon X \to Y$ be a continuous map between topological spaces $X$, $Y$. Recall, that there is an induced equivalence relation $\sim_f$ on $X$: $x \sim_f y$ if and only if $f(x) = f(y)$. There is an induced quotient map $f^{\mathrm{q}} \colon X \to X/{\sim_f}$ and an induced map $f^{\mathrm{i}} \colon X/{\sim_f} \to Y$ such that $f^{\mathrm{i}} \circ f^{\mathrm{q}} = f$. We call $f^{\mathrm{q}}$ and $f^{\mathrm{i}}$ the *quotient part* of $f$ and the *injective part* of $f$, respectively.

Let $\langle f_i\colon X_i \to Y\rangle_{i\in I}$ be a family of continuous maps. Recall, that there is a canonical map $\nabla_{i\in I} f_i\colon \sum_{i\in I} X_i \to Y$ called the *codiagonal sum* and defined by the equalities $(\nabla_{i\in I} f_i)\circ e_j = f_j$ for $j \in I$ where $e_j\colon X_j \to \sum_{i\in I} X_i$ are the canonical embeddings.

**Definition 2.9.** Let $X$ be a topological space and $\mathcal{F} := \langle X_i : i \in I\rangle$ a family of its subspaces. The *gluing structure induced by* $\mathcal{F}$ is $\mathcal{G} := \langle \mathcal{F}, \sim_f\rangle$ where $f := \nabla_{i\in I} e_i\colon \sum_{i\in I} X_i \to X$ and $e_i\colon X_i \to X$ are the embeddings for $i \in I$. The gluing structure $\mathcal{G}$ induces the glued sum $X_{\mathcal{G}} = \sum_{i\in I} X_i/{\sim_f}$ by Definition 2.1. We say that $\langle X, \mathcal{F}\rangle$ is an *inner tree sum*, or that $X$ is an *inner tree sum of* $\mathcal{F}$, if $\mathcal{G}$ induces a tree sum and $f^{\mathrm{i}}\colon X_{\mathcal{G}} \to X$ is a homeomorphism.

The family $\mathcal{F}$ also induces a set $S_{\mathcal{F}} := \{x \in X : |\{i \in I : x \in X_i\}| \geq 2\}$ and a graph $G_{\mathcal{F}}$ on $I \sqcup S_{\mathcal{F}}$ where $\langle s, i\rangle$ is an edge from $s \in S_{\mathcal{F}}$ to $i \in I$ if and only if $s \in X_i$. Note that $S_{\mathcal{F}}$ and $G_{\mathcal{F}}$ are canonically isomorphic to $S_{\mathcal{G}}$ and $G_{\mathcal{G}}$, respectively. We often identify $\mathcal{F}$ with $\mathcal{G}$, and we write $S_X$ and $G_X$ instead of $S_{\mathcal{F}}$ and $G_{\mathcal{F}}$ when the family $\mathcal{F}$ is implied.

**Remark 2.10.** We need the outer tree sum to construct bigger spaces from summands, but when a bigger space is already constructed, we usually assume that the summands are subspaces of the sum, and we switch to the inner view.

Even though we define the inner tree sum so that the connection with the outer tree sum is clear, the following characterization is easier to work with.

**Proposition 2.11.** Let $X$ be a topological space and $\mathcal{F} := \langle X_i : i \in I\rangle$ a family of its subspaces. $X$ is an inner tree sum of $\mathcal{F}$ if and only if the following conditions hold.

(i) $\bigcup_{i\in I} X_i = X$.

(ii) $X$ is inductively generated by the family $\mathcal{F}$.

(iii) The undirected version of $G_{\mathcal{F}}$ is a tree.

*Proof.* Let $e_i\colon X_i \to X$ and $e'_i\colon X_i \to \sum_{j\in I} X_j$ be the canonical embeddings for every $i \in I$. Let us consider the map $f := \nabla_{i\in I} e_i\colon \sum_{i\in I} X_i \to X$. Clearly, $f^{\mathrm{i}}$ is bijective iff $f$ is surjective iff $\bigcup_{i\in I} X_i = X$. Note that $\sum_{i\in I} X_i/{\sim_f}$ is inductively generated by the family $\langle f^{\mathrm{q}} \circ e'_i : i \in I\rangle$. By the universal property of inductive generation, a bijective $f^{\mathrm{i}}$ is a homeomorphism if and only if $X$ is inductively generated by the family $\langle f^{\mathrm{i}} \circ f^{\mathrm{q}} \circ e'_i = f \circ e'_i = e_i : i \in I\rangle$. Finally, $\sum_{i\in I} X_i/{\sim_f}$ is a tree sum if and only if $G_{\mathcal{F}}$ is a tree since $G_{\mathcal{F}}$ is canonically isomorphic to the gluing graph of $\sum_{i\in I} X_i/{\sim_f}$. □

**Observation 2.12.** Let $X$ be a topological space and $\mathcal{F} := \langle X_i : i \in I\rangle$ a family of its subspaces. If $\langle x_j : j \in J\rangle$ is a family of points in $\bigcup_{i\in I} X_i$ and $\mathcal{F}' := \mathcal{F} \sqcup \langle \{x_j\} : j \in J\rangle$, then $X$ is a tree sum of $\mathcal{F}$ if and only if $X$ is a tree sum of $\mathcal{F}'$ (i.e. one-point spaces in gluing structures are essentially irrelevant).

*Proof.* Clearly, $\bigcup \operatorname{rng}(\mathcal{F}) = \bigcup \operatorname{rng}(\mathcal{F}')$. Also, $X$ is inductively generated by $\mathcal{F}$ if and only if it is inductively generated by $\mathcal{F}'$ since every member of $\mathcal{F}'$ is contained in a member of $\mathcal{F}$. Finally, $G_{\mathcal{F}}$ is a tree if and only if $G_{\mathcal{F}'}$ is a tree. We have

$S_{\mathcal{F}'} = S_{\mathcal{F}} \cup \{x_j : j \in J\}$, and the vertices of $G_{\mathcal{F}'}$ are $I \sqcup S_{\mathcal{F}'} \sqcup J$. Also, $G_{\mathcal{F}}$ is the subgraph of $G_{\mathcal{F}'}$ induced by $I \sqcup S_{\mathcal{F}}$. For every $s \in S_{\mathcal{F}'} \setminus S_{\mathcal{F}}$ there is exactly one $i \in I$ such that $s \in X_i$, and the graph $G_{\mathcal{F}'}$ adds $s$ as a new gluing vertex and $\langle s, i \rangle$ as a new edge. The graph $G_{\mathcal{F}'}$ also adds every $j \in J$ as a new vertex and $\langle x_j, j \rangle$ as a new edge. These changes clearly do not affect, whether the graph is a tree. □

## 2.2 Tree subsums and branches

**Definition 2.13.** Let $X$ be a tree sum of a family of its subspaces $\langle X_i : i \in I \rangle$ and let $Y \subseteq X$. We often use the following notation.

- $I_Y := \{i \in I : Y \cap X_i \neq \emptyset\}$ and $I_x := I_{\{x\}}$ for $x \in X$.
- $S_Y := S_X \cap Y$.
- $G_Y$ denotes the subgraph of $G_X$ induced by $I_Y \sqcup S_Y$.
- $\mathcal{F}_Y := \langle Y \cap X_i : i \in I_Y \rangle$.

We say that $Y$ is a *tree subsum* of $X$ if it is an inner tree sum of the family $\mathcal{F}_Y$. Note that $S_Y = S_{\mathcal{F}_Y}$ and $G_Y = G_{\mathcal{F}_Y}$, so the notation is consistent with Definition 2.9.

**Proposition 2.14.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$ and $Y \subseteq X$. The following conditions are equivalent.

(i) $Y$ is a tree subsum of $X$.

(ii) $G_Y$ is connected, i.e. it is a subtree of $G_X$.

(iii) $q_i[Y] = Y \cap X_i$ for every $i \in I_Y$.

*Proof.* Let $Y_i$ denote $Y \cap X_i$ for every $i \in I$.

(i) $\implies$ (ii) is trivial.

(ii) $\implies$ (i). By Proposition 2.11 it remains to show that $Y$ is inductively generated by the subspaces $\langle Y_i : i \in I_Y \rangle$. Let $U \subseteq Y$ be such that for every $i \in I_Y$ the set $U \cap Y_i$ is open in $Y_i$, i.e. there is $U_i \subseteq X_i$ open in $X_i$ such that $U_i \cap Y_i = U \cap Y_i$.

Let us put $W := \bigcup_{i \in I_Y} U_i$. We have $W \cap Y = \bigcup_{i \in I_Y} U_i \cap Y = \bigcup_{i \in I_Y} U_i \cap Y_i = \bigcup_{i \in I_Y} U \cap Y_i = U$. But $W$ does not have to be open in $X$. For every $i \in I_U$ we consider the set $V_i := \bigcup\{q_i^{-1}(s) : s \in S_{U_i} \setminus Y_i\}$. These are all points in the summands of $X$ attached to $X_i$ via some gluing point in $U_i \setminus Y$. For every $x \in V_i$ there is $j \in I \setminus \{i\}$ such that $x \in X_j$ and a unique path from $j$ to $i$ in $G_X$. This path goes through some $s \in S_{U_i} \setminus Y_i$. Since $G_Y$ is connected and $i \in G_Y$ and $s \notin G_Y$, we have $j \notin G_Y$ and $x \notin Y$. Therefore, $V_i \cap Y = \emptyset$.

Let us put $W' := W \cup \bigcup_{i \in I_U} V_i$. We have $W' \cap Y = W \cap Y = U$. To show that $W'$ is open in $X$ it is enough to observe that $W' \cap X_i$ is open in $X_i$ for every $i \in I$. Let $i \in I_U$; from the definition of $V_i$, we have $V_i \cap X_j \in \{\emptyset, X_j\}$ for every $j \neq i$, and $V_i \cap X_i = S_{U_i} \setminus Y_i \subseteq U_i$. Therefore, $W' \cap X_i \in \{\emptyset, X_i\}$ for every $i \in I \setminus I_U$, and $W' \cap X_i = U_i$ for every $i \in I_U$.

(ii) $\implies$ (iii). Let $i \in I_Y$. We have $q_i[Y] = Y_i \cup \{s \in S_{X_i} : Y \cap q_i^{-1}(s) \setminus \{s\} \neq \emptyset\}$. Let $s \in S_{X_i}$ and $x \in q_i^{-1}(s) \setminus \{s\}$. There is $j \in I$ such that $x \in X_j$ and the path from $i$ to $j$ in $G_X$ goes through $s$. We have $i \in G_Y$, so if $x \in Y$, then $j \in G_Y$ and $s \in G_Y$ by the connectedness of $G_Y$, and hence $s \in Y_i$.

(iii) $\implies$ (ii). Let $i \in G_Y$, $s \in S_{X_i}$, and $j \in G_Y$ such that the path from $i$ to $j$ in $G_X$ goes through $s$. It is enough to show that $s \in Y$. We have that $\{s\} = q_i[Y_j] \subseteq q_i[Y] = Y_i \subseteq Y$. □

**Proposition 2.15.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$. Let $\mathcal{F} := \langle Y_i : i \in I_{\mathcal{F}}\rangle$ be a family such that $I_{\mathcal{F}} \subseteq I$ and $Y_i \subseteq X_i$ for every $i \in I_{\mathcal{F}}$. Let $Y := \bigcup_{i \in I_{\mathcal{F}}} Y_i$.

(i) If $G_{\mathcal{F}}$ is connected, then $Y \cap X_i = Y_i$ for every $i \in I_{\mathcal{F}}$.

(ii) If $G_{\mathcal{F}}$ is connected, then $|Y \cap X_i| \leq 1$ for every $i \in I \setminus I_{\mathcal{F}}$.

(iii) If $Y$ is a tree sum of $\mathcal{F}$, then $Y$ is a tree subsum of $X$.

*Proof.*

(i) Let $i \in I_{\mathcal{F}}$. Clearly, $Y \cap X_i = Y_i \cup (S_Y \cap X_i)$. For every $s \in S_Y \cap X_i \setminus Y_i$ there is $j \in I_{\mathcal{F}} \setminus \{i\}$ such that $s \in Y_j$. Since $G_{\mathcal{F}}$ is connected and it is an induced subgraph of $G_X$, which is a tree, the path $\langle i, s, j\rangle$ in $G_X$ is a path in $G_{\mathcal{F}}$ as well. Hence, $s \in S_{\mathcal{F}}$ and $s \in Y_i$.

(ii) Let $i \in I \setminus I_{\mathcal{F}}$. Suppose that $s \neq s' \in Y \cap X_i$. Let $j, j' \in I_{\mathcal{F}}$ be such that $s \in Y_j \cap X_i$ and $s' \in Y_{j'} \cap X_i$. We have that $j \neq j'$ since otherwise $\langle i, s, j\rangle$ and $\langle i, s', j\rangle$ would be two different paths in $G_X$. Hence, we have a path $\langle j, s, i, s', j'\rangle$ in $G_X$. But since $G_{\mathcal{F}}$ is a connected subgraph of $G_X$, there is another path from $j$ to $j'$. That is a contradiction since $G_X$ is a tree.

(iii) We are comparing the families $\mathcal{F}$ and $\mathcal{F}_Y = \langle Y \cap X_i : i \in I_Y\rangle$. We may assume that every $Y_i \neq \emptyset$, otherwise we would have $\mathcal{F} = \langle\emptyset\rangle$, $Y = \emptyset$, $\mathcal{F}_Y = \langle\rangle$, and the claim would hold. By that assumption, $I_{\mathcal{F}} \subseteq I_Y$. Since $G_{\mathcal{F}}$ is a tree, we have $\mathcal{F} = \mathcal{F}_Y{\restriction}_{I_{\mathcal{F}}}$ by (i) and $|Y \cap X_i| = 1$ for $i \in I_Y \setminus I_{\mathcal{F}}$ by (ii). Therefore, we may use Observation 2.12, and $Y$ is a tree sum of $\mathcal{F}_Y$ since it is a tree sum of $\mathcal{F}$. □

**Lemma 2.16.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$. Let $\mathcal{F} := \langle Y_j : j \in J\rangle$ be a family of subspaces of $X$. If for every $i \in I$ there is $j \in J$ such that $X_i \subseteq Y_j$, then $\bigcup_{j \in J} Y_j = X$ and $X$ is inductively generated by $\mathcal{F}$. Therefore, $X$ is a tree sum of $\mathcal{F}$ if and only if $G_{\mathcal{F}}$ is a tree.

*Proof.* We use Proposition 2.11. Clearly, we have $\bigcup_{j \in J} Y_j \supseteq \bigcup_{i \in I} X_i = X$. Also, if a set $U \subseteq X$ is such that $U \cap Y_j$ is open in $Y_j$ for every $j \in J$, then $U \cap X_i$ is open in $X_i$ for every $i \in I$, and hence $U$ is open in $X$, and hence $X$ is inductively generated by $\mathcal{F}$. The conclusion follows again from Proposition 2.11. □

**Definition 2.17.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$. For every $x \in X$ we define its *branches* $\langle B_{x,i} : i \in I_x\rangle$ by the formula $B_{x,i} := \{x\} \cup q_i^{-1}[X_i \setminus \{x\}]$. That is, for $x \in S_X$ we have $B_{x,i} = \bigcup\{X_j : j \in J_{x,i}\}$ where $J_{x,i}$ is the set of all indices $j \in I$ such that in the path from $x$ to $j$ in $G_X$ the edge from $x$ goes to $i$.

Note that if $x \in S_X$ we have $|I_x| \geq 2$ and $B_{x,i} \cap B_{x,j} = \{x\}$ for every $i \neq j \in I_x$, whereas if $x \in X \setminus S_X$ there is only one $i$ in $I_x$ and $B_{x,i} = X$.

**Observation 2.18.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$, let $x \in X$, and let $\mathcal{B}_x := \langle B_{x,i} : i \in I_x\rangle$ be the enumeration of branches at $x$. We have that every $B_{x,i}$ is a tree subsum of $X$ and that $X$ is a tree sum of $\mathcal{B}_x$.

*Proof.* For $x \in X \setminus S_X$ this is clear, so let $x \in S_X$. For every $j \in I$ such that $B_{x,i} \cap X_j \neq \emptyset$ there is a path $\langle x, i_0, s_0, \dots, j_n = j\rangle$ in $G_X$. Either $i_0 = i$ and $X_{j_k} \subseteq B_{x,i}$ for every $k \leq n$, or $i_0 = j$. In both cases the path lies in $G_{B_{x,i}}$, and hence the graph is connected and $B_{x,i}$ is a tree subsum of $X$ by Proposition 2.14. For every $i \neq j \in I_x$ we have $B_{x,i} \cap B_{x,j} = \{x\}$, and hence $G_{\mathcal{B}_x}$ is a tree. Since every summand $X_j$ lies in some $B_{x,i}$, we have that $X$ is a tree sum of $\mathcal{B}_x$ by Lemma 2.16. □

## 2.3 Separation of tree sums

Now we will introduce an alternative description of standard separation axioms that is based on existence of continuous maps into special topological spaces. We do this in order to prove preservation of separation axioms for tree sums in a uniform and concise way.

**Definition 2.19.** We say that $\mathcal{S}$ is a *monotone separation scheme* if $\mathcal{S} = \langle Y_\mathcal{S}, \leq_\mathcal{S}\rangle$ where $Y_\mathcal{S}$ is a topological space containing the points 0 and 1, and $\leq_\mathcal{S}$ is a linear order on $Y_\mathcal{S}$ such that 0 is the minimum, 1 is the maximum, and for every $y \in Y_\mathcal{S}$ we have $Y_\mathcal{S}/(\leftarrow, y] \cong [y, \rightarrow)$ via the obvious canonical map (this last condition is motivated by Observation 2.20).

Let $X$ be a topological space. We say that a pair $A, B \subseteq X$ is $\mathcal{S}$-separated if there is a continuous function $f\colon X \to Y_\mathcal{S}$ such that $f[A] \subseteq \{0\}$ and $f[B] \subseteq \{1\}$. We also say that

- $X$ is $\mathcal{S}^{\mathrm{P}}$-separated if for every $x \neq y \in X$ either $\{x\}, \{y\}$ or $\{y\}, \{x\}$ is $\mathcal{S}$-separated,
- $X$ is $\mathcal{S}^{\mathrm{PP}}$-separated if every pair of distinct points of $X$ is $\mathcal{S}$-separated,
- $X$ is $\mathcal{S}^{\mathrm{PC}}$-separated if every point and every closed set not containing that point are $\mathcal{S}$-separated.

We consider the following monotone separation schemes:

- $\mathcal{S}_1$ is the Sierpiński space on $\{0 < 1\}$ with isolated point 1.
- $\mathcal{S}_2$ is the space $\{0 < \frac{1}{2} < 1\}$ with the topology generated by the two singletons $\{0\}$, $\{1\}$.
- $\mathcal{S}_{2\frac{1}{2}}$ is the space $\{0 < \bar{0} < \frac{1}{2} < \bar{1} < 1\}$ with the topology generated by the sets $\{0\}$, $\{0, \bar{0}, \frac{1}{2}\}$, $\{\frac{1}{2}, \bar{1}, 1\}$, $\{1\}$.
- $\mathcal{S}_\mathrm{f}$ is $[0, 1]$ where the order is inherited from $\mathbb{R}$.
- $\mathcal{S}_\mathrm{c}$ is the discrete space $\{0 < 1\}$.

**Observation 2.20.** Let $\mathcal{S}$ be a monotone separation scheme, $X$ a topological space. If $A, B \subseteq X$ are $\mathcal{S}$-separated, then for every $y \in Y_\mathcal{S}$ there is a continuous map $f\colon X \to Y_\mathcal{S}$ such that $f[A] \subseteq \{y\}$, $f[B] \subseteq \{1\}$.

*Proof.* Let $y \in Y_{\mathcal{S}}$, let $q\colon Y_{\mathcal{S}} \to [y, \to)$ be the quotient map induced by the canonical homeomorphism $Y_{\mathcal{S}}/(\leftarrow, y] \cong [y, \to)$. If $f_0\colon X \to Y_{\mathcal{S}}$ $\mathcal{S}$-separates $A, B$, then for $f := q \circ f_0\colon X \to [y, \to) \subseteq Y_{\mathcal{S}}$ we have $f[A] \subseteq \{q(0)\} = \{y\}$, $f[B] \subseteq \{q(1)\} = \{1\}$. □

**Observation 2.21.** Let $X$ be a topological space.

- $X$ is $T_0$ if and only if it is $\mathcal{S}_1^{\mathrm{P}}$-separated.
- $X$ is symmetric (i.e. for every $x \neq y \in X$, if there is open $U_x$ such that $x \in U_x \not\ni y$, then there is open $U_y$ such that $x \notin U_y \ni y$) if and only if for every point $x$ disjoint from a closed set $F$ there is an open set $U$ such that $x \notin U \supseteq F$, that is if and only if $X$ is $\mathcal{S}_1^{\mathrm{PC}}$-separated.
- $X$ is $T_1$ if and only if it is $\mathcal{S}_1^{\mathrm{PP}}$-separated.
- $X$ is $T_2$ or Hausdorff if and only if it is $\mathcal{S}_2^{\mathrm{PP}}$-separated.
- $X$ is $T_{2\frac{1}{2}}$ or Urysohn (i.e. for every $x \neq y \in X$ there are open sets $U_x \ni x$ and $U_y \ni y$ such that $\overline{U_x} \cap \overline{U_y} = \emptyset$) if and only if it is $\mathcal{S}_{2\frac{1}{2}}^{\mathrm{PP}}$-separated.
- $X$ is functionally $T_2$ (i.e. for every $x \neq y \in X$ there is a continuous function $f\colon X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$) if and only if it is $\mathcal{S}_{\mathrm{f}}^{\mathrm{PP}}$-separated.
- $X$ is totally separated (i.e. for every $x \neq y \in X$ there is a clopen set $U \subseteq X$ such that $x \in U \not\ni y$) if and only if it is $\mathcal{S}_{\mathrm{c}}^{\mathrm{PP}}$-separated.
- $X$ is regular if and only if it is $\mathcal{S}_2^{\mathrm{PC}}$-separated.
- $X$ is completely regular if and only if it is $\mathcal{S}_{\mathrm{f}}^{\mathrm{PC}}$-separated.
- $X$ is zero-dimensional if and only if it is $\mathcal{S}_{\mathrm{c}}^{\mathrm{PC}}$-separated.

**Observation 2.22.** For every monotone separation scheme $\mathcal{S}$ the properties of being $\mathcal{S}^{\mathrm{P}}$-separated, $\mathcal{S}^{\mathrm{PP}}$-separated, and $\mathcal{S}^{\mathrm{PC}}$-separated are hereditary.

*Proof.* Let $X \subseteq Y$ be topological spaces. If $x, y$ are distinct points of $X$, then they are distinct points of $Y$. If $x$ is a point not in a closed set $F$ in $X$, then $x \notin \mathrm{cl}_Y(F)$. Hence, we can move the situation to $Y$. If $f$ $\mathcal{S}$-separates the corresponding sets in $Y$, then $f{\restriction}_X$ $\mathcal{S}$-separates the corresponding sets in $X$. □

**Proposition 2.23.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$ and let $\mathcal{S}$ be a monotone separation scheme. The space $X$ is $\mathcal{S}^{\mathrm{P}}$-, $\mathcal{S}^{\mathrm{PP}}$-, or $\mathcal{S}^{\mathrm{PC}}$-separated if and only if all the spaces $X_i$ are $\mathcal{S}^{\mathrm{P}}$-, $\mathcal{S}^{\mathrm{PP}}$-, or $\mathcal{S}^{\mathrm{PC}}$-separated, respectively.

*Proof.* "$\Longrightarrow$" follows from Observation 2.22. "$\Longleftarrow$" for $\mathcal{S}^{\mathrm{P}}$ and $\mathcal{S}^{\mathrm{PP}}$ follows from the fact that for every two points $x \neq y \in X$ there is $i \in I$ such that $q_i(x) \neq q_i(y)$ since if a continuous map $f\colon X_i \to Y_{\mathcal{S}}$ $\mathcal{S}$-separates $\{q_i(x)\}$, $\{q_i(y)\}$, then $f \circ q_i\colon X \to Y_{\mathcal{S}}$ $\mathcal{S}$-separates $\{x\}$, $\{y\}$.

So let $x \neq y \in X$. If $x \notin S_X$, let $i \in I$ be the index such that $x \in X_i$. If $y \in X_i$, then $q_i(y) = y \neq x = q_i(x)$; if $y \notin X_i$, then $q_i(y) \in S_X \not\ni x = q_i(x)$. If $y \notin S_X$, we proceed symmetrically. If $x, y \in S_X$, consider the only path in $G_X$ from $x$ to $y$, going through vertices $x, i_0, s_0, \dots, i_n, y$. Then we have $q_{i_0}(x) = x \neq s_0 = q_{i_0}(y)$.

To prove "$\Longleftarrow$" for $\mathcal{S}^{\mathrm{PC}}$ let $x \in X$, $x \notin F \subseteq X$ closed. If $x \in S_X$, we put $S' := S_X$ and $G' := G_X$; if $x \notin S_X$, we put $S' := S_X \cup \{x\}$ and define $G'$ as the graph on $I \sqcup S'$ extending $G_X$ with the edge $\langle x, i_0, x\rangle$ where $i_0 \in I$ is the index such that $x \in X_{i_0}$. We also define a strict partial order $<$ on $G'$: $a < b$ if and only if the path from $x$ to $a$ is a strict initial segment of the path from $x$ to $b$. Basically, we are just rooting the tree at $x$ in order to perform an inductive construction.

We will define continuous maps $f_i \colon X_i \to Y_{\mathcal{S}}$ for $i \in I$ and values $y_s \in Y_{\mathcal{S}}$ for $s \in S'$. We define $y_x := 0$. If $s \in S'$ is a $<$-successor of $i \in I$, we define $y_s := f_i(s)$. If $i \in I$ is a $<$-successor of $s \in S'$, we define $f_i$ as a continuous map such that $f_i(s) = y_s$ and $f_i[F \cap X_i] \subseteq \{1\}$, which exists by Observation 2.20. By the construction, $f := \bigcup_{i \in I} f_i \colon X \to Y_{\mathcal{S}}$ is continuous and $\mathcal{S}$-separates $\{x\}$, $F$. $\square$

**Corollary 2.24.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$. The space $X$ is separated if and only if all the spaces $X_i$ are separated with "separated" meaning $T_0$, symmetric, $T_1$, $T_2$, $T_{2\frac{1}{2}}$, functionally $T_2$, totally separated, regular, completely regular, or zero-dimensional.

**Definition 2.25.** Let $X$ be a topological space. We say that $A \subseteq X$ is a *$T_{\frac{1}{2}}$-subset* of $X$ if every point of $A$ is closed or isolated in $X$. Equivalently, $X$ is $T_{\frac{1}{2}}$ at every point of $A$.

In order to take care of the $T_{\frac{1}{2}}$ separation axiom we need to introduce the following condition.

**Definition 2.26.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$. We say that the corresponding gluing is *$T_{\frac{1}{2}}$-compatible* if we never glue a non-isolated closed point to a non-closed isolated point, i.e. there are no $s \in S_X$, $i, j \in I_s$ such that $s$ is non-isolated and closed in $X_i$ and non-closed isolated in $X_j$. Note that if the spaces $X_i$ are $T_{\frac{1}{2}}$, this is equivalent to never gluing a non-closed point to a non-isolated point.

**Proposition 2.27.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$.

(i) $S_X$ is a $T_{\frac{1}{2}}$-subset of $X$ if and only if every $S_X \cap X_i$ is a $T_{\frac{1}{2}}$-subset of $X_i$ and the gluing is $T_{\frac{1}{2}}$-compatible.

(ii) The space $X$ is $T_{\frac{1}{2}}$ if and only if all spaces $X_i$ are $T_{\frac{1}{2}}$ and the gluing is $T_{\frac{1}{2}}$-compatible.

*Proof.*

(i) Clearly, if $S_X$ is a $T_{\frac{1}{2}}$-subset of $X$, then $S_X \cap X_i$ is a $T_{\frac{1}{2}}$-subset of $X_i$ for every $i \in I$. Also, under this condition the gluing is $T_{\frac{1}{2}}$-compatible if and only if every $s \in S_X$ is closed in every $X_i$ or isolated in every $X_i$ for $i \in I_s$. In other words, if and only if $S_X$ is a $T_{\frac{1}{2}}$-subset of $S_X$.

(ii) The space $X$ is $T_{\frac{1}{2}}$ if and only if both $S_X$ and $X \setminus S_X$ are $T_{\frac{1}{2}}$-subsets of $X$. The same holds for spaces $X_i$. It is enough to use (i) and observe that a point in $X \setminus S_X$ is closed or isolated in $X$ if and only if it is so in the space $X_i$ that contains it. $\square$

## 2.4 Neighborhood-related properties and I-subsets

Let us start with two lemmata for building neighborhoods in tree sums.

**Lemma 2.28.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$, let $U_i \subseteq X_i$ for every $i \in I$, and let $U := \bigcup_{i \in I} U_i$. If every $U_i$ is open in the corresponding $X_i$ and every $s \in S_U$ is either isolated or $s \in \bigcap_{i \in I_s} U_i$, then $U$ is open in $X$.

*Proof.* It is enough to show that every $U \cap X_i$ is open in the corresponding $X_i$. Clearly, $U \cap X_i = U_i \cup S_{U \cap X_i}$. By our assumptions, every gluing point in $U \cap X_i$ is either isolated or already contained in $U_i$, and hence $U \cap X_i$ is open in $X_i$. □

**Lemma 2.29.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$ and $x \in X$. Let $S'$ denote the set of all non-isolated gluing points. If $\langle U_i : i \in I_x \rangle$ is a family such that for every $i \in I_x$ we have $x \in U_i \subseteq X_i$ and $U_i$ is open in $X_i$, and $V \subseteq X$ is open such that $x \in V$ and $V \cap S' \subseteq \{x\}$, then $W := V \cap \bigcup_{i \in I_x} U_i$ is an open neighborhood of $x$ in $X$.

*Proof.* The claim follows from Lemma 2.28 applied to the family $\langle V \cap U_i : i \in I \rangle$ where we additionally put $U_i := \emptyset$ for $i \in I \setminus I_x$. □

Let us define the notion of I-subset that naturally occurs in several following propositions.

**Definition 2.30.** Let $X$ be a topological space. We say that $A \subseteq X$ is an I-*subset* of $X$ if it is a union of an open discrete subset and a closed discrete subset of $X$. Equivalently, the points of $A$ that are not isolated in $X$ form a closed discrete subset of $X$. The name is derived from the related concept of I-space introduced in [2, Definition 1.4].

**Definition 2.31.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$. We say that the corresponding gluing is I-*compatible* if we never glue a non-isolated closed point to an isolated point, i.e. there are no $s \in S_X$, $i, j \in I_s$ such that $s$ is non-isolated and closed in $X_i$ and isolated in $X_j$. Note that if the spaces $X_i$ are $T_{\frac{1}{2}}$, this is equivalent to never gluing an isolated point to a non-isolated point.

Note that the "I" in the notions of I-subset, I-space, and I-compatibility is a constant symbol referring to "isolated" rather than a mathematical variable $I$. In particular, it is not related to the index set $I$ that is sometimes present.

**Observation 2.32.** In a topological space every I-subset is a $T_{\frac{1}{2}}$-subset.

**Observation 2.33.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$. If the gluing is I-compatible, then it is $T_{\frac{1}{2}}$-compatible. If, additionally, no space $X_i$ contains a clopen point, then the other implication holds as well.

*Proof.* The claim follows from the definition. If there are no clopen points in spaces $X_i$, then every isolated point is non-closed. □

**Observation 2.34.** Let $X$ be a topological space inductively generated by a family of its subspaces $\langle X_i : i \in I \rangle$. Let $A \subseteq X$ and let $A_i := A \cap X_i$ for every $i \in I$.

(i) $A$ is closed discrete if and only if every $A_i$ is closed discrete in the corresponding $X_i$.

(ii) $A$ is open discrete if and only if every $A_i$ is open discrete in the corresponding $X_i$.

*Proof.* Clearly, if $A$ is closed discrete in $X$, then so is every $A_i$ in $X_i$. For the other implication let every $A_i$ be closed discrete in $X_i$. For every $B \subseteq A$ and $i \in I$ we have that $B \cap X_i$ is closed in $X_i$ since $B \cap X_i \subseteq A_i$ and $A_i$ is closed discrete. Therefore, every $B \subseteq A$ is closed in $X$, and hence $A$ is closed discrete in $X$. The proof of (ii) is analogous. □

**Proposition 2.35.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$.

(i) The set $S_X$ is discrete if and only if $S_X \cap X_i$ is discrete for every $i \in I$.

(ii) If $S_X$ is an I-subset of $X$, then $S_X \cap X_i$ is an I-subset of $X_i$ for every $i \in I$ and the gluing is $T_{\frac{1}{2}}$-compatible.

(iii) If $S_X \cap X_i$ is an I-subset of $X_i$ for every $i \in I$ and the gluing is I-compatible, then $S_X$ is an I-subset of $X$.

*Proof.*

(i) Clearly, if $S_X$ is discrete, then $S_X \cap X_i$ is discrete for every $i \in I$. For the other implication let $s \in S_X$. Since $S_X \cap X_i$ is discrete for every $i \in I$, then for every $i \in I_s$ there is $U_i$ open in $X_i$ such that $U_i \cap S_X = \{s\}$. Consider $U := \bigcup_{i \in I_s} U_i$. Then $U \cap S_X = \{s\}$ and $U$ is open since $U \cap X_i = U_i$ if $i \in I_s$, $\emptyset$ otherwise.

(ii) Clearly, $S_X \cap X_i$ is an I-subset of $X_i$ for every $i \in I$. The rest follows from Observation 2.32 and Proposition 2.27 (i).

(iii) Let $S'$ be the set of all points of $S_X$ not isolated in $X$ and let $S'_i$ be the set of all points of $S_X \cap X_i$ not isolated in $X_i$ for every $i \in I$. For every $s \in S'$ there is $i_s \in I_s$ such that $s$ is not isolated in $X_{i_s}$. The point $s$ is also closed in $X_{i_s}$ since $S'_{i_s}$ is closed discrete. Since the gluing is I-compatible, $s$ is not isolated in any $X_i$ for $i \in I$. Therefore, $S' \cap X_i \subseteq S'_i$ for every $i \in I$, and since every $S'_i$ is closed discrete in $X_i$, the set $S'$ is closed discrete in $X$ by Observation 2.34. □

The following examples show that the claims in Proposition 2.35 are sharp.

**Example 2.36.** Let $X$ be a wedge sum of spaces $\langle X_i : i \in I \rangle$, i.e. $S_X = \{x\}$ for some $x \in X$. If $x$ is a closed point in every $X_i$, then $S_X$ is clearly an I-subset of $X$. On the other hand, if additionally $x$ is clopen in some but not all spaces $X_i$, then we glued an isolated point to a closed non-isolated point, so the gluing is not I-compatible.

**Example 2.37.** Let us consider a tree sum $X = \sum_{n \leq \omega} X_n / {\sim}$ where $X_n$ for $n < \omega$ is the Sierpiński space on $\{0, 1\}$ with isolated point 1, the space $X_\omega$ is the convergent sequence $\omega + 1$, and we glue $\langle \omega, n \rangle \sim \langle n, 0 \rangle$ for every $n < \omega$, i.e. we glue the $n$-th member of the sequence with the non-isolated point of the corresponding Sierpiński space.

We have that the gluing is $T_{\frac{1}{2}}$-compatible, and so $X$ is $T_{\frac{1}{2}}$ by Proposition 2.27. We also have that $S_X \cap X_n$ is an I-subset of $X_n$ for every $n \leq \omega$, but $S_X$ is not an I-subset of $X$ – it contains no isolated point of $X$ and it is discrete but not closed.

Now we use the condition of $S_X$ being an I-subset of $X$ as an assumption.

**Proposition 2.38.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$. If $S_X$ is an I-subset of $X$, then $X$ is hereditarily inductively generated by the inclusions of the spaces $X_i$.

*Proof.* Let $U \subseteq A \subseteq X$. Put $A_i := A \cap X_i$ for $i \in I$. We want to show that if $U \cap A_i$ is open in $A_i$ for every $i \in I$, then $U$ is open in $A$. Let $x \in U$. For every $i \in I_x$ there is $U_i$ open in $X_i$ such that $U_i \cap A_i = U \cap A_i$. Let $S'$ be the set of all non-isolated gluing points. Since $S'$ is closed discrete, there is an open set $V \subseteq X$ containing $x$ such that $V \cap S' \subseteq \{x\}$. By Lemma 2.29 $W := V \cap \bigcup_{i \in I_x} U_i$ is open in $X$, and we have $x \in W \cap A \subseteq \bigcup_{i \in I_x} U_i \cap A = \bigcup_{i \in I_x} U_i \cap A_i = \bigcup_{i \in I_x} U \cap A_i \subseteq U$. □

**Proposition 2.39.** Let $\langle X, \tau \rangle := \sum_{i \in I} \langle X_i, \tau_i \rangle / \sim$ be a tree sum, let $\mathcal{A} \subseteq \mathcal{P}(X)$. We put $\tau^* := \tau \vee \mathcal{A}$, $\tau_i^* := \tau_i \vee \{A \cap X_i : A \in \mathcal{A}\}$. If we have that

(i) $S_X$ is an I-subset of $\langle X, \tau \rangle$,

(ii) for every $x \in S_X$ there is a $\tau$-open set $G_x$ such that $\{A \in \mathcal{A} : x \in A \not\supseteq G_x\}$ is finite;

then $\langle X, \tau^* \rangle = \sum_{i \in I} \langle X_i, \tau_i^* \rangle / \sim$, i.e. such expansion of a tree sum is a tree sum of the corresponding expansions.

*Proof.* Clearly, all the maps $e_i \colon \langle X_i, \tau_i^* \rangle \to \langle X, \tau^* \rangle$ are continuous, and hence we have that $\mathrm{id}_X \colon \sum_{i \in I} \langle X_i, \tau_i^* \rangle / \sim \; \to \langle X, \tau^* \rangle$ is continuous by the inductive generation. To prove the equality it is enough to show that $\tau^*$ is inductively generated by maps $e_i \colon \langle X_i, \tau_i^* \rangle \to X$. So let $U \subseteq X$ be such that $U \cap X_i$ is $\tau_i^*$-open for every $i \in I$. We will show that $U$ is $\tau^*$-open.

Let $x \in U \setminus S'$ where $S'$ denotes the set of all non-isolated gluing points. If $x$ is an isolated gluing point, then we are done, otherwise let $i$ be the only $i \in I$ such that $x \in X_i$. Let $U_i$ be a $\tau_i$-open set and $\mathcal{B} \subseteq \mathcal{A}$ a finite family such that $x \in U_i \cap \bigcap \mathcal{B} \subseteq U$. We have that $U_i \setminus S'$ is $\tau$-open since it is $\tau_i$-open and for every $j \neq i$ it holds that $(U_i \setminus S') \cap X_j$ is either empty or an isolated gluing point connecting $X_i$ with $X_j$. Therefore, $W_x := U_i \cap \bigcap \mathcal{B} \setminus S'$ is a $\tau^*$-neighborhood of $x$ in $U$.

Let $x \in U \cap S'$. We put $B := \bigcap \{A \in \mathcal{A} : x \in A \not\supseteq G_x\}$, which is $\tau^*$-open since the set is finite. For every $i \in I_x$ there is an $\tau_i$-open set $U_i \subseteq G_x$ such that $x \in U_i \cap B \subseteq U$. There is also a $\tau$-open set $V$ such that $V \cap S' = \{x\}$. By Lemma 2.29 $\bigcup_{i \in I_x} U_i \cap V$ is a $\tau$-neighborhood of $x$, so $W_x := \bigcup_{i \in I_x} U_i \cap V \cap B$ is a $\tau^*$-neighborhood of $x$ in $U$. □

## 2.5 Connectedness-related properties

Now we focus on connectedness-related properties of tree sums.

**Proposition 2.40.** A tree sum $X$ of spaces $\langle X_i : i \in I \rangle$ is connected if and only if all the spaces $X_i$ are connected.

*Proof.* "$\Longrightarrow$". By Proposition 2.7 all the spaces $X_i$ are quotients of the connected space $X$. "$\Longleftarrow$". Every component of connectedness contains all spaces $X_i$ that it intersects. Hence, it contains whole $X$ because every two spaces $X_i$, $X_j$ are connected via a path in $G_X$. □

**Observation 2.41.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$. Let $x \in X$ and let $\langle B_i : i \in I_x \rangle$ be the branches at $x$. If $x$ is closed or isolated in $X$, then $\langle B_i \setminus \{x\} : i \in I_x \rangle$ is a clopen decomposition of $X \setminus \{x\}$.

*Proof.* If $x$ is closed, then every $B_i \setminus \{x\}$ is open in $X$ since the only space $X_j$ such that $(B_i \setminus \{x\}) \cap X_j \notin \{\emptyset, X_j\}$ is $X_i$ where the intersection is $X_i \setminus \{x\}$, which is open. Similarly, if $x$ is isolated, then every $B_i$ is open in $X$ since $B_i \cap X_j \in \{\emptyset, \{x\}, X_j\}$ for every $j \in I$. In both cases, $B_i \setminus \{x\}$ is clopen in $X \setminus \{x\}$. □

**Observation 2.42.** Let $X$ be a topological space, let $x \in X$, and let $\langle X_i : i \in I \rangle$ be a clopen decomposition of $X \setminus \{x\}$. One of the following situations happens.

(i) The point $x$ is closed in $X$ and every $X_i$ is open in $X$.

(ii) The point $x$ is isolated in $X$ and every $X_i$ is closed in $X$.

(iii) There is $i \in I$ such that $x$ is neither closed nor isolated in $X_i \cup \{x\}$ while $X_j$ is clopen for every $j \in I \setminus \{i\}$.

*Proof.* Since $\langle X_i : i \in I \rangle$ is a clopen decomposition of $X \setminus \{x\}$, there are sets $\langle U_i : i \in I \rangle$ open in $X$ such that for every $i \in I$ we have $U_i \setminus \{x\} = X_i$. If we may choose $U_i = X_i$ for every $i \in I$, we are in situation (i). Otherwise, there is $i \in I$ such that $U_i = X_i \cup \{x\}$ and $X_i$ is not open in $X$. If there is $j \in I \setminus \{i\}$ such that we may choose $U_j = X_j \cup \{x\}$, we are in situation (ii) since $\{x\} = U_i \cap U_j$ and $X \setminus X_k = \bigcup \{U_l : l \in I \setminus \{k\}\}$ for every $k \in I$. If there is no such $j$, then $U_j = X_j$ for every $j \in I \setminus \{i\}$, the point $x$ is not isolated in $X$, and $\langle U_i, U_j : j \in I \setminus \{i\} \rangle$ is a clopen decomposition of $X$. Hence, we are in situation (iii). □

**Proposition 2.43.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$ such that every gluing point is closed or isolated, i.e. $S_X$ is a $T_{\frac{1}{2}}$-subset of $X$. Let $C \subseteq X$ and $C_i := C \cap X_i$ for every $i \in I_C$. The set $C$ is connected if and only if every $C_i$ is connected and $G_C$ is connected. That is, connected subspaces of $X$ are exactly tree subsums of connected subspaces.

*Proof.* Suppose that $C$ is connected. Let $s \in S_X \setminus C$ and let $\langle B_i : i \in I_s \rangle$ be the branches of $X$ at $s$. Since $s$ is closed or isolated, it follows from Observation 2.41 that every $B_i \setminus \{s\}$ is clopen in $X \setminus \{s\}$, and hence $C \subseteq B_i \setminus \{s\}$ for some $i \in I_s$. Therefore, $G_C$ is a connected graph.

If $G_C$ is connected, then $C$ is a tree sum of $\langle C_i : i \in I_C \rangle$ by Proposition 2.14, and the claim follows from Proposition 2.40 applied to $C$ and $\langle C_i : i \in I_C \rangle$. □

## 2.6 Maximal connectedness of tree sums

Now we finally use the machinery built in the previous sections to prove the theorems about maximal connectedness in tree sums of topological spaces.

**Theorem 2.44.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$ such that the set of all non-isolated gluing points is closed discrete, i.e. $S_X$ is an I-subset of $X$.

(i) If the spaces $X_i$ are maximal connected, then $X$ is maximal connected.

(ii) If the spaces $X_i$ are strongly connected, then $X$ is strongly connected.

(iii) If the spaces $X_i$ are essentially connected, then $X$ is essentially connected.

*Proof.* Let $\tau$ be the topology on $X$, $\tau_i$ the topology on $X_i$ for every $i \in I$.

(i) By Proposition 2.40 $X$ is connected. Let $A \subseteq X$ be non-$\tau$-open. Consider $\tau^* := \tau \vee \{A\}$ and $\tau_i^* := \tau_i \vee \{A \cap X_i\}$ for $i \in I$. Since $A$ is not $\tau$-open, there is $i \in I$ such that $A \cap X_i$ is not $\tau_i$-open, and hence $\tau_i^*$ is disconnected since $\tau_i$ is maximal connected. By Proposition 2.39 $\langle X, \tau^*\rangle$ is a tree sum of the spaces $\langle X_i, \tau_i^*\rangle$. Therefore, it is disconnected by Proposition 2.40.

(ii) Let $\tau_i^*$ be a maximal connected expansion of $\tau_i$ for every $i \in I$, let $\langle X, \tau^*\rangle$ be the corresponding tree sum. Clearly, $\tau^*$ is an expansion of $\tau$. Since $S_X$ is an I-subset of $\langle X, \tau\rangle$, it is an I-subset of $\langle X, \tau^*\rangle$ as well, and hence by (i) $\tau^*$ is maximal connected.

(iii) Again, $X$ is connected by Proposition 2.40. By Observation 1.4 it is enough to test essential connectedness only on expansions by finite families. Let $C$ be a connected subset of $\langle X, \tau\rangle$, let $\tau^*$ be a connected expansion of $\tau$ by a finite family, and let $\tau_i^* := \tau^*\restriction_{X_i}$ for $i \in I$. $\langle X, \tau^*\rangle$ is a tree sum of spaces $\langle X_i, \tau_i^*\rangle$ by Proposition 2.39. By Proposition 2.43 every $\tau_i^*$ is connected and every $C_i := C \cap X_i$ is $\tau_i$-connected. By the essential connectedness every $C_i$ is $\tau_i^*$-connected and hence $\tau^*$-connected. Therefore, $C$ is $\tau^*$-connected again by Proposition 2.43. □

**Corollary 2.45.** The following spaces are strongly connected: $\mathbb{R}^\kappa$ (or generally any real topological vector space), $[0,1]^\kappa$, $[0,1)^\kappa$ for $\kappa \geq 1$, spheres, and many others for which the technique described in the proof can be adapted.

*Proof.* Let us consider the space $\mathbb{R}^\kappa$. Let $\{L_i : i \in I\}$ be the family of all lines through the origin in $\mathbb{R}^\kappa$, and let $\tau$ be the topology on $\mathbb{R}^\kappa$ inductively generated by the lines (this idea is due to [8, Corollary 5A]). Clearly, $\tau$ refines the standard topology on $\mathbb{R}^\kappa$. By Proposition 2.11 $\langle \mathbb{R}^\kappa, \tau\rangle$ is a tree sum of the lines $L_i$. By Theorem 1.14 the lines are strongly connected, and hence $\langle \mathbb{R}^\kappa, \tau\rangle$ is strongly connected by Theorem 2.44.

The situation with the other spaces is analogous. The general idea is to cut a space so one gets a tree sum of real intervals that refines the original topology. For example a sphere is cut into meridians glued together at one pole with the other pole attached to one of the meridians. □

**Question 2.46.** Is every CW complex strongly connected?

**Observation 2.47.** Let $\langle X, \tau \rangle$ be a topological space, $\tau^*$ a connected expansion of $\tau$, $x \in X$. We denote the family of all $\tau$-connected components of $X \setminus \{x\}$ by $\mathcal{C}_x$ and the family of all $\tau^*$-connected components of $X \setminus \{x\}$ by $\mathcal{C}^*_x$. If $\langle X, \tau \rangle$ is essentially connected, then $\mathcal{C}_x = \mathcal{C}^*_x$. Hence, if $\langle X, \tau \rangle$ is a topological realization of a tree graph, then any maximal connected expansion $\tau^*$ still possesses the structure of the graph: the vertices of degree $\neq 2$ can be easily identified and the edges remain connected by essential connectedness.

Let us focus on the question, whether the assumption of $S_X$ being an I-subset of $X$ is necessary in Theorem 2.44.

**Example 2.48.** Not every tree sum of copies of the Sierpiński space is maximal connected. Let $X_1$ be the Sierpiński space on $\{0, 1\}$ with isolated point 1 and $X_2$ the Sierpiński space on $\{1, 2\}$ with isolated point 2. Consider $X := (X_1 \oplus X_2)/{\sim}$ where $\sim$ glues the points 1 together. The specialization order (Definition 3.6) on $X$ is $0 < 1 < 2$ and the gluing is not $T_{\frac{1}{2}}$-compatible, and hence $X$ is not maximal connected since it is even not $T_{\frac{1}{2}}$.

Also, the set $C := \{0, 2\} \subseteq X$ is connected, but the graph $G_C$ is not connected. This shows that the assumption about closed or isolated gluing points in Proposition 2.43 is necessary.

**Example 2.49.** Not every tree sum of maximal connected spaces is maximal connected or even essentially connected. Let $\langle [0, 1]_x : x \in [0, 1] \rangle$ be copies of the real interval $[0, 1]$ with a maximal connected expansion of the standard topology. Consider a comb-like space $\langle X, \tau \rangle := \sum_{x \in [0,1]} [0, 1]_x/{\sim}$ where $\sim$ glues together points $\langle 0, x \rangle \sim \langle x, 1 \rangle$ for $x > 0$. $\langle X, \tau \rangle$ is a tree sum of the maximal connected intervals, but it is not maximal connected itself.

Let $A := \{\langle 0, 0 \rangle\} \cup \bigcup_{x>0} [0, 1)_x$, and $\tau^* := \tau \vee \{A\}$. Clearly, $\tau^*$ is a strict expansion of $\tau$ since $A \cap [0, 1]_0 = \{\langle 0, 0 \rangle\}$, which is not $\tau$-open in $[0, 1]_0$. By Lemma 1.7, the set $X \setminus \{\langle 0, 0 \rangle\}$ is $\tau^*$-connected since it is $\tau$-connected and $(X \setminus \{\langle 0, 0 \rangle\}) \cap A = \bigcup_{x>0} [0, 1)_x$, which is $\tau$-open. We also have that $\langle 0, 0 \rangle$ is in $\tau^*$-closure of $\bigcup_{x>0} [0, 1)_x$. Together, $\tau^*$ is still connected. But since $[0, 1]_0$ becomes disconnected in $\tau^*$, $\tau$ is not even essentially connected.

**Question 2.50.** Is the space $X$ form the previous example strongly connected? Is every tree sum of maximal connected spaces strongly connected?

The fact that the space from the previous example is not even essentially connected is not a coincidence as the following observation shows.

**Observation 2.51.** Let $X$ be a topological space whose topology is inductively generated by a family of maximal connected subspaces $\langle X_i : i \in I \rangle$. If $X$ is essentially connected, then it is maximal connected.

*Proof.* Let $\tau$ be the topology on $X$. If $X$ is not maximal connected, then there is a non-open set $A \subseteq X$ such that $\tau^* := \tau \vee \{A\}$ is still connected. By the inductive generation there is $i \in I$ such that $A \cap X_i$ is not open in $X_i$, and hence $X_i$ is $\tau$-connected but not $\tau^*$-connected, so $\langle X, \tau \rangle$ is not essentially connected. $\square$

**Proposition 2.52.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I\rangle$ and let $S$ be the set of all points $s \in S_X$ such that there are $i \neq j \in I$ such that $\{s\}$ is nowhere dense in both $X_i$ and $X_j$, i.e. $S$ is the set of all points that are nowhere dense in at least two summands. If $X$ is nodec, then $S$ is closed discrete.

*Proof.* By Observation 2.34 it is enough to show that $S \cap X_i$ is closed discrete in $X_i$ for every $i \in I$. Let $i \in I$ and for every $s \in S \cap X_i$ let $\langle B_{s,j} : j \in I_s\rangle$ be the enumeration of branches of $X$ at $s$. By the definition of $S$ there is $j \in I_s \setminus \{i\}$ such that $\{s\}$ is nowhere dense in $X_j$. Let $U_s := B_{s,j} \setminus \{s\}$.

*$U_s$ is open in $X$.* Since $\{s\}$ is nowhere dense in $X_j$ and hence in $B_{s,j}$, it is closed in $B_{s,j}$, which is nodec as a subspace of $X$ (Proposition 1.10). Hence, $U_s$ is open in $B_{s,j}$. Since we also have $U_s \cap B_{s,k} = \emptyset$ for every $k \in I_s \setminus \{j\}$ and $X$ is a tree sum of $\langle B_{s,j} : j \in I_s\rangle$ by Observation 2.18, we have proved the claim.

We also have that $s \in \overline{U_s}$ since $s$ is not isolated in $B_{s,j}$. Finally, let $U := \bigcup\{U_s : s \in S \cap X_i\}$. We have $S \cap X_i \subseteq \overline{U} \setminus U$, and the latter is a closed discrete subset of $X$ by an equivalent condition in Definition 1.8. □

**Theorem 2.53.** Let $X$ be a tree sum of nondegenerate spaces $\langle X_i : i \in I\rangle$. The following conditions are equivalent.

(i) The space $X$ is maximal connected.

(ii) The spaces $X_i$ are maximal connected and $S_X$ is an I-subset of $X$.

(iii) The spaces $X_i$ are maximal connected, $S_{X_i}$ is an I-subset of $X_i$ for every $i \in I$, and the gluing is $T_{\frac{1}{2}}$-compatible or, equivalently, I-compatible.

(iv) The spaces $X_i$ are maximal connected and $X$ is essentially connected.

*Proof.*

(i) $\implies$ (ii). Every $X_i$ is a connected subspace of $X$ by Proposition 2.40, and hence is maximal connected by Proposition 1.12 (i). Let $S$ denote the set of all non-isolated gluing points of $X$. For every $s \in S$ and $i \in I_s$ we have that $s$ is closed in $X$ since $X$ is $T_{\frac{1}{2}}$. Hence, $s$ is non-isolated in $X_i$ since otherwise it would be clopen in connected nondegenerate space $X_i$. Therefore, $\{x\}$ is nowhere dense in $X_i$, and we may use Proposition 2.52 to show that $S$ is closed discrete in $X$, and hence $S_X$ is an I-subset of $X$.

(ii) $\implies$ (i) is Theorem 2.44 (i).

(ii) $\iff$ (iii). The equivalence of $T_{\frac{1}{2}}$-compatibility and I-compatibility follows from Observation 2.33. Therefore, the claim follows from Proposition 2.35.

(i) $\iff$ (iv). One implication is trivial, the other follows from Observation 2.51. □

**Corollary 2.54.** Let $X$ be a tree sum of nondegenerate spaces $\langle X_i : i \in I\rangle$. The space $X$ is $T_1$ maximal connected if and only if the spaces $X_i$ are $T_1$ maximal connected and the gluing set $S_X$ is closed discrete.

*Proof.* Follows immediately from Proposition 2.24, Theorem 2.53, and the fact that in a nondegenerate connected $T_1$ space there are no isolated points, so every I-subset is closed discrete. □

# 3 Finitely generated spaces

Maximal connected spaces in the class of finitely generated spaces were first characterized by Thomas in [14, Theorem 5]. He also proposed a way to visualize them. Later, Kennedy and McCartan in [10] characterized finitely generated maximal connected topologies in the lattice $\mathcal{T}(X)$ as joins of two topologies of special form based on the notion of a *final A-degenerate cover* where $A \subseteq X$.

In this section we reformulate the characterization in the language of specialization preorder and graphs – finitely generated maximal connected spaces correspond to tree graphs having a fixed bipartition where the correspondence is derived from the graphs of their specialization preorders (Proposition 3.11 and Corollary 3.12). We also reformulate the characterization in the language of tree sums – they are exactly $T_{\frac{1}{2}}$-compatible tree sums of copies of the Sierpiński space (Corollary 3.14), and we propose another method for their visualization.

**Notation 3.1.** Let $X$ be a topological space and $x \in X$. We put

$$x^\circ := \bigcap\{U \subseteq X : U \text{ open neighborhood of } x\}.$$

**Observation 3.2.** Let $X$ be a topological space.

- Let $x \in X$. The set $x^\circ$ is the only candidate for a minimal neighborhood of $x$. Hence, $x$ has a minimal neighborhood if and only if $x^\circ$ is open.
- For every, $x, y \in X$ we have $\overline{\{x\}} \subseteq \overline{\{y\}} \iff x \in \overline{\{y\}} \iff x^\circ \ni y \iff x^\circ \supseteq y^\circ$.
- $X$ is $T_1$ if and only if $\overline{\{x\}} = \{x\}$ for every $x \in X$ if and only if $x^\circ = \{x\}$ for every $x \in X$.
- $X$ is symmetric if and only if $\overline{\{x\}} = x^\circ$ for every $x \in X$.
- For every $x \in X$ the sets $\overline{\{x\}}$ and $x^\circ$ are connected.

**Definition 3.3.** Recall that a topological space $X$ is called *finitely generated* (or *Alexandrov discrete*) if for every $x \in \overline{A}$ in $X$ there exists a finite set $F \subseteq A$ such that $x \in \overline{F}$, equivalently if every intersection of open sets is open, equivalently if every point $x$ has a minimal neighborhood (which is $x^\circ$), equivalently if $X$ is inductively generated by the family of all finite subspaces.

**Observation 3.4.** Every finitely generated $T_{\frac{1}{2}}$ space is submaximal. Hence, a finitely generated space is $T_{\frac{1}{2}}$ if and only if it is submaximal.

*Proof.* Let $X$ be a finitely generated $T_{\frac{1}{2}}$ space. If $D \subseteq X$ dense, then it contains all isolated points, so $\{x\}$ is closed for each $x \in X \setminus D$ because $X$ is $T_{\frac{1}{2}}$, and $X \setminus D$ is closed because $X$ is finitely generated. The last claim follows from Proposition 1.9. □

**Observation 3.5.** Every finitely generated space $X$ is locally connected. In particular, components of connectedness are exactly nonempty clopen connected subsets.

*Proof.* If $x \in U \subseteq X$ for some $U$ open, we have $x \in x^\circ \subseteq U$ and $x^\circ$ is open connected. □

**Definition 3.6.** Recall that for every topological space $X$ the *specialization preorder* is defined on its points by the formula

$$x \leq y \;:\!\iff\; \overline{\{x\}} \subseteq \overline{\{y\}} \iff x^\circ \supseteq y^\circ.$$

The following proposition lists some well-known properties of the specialization preorder. We include a proof for the sake of completeness.

**Proposition 3.7.** Let $X$ be a topological space, $\leq$ the specialization preorder on $X$.

(i) Every open set is an upper set. Every closed set is a lower set.

(ii) The converse of (i) holds precisely for finitely generated spaces.

(iii) The construction of the specialization preorder provides a $1:1$ correspondence between finitely generated spaces and preorders.

(iv) The specialization preorder is an order if and only if $X$ is $T_0$.

(v) Every isolated point is a maximal element. Every closed point is a minimal element. If $X$ is $T_0$ the converse also holds.

(vi) $X$ is $T_{\frac{1}{2}}$ if and only if $\leq$ is an order with at most two levels.

*Proof.*

(i) If $F \subseteq X$ is closed and $x \leq y \in F$, then $x \in \overline{\{y\}} \subseteq F$. Dually for an open set.

(ii) An intersection of open sets is an upper set as an intersection of upper sets, and hence it is open if upper sets are open. If $X$ is finitely generated and $U \subseteq X$ is an upper set, then we have $x \in x^\circ \subseteq U$ for every $x \in U$, and hence $U$ is open. Dually for closed sets.

(iii) By (ii) we know that the construction is injective. We need to show that it is surjective, i.e. for every preordered set $\langle X, \leq \rangle$ there is a finitely generated topology on $X$ such that $\leq$ is its specialization preorder. It is enough to consider the set of all $\leq$-upper sets as the desired topology. Then $y \in x^\circ$ if and only if $y$ is in the principal upper set generated by $x$, i.e. $y \geq x$.

(iv) $\leq$ is an order if and only if $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$ implies $x = y$ for every $x, y \in X$.

(v) If $x$ is a closed point, then $y \leq x \iff y \in \overline{\{x\}} = \{x\} \implies y = x$, and hence it is minimal. If $X$ is $T_0$, then $\leq$ is an order, and if $x$ is minimal, then $y \in \overline{\{x\}} \iff y \leq x \implies y = x$, and hence $\{x\}$ is closed. Dually for an isolated point and a maximal element.

(vi) If $X$ is $T_{\frac{1}{2}}$, then $\leq$ is an order by (iv) and it has at most two levels by (v). If $\leq$ is an order with at most two levels, then $X$ is $T_0$ by (iv), and every point is isolated or closed by (v). □

**Definition 3.8.** Let $X$ be a finitely generated $T_{\frac{1}{2}}$ space and $\leq$ its specialization preorder. We define its *specialization graph* $G_X^<$ as follows. Vertices are the points of $X$, and there is a directed edge $\langle x, y\rangle$ in the graph if and only if $x < y$.

**Proposition 3.9.** The map $X \mapsto G_X^<$ provides a $1:1$ correspondence between finitely generated $T_{\frac{1}{2}}$ spaces and directed graphs with oriented paths of length at most one. On a fixed base set, finer topologies correspond to graphs with less edges.

*Proof.* Clearly, the map $X \mapsto G_X^<$ factorizes through the construction of specialization preorder, and we have the correspondence between finitely generated $T_{\frac{1}{2}}$ spaces and orders with at most two levels by Proposition 3.7 (iii), (vi). Hence, it is enough to establish the correspondence between orders with at most two levels and directed graphs with directed paths of length at most one.

A directed edge $\langle x, y\rangle$ is in $G_X^<$ if and only if $x$ is a closed point, $y$ is an isolated point, and $x \in \overline{\{y\}}$. In that case, $x$ is $\leq$-minimal and $y$ is $\leq$-maximal, and clearly there cannot be oriented paths of length $> 1$ in $G_X^<$.

On the other hand, we may start with an arbitrary directed graph $G$ with oriented paths of length at most one and interpret it as a strict part of an order with at most two levels. □

**Lemma 3.10.** Every simple expansion of a finitely generated space is finitely generated. Hence, every simple expansion of a finitely generated $T_{\frac{1}{2}}$ space is finitely generated and $T_{\frac{1}{2}}$.

*Proof.* Let $\langle X, \tau\rangle$ be a finitely generated topological space, let $A \subseteq X$, and $\tau^* := \tau \vee \{A\}$. Every family $\mathcal{U}$ of $\tau^*$-open sets is of form $\{(U_i \cup A) \cap V_i : i \in I\}$ where all the sets $U_i$, $V_i$ are $\tau$-open. Hence, $\bigcap \mathcal{U} = \bigcap_{i\in I}(U_i \cup A) \cap V_i = ((\bigcap_{i\in I} U_i) \cup A) \cap (\bigcap_{i\in I} V_i)$ is $\tau$-open. □

**Proposition 3.11.** Let $X$ be a finitely generated $T_{\frac{1}{2}}$ space.

(i) Connected components of $X$ are exactly undirected connected components of $G_X^<$. Hence, $X$ is connected if and only if $G_X^<$ is connected (as undirected graph).

(ii) $X$ is maximal connected if and only if $G_X^<$ is a tree (as undirected graph).

*Proof.*

(i) Let $x \in X$. The connected component of $X$ containing $x$ is the lower upper set generated by $\{x\}$ because that is the smallest clopen set containing $x$ (we use Proposition 3.5). That is exactly the component of undirected connectedness of $G_X^<$.

(ii) We use the correspondence from Proposition 3.9. Connected topologies correspond to connected graphs, and trees are exactly the minimal connected graphs. We also need Lemma 3.10 and the fact that maximal connectedness can be tested on simple expansions (Observation 1.4). □

**Corollary 3.12.** Finitely generated maximal connected spaces correspond to tree graphs with a fixed bipartition.

*Proof.* We can use Proposition 3.11 (ii) because every maximal connected space is $T_{\frac{1}{2}}$ by Proposition 1.9, and because we can equivalently describe directed graphs with directed paths of length at most one as undirected graphs with a fixed bipartition. □

**Proposition 3.13.** Let $X$ be a tree sum of spaces $\langle X_i : i \in I \rangle$. The space $X$ is finitely generated if and only if all spaces $X_i$ are finitely generated.

*Proof.* One implication is clear since every subspace of a finitely generated space is itself finitely generated. On the other hand, if every $X_i$ is finitely generated, it is inductively generated by its finite subspaces, and since $X$ is inductively generated by the spaces $X_i$, it is, by transitivity, inductively generated by some of its finite subspaces and hence by all of its finite subspaces. □

**Corollary 3.14.** Besides the one-point space, finitely generated maximal connected spaces are exactly $T_{\frac{1}{2}}$-compatible tree sums of copies of the Sierpiński space.

*Proof.* Clearly, the Sierpiński space is maximal connected. Hence, every $T_{\frac{1}{2}}$-compatible tree sum of copies of the Sierpiński space is maximal connected by Theorem 2.53, and it is finitely generated by Proposition 3.13.

On the other hand, let $X$ be a finitely generated maximal connected space. Clearly, every edge of $G_X^{<}$ corresponds to a subspace of $X$ homeomorphic to the Sierpiński space. By Proposition 3.11 the corresponding gluing graph is a tree, and the subspaces cover whole $X$ unless $X$ is a one-point space. Also, the Sierpiński subspaces inductively generate the topology. If $x \in \overline{A} \setminus A$ for some $A \subseteq X$, then there is $y \in A$ such that $x \in \overline{\{y\}}$, so $x < y$ and $\{x, y\}$ is the witnessing Sierpiński subspace. Altogether, $X$ a tree sum of the Sierpiński subspaces by Proposition 2.11, and the gluing is $T_{\frac{1}{2}}$-compatible again by Theorem 2.53. □

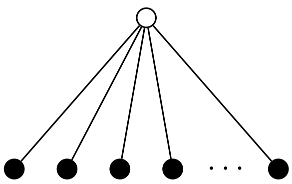
**Notation 3.15.** We propose to visualize finitely generated maximal connected spaces as follows. We just draw the specialization graph, and instead of orienting the edges we distinguish two kinds of vertices: the vertices corresponding to isolated points shall be drawn as open dots while the vertices corresponding to closed points as solid dots. Whenever it is suitable, we draw the isolated vertices above the closed vertices in order to stress the specialization order.

In the original paper [14] Thomas used a similar but different visualization. Instead of drawing an edge from a closed point to every isolated point in its smallest neighborhood, Thomas represents the smallest neighborhood by a line segment containing all the points.

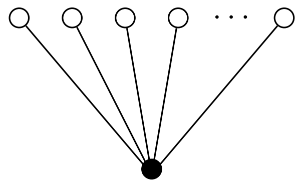
We think our visualization is less restrictive, and it deals with the duality on finitely generated spaces well – it is enough to switch the colors of isolated and closed vertices.

**Example 3.16.** We give some examples of maximal connected finitely generated spaces. Visualizations of these spaces are given in Figure 2.

- Clearly, the empty space, the one-point space, and the Sierpiński space are maximal connected.
- If $X$ is a set, $x \in X$ and $A \subseteq X$ is open if and only if $x \in A$, we obtain a space with so-called *included point topology*, also called *principal ultrafilter space*.
- If $X$ is a set, $x \in X$ and $A \subseteq X$ is open if and only if $x \notin A$, we obtain a space with so-called *excluded point topology*, also called *principal ultraideal space*.
- Let us consider the set of all integers $\mathbb{Z}$ with the topology generated by open sets $\{\{2k-1,\, 2k,\, 2k+1\} : k \in \mathbb{Z}\}$. We obtain a finitely generated maximal connected space called *Khalimsky line* or *digital line*.

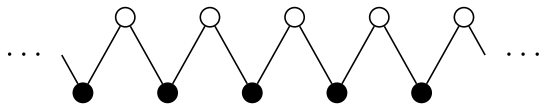


A principal ultrafilter space.



A principal ultraideal space.

The Sierpiński space.



The Khalimsky line.

Figure 2: Examples of finitely generated maximal connected spaces.

Let us conclude with Figure 3, which shows all nondegenerate finitely generated maximal connected spaces with at most five points, using our visualization. This may be compared with the corresponding picture in [14]. In our picture, the duality of finitely generated spaces (by considering the dual specialization preorder, in our case just by swapping isolated and closed points) is apparent.

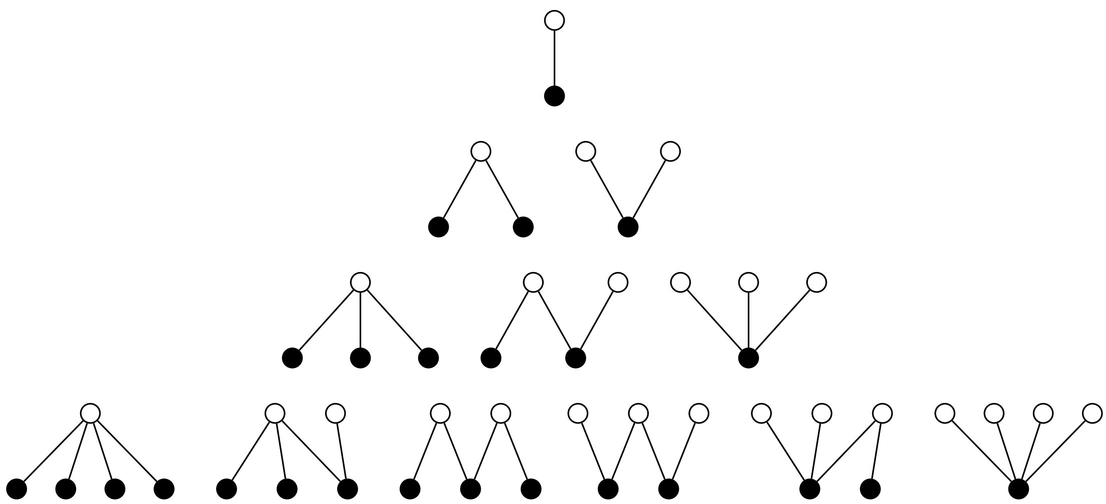


Figure 3: All nondegenerate maximal connected spaces with at most five elements.

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# III. Compactifiable classes of compacta

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*Dedicated to the memory of Petr Simon,*
*member of Seminar on Topology at Charles University.*

**Abstract**

We introduce the notion of compactifiable classes – these are classes of metrizable compact spaces that can be up to homeomorphic copies “disjointly combined” into one metrizable compact space. This is witnessed by so-called compact composition of the class. Analogously, we consider Polishable classes and Polish compositions. The question of compactifiability or Polishability of a class is related to hyperspaces. Strongly compactifiable and strongly Polishable classes may be characterized by the existence of a corresponding family in the hyperspace of all metrizable compacta. We systematically study the introduced notions – we give several characterizations, consider preservation under various constructions, and raise several questions.

*Classification:* 54D80, 54H05, 54B20, 54E45, 54F15.

*Keywords:* Compactifiable class, Polishable class, homeomorphism equivalence, metrizable compactum, Polish space, hyperspace, complexity, universal element, common model, inverse limit.

## 1 Introduction

Let us consider two classes $\mathcal{C}$ and $\mathcal{D}$ of topological spaces (not necessarily closed under homeomorphic copies). We say that these classes are *equivalent* (and we write $\mathcal{C} \cong \mathcal{D}$) if every space in $\mathcal{C}$ is homeomorphic to a space in $\mathcal{D}$ and vice versa.

Given a class $\mathcal{C}$ of metrizable compacta, we are interested whether $\mathcal{C}$ (up to the equivalence) can be disjointly composed into one metrizable compactum such that the corresponding quotient space is also a metrizable compactum. In our terminology introduced below, we ask whether the class $\mathcal{C}$ is *compactifiable.* If $\mathcal{C}$ is a class of continua, this is equivalent to finding a metrizable compactum whose set of connected components is equivalent to $\mathcal{C}$ (see Observation 2.12).

Original motivation comes from our interest in spirals [2] and from the construction of Minc [13], who for each nondegenerate metric continuum $X$ constructed a metrizable compactum $K$ whose components form a pairwise nonhomeomorphic family of spirals over $X$ with the decomposition space being $2^\omega$,

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and asked [13, Question 1] whether there is a metrizable compactum $K$ whose set of components is equivalent to the class of all spirals over $X$, i.e. whether the class of all spirals over $X$ is compactifiable. So compactifiability of a class may be viewed as a dual condition to the existence of a metrizable compactum whose components from a pairwise non-homeomorphic subfamily of the class. Minc [13, Question 2] also asked whether both conditions may be realized at the same time and/or whether the resulting decomposition may be continuous. This latter property corresponds to our notion of *strongly compactifiable classes*.

In Section 2 of our paper we define *compactifiable* and *Polishable* classes and their witnessing *compositions*. We consider several basic constructions of compositions, and we obtain several conditions equivalent to compactifiability and Polishability (Theorem 2.10 and 2.11).

In Section 3 we study connections between compactifiable or Polishable classes and hyperspaces. The Hilbert cube $[0,1]^\omega$ is universal for metrizable compacta, so a class of metrizable compacta may be realized as a subset of the hyperspace $\mathcal{K}([0,1]^\omega)$. We define *strongly compactifiable* and *strongly Polishable* classes, and characterize them by the existence of an equivalent family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ of a suitable complexity – closed or equivalently $F_\sigma$ for strong compactifiability and $G_\delta$ or equivalently analytic for strong Polishability (Theorem 3.13 and 3.14). Note that if a class $\mathcal{C}$ closed under homeomorphic copies is strongly compactifiable, $\mathcal{C} \cap \mathcal{K}([0,1]^\omega)$ is not necessarily closed – there is only an equivalent closed family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$. This leads to considering descriptive complexity of subsets of $\mathcal{K}([0,1]^\omega)$ up to the equivalence. The first author further develops this topic in [1].

In Section 4 we study preservation of the properties under various constructions, and consequently we obtain several examples. Among other results we prove the following. The four introduced properties are stable under countable unions. Every hereditary class of metrizable compacta or continua with a universal element is strongly compactifiable, and every class of metrizable compacta (resp. continua) closed under continuous images with a common model (i.e. a member of the class that continuously maps onto every other member of the class) is strongly Polishable (resp. compactifiable). For every strongly Polishable class $\mathcal{C}$ closed under homeomorphic copies and every Polish space $X$, the set $\mathcal{C} \cap \mathcal{K}(X)$ is analytic – this gives a necessary condition.

We may view the properties of being strongly compactifiable, compactifiable, strongly Polishable, and Polishable as degrees of complexity – classes of metrizable compacta that are compactifiable are "more comprehensible" than classes that are not compactifiable. A different measure of complexity of a class $\mathcal{C}$ of metrizable compacta is the complexity of the corresponding classification problem, i.e. the *Borel reducibility* [7, Chapter 5] of the homeomorphism relation $\cong_{\mathcal{C}}$. However, we first need to realize $\mathcal{C}$ as a standard Borel space in a natural way, e.g. as a subset of the hyperspace $\mathcal{K}([0,1]^\omega)$. That means this notion formally depends on the choice of such natural coding, even though it is a common belief that the particular natural coding does not matter in fact. See for example [7, Theorem 14.1.3].

Another inspiration for our study was the construction of a universal arc-like continuum [14, Theorem 12.22]. In Section 5 we modify this construction and prove that for every countable family $\mathcal{P}$ of metrizable compacta, the class of all $\mathcal{P}$-like spaces is compactifiable. We also argue that a compact composition may be viewed as a weaker form of a universal element for the class.

Several questions remain open. We do not have any particular example distinguishing between the four properties (Question 3.24), we have just some candidates. Also, the compactifiability of spirals remains open.

# 2 Compositions

In this section we formally define *compactifiable* and *Polishable classes* and the witnessing *compositions*. We describe several constructions of compositions and give some characterizations of compactifiability and Polishability. We also observe that compactifiable and Polishable classes are stable under countable unions. In particular, every countable class of metrizable compacta is compactifiable.

The idea of disjointly composing topological spaces is captured by the following notion.

**Definition 2.1.** A *composition* $\mathcal{A}$ consists of a continuous map $q\colon A \to B$ between topological spaces. In this context, $A$ is called the *composition space*, $B$ is called the *indexing space*, and $q$ is called the *composition map*. The idea is that the composition map $q$ captures how its fibers are composed in the composition space $A$. The notation $\mathcal{A}(q\colon A \to B)$ means that $\mathcal{A}$ is a composition with composition space $A$, indexing space $B$, and composition map $q$.

The following language gives us some flexibility when working with compositions.

- $\mathcal{A}$ is a *composition of an indexed family of topological spaces* $\langle A_b\rangle_{b\in B}$ if $q^{-1}(b) = A_b$ for every $b \in B$. Of course the family $\langle A_b\rangle_{b\in B}$ is a decomposition of $A$ (i.e. $A_b \cap A_{b'} = \emptyset$ for every $b \neq b' \in B$ and $\bigcup_{b\in B} A_b = A$) and is determined by $\mathcal{A}$. On the other hand, every decomposition $\langle A_b\rangle_{b\in B}$ of a topological space $A$ induces the unique map $q\colon A \to B$ with fibers $\langle A_b\rangle_{b\in B}$ and the composition $\mathcal{A}(q\colon A \to B)$ if the map $q$ is continuous.
- $\mathcal{A}$ is a *composition of an indexed family of embeddings* $\langle e_b\colon A_b \hookrightarrow A\rangle_{b\in B}$ if $q^{-1}(b) = \operatorname{rng}(e_b)$ for every $b \in B$. Again, $\langle \operatorname{rng}(e_b)\rangle_{b\in B}$ is necessarily a decomposition of $A$.
- $\mathcal{A}$ is a *composition of a class of topological spaces* $\mathcal{C}$ if the family $\{q^{-1}(b) : b \in B\}$ is equivalent to $\mathcal{C}$.

We are interested in the following special types of compositions.

- $\mathcal{A}$ is a *compact composition* if both $A$ and $B$ are metrizable compacta.
- $\mathcal{A}$ is a *Polish composition* if both $A$ and $B$ are Polish spaces.

**Remark 2.2.** In [13] P. Minc constructed a compact composition of a $2^\omega$-indexed family of pairwise non-homeomorphic compactifications of a ray with remainders being copies of an arbitrary fixed nondegenerate metrizable continuum.

**Remark 2.3.** Given a composition $\mathcal{A}(q\colon A \to B)$ of a family $\langle A_b \rangle_{b\in B}$, the spaces $A_b$ are all nonempty if and only if the composition map $q$ is surjective.

**Definition 2.4.** A class $\mathcal{C}$ of topological spaces is called *compactifiable* (resp. *Polishable*) if there is a compact (resp. Polish) composition of $\mathcal{C}$, i.e. if there is a continuous map $q\colon A \to B$ between metrizable compacta (resp. Polish spaces) such that $\{q^{-1}(b) : b \in B\} \cong \mathcal{C}$. Note that the spaces $q^{-1}(b)$ are necessarily metrizable compacta (resp. Polish spaces).

**Construction 2.5** (rectangular composition)**.** Let $A$, $B$ be topological spaces and let $F \subseteq A \times B$. By $F^b$ we denote the subset of $A$ corresponding to the section of $F$ through $b$, i.e. $F^b = \{a \in A\colon \langle a, b\rangle \in F\}$. For every $b \in B$ let $e_b$ denote the canonical embedding $F^b \to F^b \times \{b\} \subseteq F$. The set $F$ induces the composition $\mathcal{A}_F(\pi_B{\restriction}_F\colon F \to B)$ of the family $\langle e_b \rangle_{b\in B}$. If the spaces $A$, $B$ are metrizable compacta (resp. Polish spaces) and the set $F$ is closed (resp. $G_\delta$) in $A \times B$, then the composition $\mathcal{A}_F$ is compact (resp. Polish).

Moreover, every composition can essentially be obtained this way. For a composition $\mathcal{A}(q\colon A \to B)$ we consider the graph of $q$, $G = \{\langle a, q(a)\rangle : a \in A\} \subseteq A \times B$, which is closed if $B$ is Hausdorff. Since $A$ is homeomorphic to $G$ and $G^b = q^{-1}(b)$ for every $b \in B$, the compositions $\mathcal{A}$ and $\mathcal{A}_G$ are essentially the same.

**Construction 2.6** (pullback composition)**.** Let $\mathcal{A}(q\colon A \to B)$ be a composition and let $f\colon B' \to B$ be a continuous map. The *pullback of* $\mathcal{A}$ *along* $f$ is the composition $\mathcal{A}'(q'\colon A' \to B')$ where $A' := \{\langle a, b'\rangle \in A \times B' : q(a) = f(b')\}$ and $q' := \pi_{B'}{\restriction}_{A'}$, so $\mathcal{A}'$ is the rectangular composition induced by $A' \subseteq A \times B'$.

If $\mathcal{A}$ is a composition of spaces $\langle A_b \rangle_{b\in B}$, then $\mathcal{A}'$ is essentially a composition of $\langle A_{f(b')} \rangle_{b'\in B'}$ since for every $b' \in B'$ we have the canonical embedding $e_{b'}\colon A_{f(b')} \to A_{f(b')} \times \{b'\} \subseteq A'$ and so $\mathcal{A}'$ is formally a composition of $\langle e_{b'} \rangle_{b'\in B'}$. This way we change the indexing space so that each space $A_b$ has $f^{-1}(b)$-many copies in $A'$.

Moreover, $A'$ is a closed subset $A \times B'$ if $B$ is Hausdorff. Hence, if $\mathcal{A}$ is a compact (resp. Polish) composition and $B'$ is a metrizable compactum (resp. a Polish space), then $\mathcal{A}'$ is a compact (resp. Polish) composition as well.

**Corollary 2.7** (subcomposition)**.** If $\mathcal{A}(q\colon A \to B)$ is a compact (resp. Polish) composition of spaces $\langle A_b \rangle_{b\in B}$ and $C \subseteq B$ is $F_\sigma$ (resp. analytic), then the class $\{A_c : c \in C\}$ is compactifiable (resp. Polishable).

*Proof.* In the compact case with closed $C \subseteq B$, it is enough to consider the *induced subcomposition* $\mathcal{A}_C(q\colon q^{-1}[C] \to C)$, which may be viewed as a special case of the pullback construction. If $C = \bigcup_{n\in\omega} C_n$ for some closed sets $C_n \subseteq B$, then $\{A_c : c \in C\}$ is a countable union of compactifiable classes, which is compactifiable as we will show later (Observation 2.14). In the Polish case, there is a Polish space $B'$ and a continuous surjection $f\colon B' \twoheadrightarrow C$, so the pullback of $\mathcal{A}$ along $f$ is a Polish composition of $\{A_{f(b')} : b' \in B'\} = \{A_c : c \in C\}$. ☐

**Remark 2.8.** We always consider an analytic set as a subset of a Polish space. By *analytic space* we mean any topological space that arises from an analytic set endowed with the corresponding subspace topology, i.e. a metrizable continuous image of a Polish space. However, in the following constructions (like in Lemma 2.9) we in fact do not need the metrizability, so the propositions would remain valid even for non-metrizable continuous images of Polish spaces.

**Lemma 2.9.** Let $A$ be a Polish space, let $B$ be an analytic space, let $F \subseteq A \times B$ be a $G_\delta$ subset, and let $\mathcal{A}_F(q\colon F \to B)$ be the corresponding rectangular composition. Moreover, let $B'$ be a Polish space and let $f\colon B' \to B$ be a continuous map. The pullback $\mathcal{A}'(q'\colon F' \to B')$ of $\mathcal{A}_F$ along $f$ is a Polish composition.

*Proof.* We need to show that the composition space $F'$ is Polish. We have $F' = \{\langle\langle a, b\rangle, b'\rangle \in (A \times B) \times B' : \langle a, b\rangle \in F \text{ and } b = f(b')\}$, which is canonically homeomorphic to $G := \{\langle a, b'\rangle \in A \times B' : \langle a, f(b')\rangle \in F\} = g^{-1}[F]$ where $g := \mathrm{id}_A \times f\colon A \times B' \to A \times B$. Since $F$ is $G_\delta$ in $A \times B$, $G$ is $G_\delta$ in the Polish space $A \times B'$. □

By combining the previous observations we obtain the following characterizations.

**Theorem 2.10.** The following conditions are equivalent for a class $\mathcal{C}$ of topological spaces.

(i) $\mathcal{C}$ is compactifiable.

(ii) There is a metrizable compactum $A$ and a closed equivalence relation $E \subseteq A \times A$ such that $\{E^a : a \in A\} \cong \mathcal{C} \setminus \{\emptyset\}$.

(iii) There is a metrizable compactum $A$, a metrizable $\sigma$-compact space $B$, and a closed set $F \subseteq A \times B$ such that $\{F^b\colon b \in B\} \cong \mathcal{C}$.

(iv) There is a closed set $F \subseteq [0, 1]^\omega \times 2^\omega$ such that $\{F^b : b \in 2^\omega\} \cong \mathcal{C}$, or $\mathcal{C} = \emptyset$.

**Theorem 2.11.** The following conditions are equivalent for a class $\mathcal{C}$ of topological spaces.

(i) $\mathcal{C}$ is Polishable.

(ii) There is a Polish space $A$ and a closed equivalence relation $E \subseteq A \times A$ such that $\{E^a : a \in A\} \cong \mathcal{C} \setminus \{\emptyset\}$.

(iii) There is a Polish space $A$, an analytic space $B$, and a $G_\delta$ set $F \subseteq A \times B$ such that $\{F^b\colon b \in B\} \cong \mathcal{C}$.

(iv) There is a $G_\delta$ set $F \subseteq [0, 1]^\omega \times \omega^\omega$ such that $\{F^b : b \in \omega^\omega\} \cong \mathcal{C}$, or $\mathcal{C} = \emptyset$.

(v) There is a closed set $F \subseteq (0, 1)^\omega \times \omega^\omega$ such that $\{F^b : b \in \omega^\omega\} \cong \mathcal{C}$, or $\mathcal{C} = \emptyset$.

*Proof of Theorem 2.10 and 2.11.*

(i) $\implies$ (ii). For a composition $\mathcal{A}(q\colon A \to B)$ of $\mathcal{C}$ it is enough to consider the equivalence $E := \{\langle a, a'\rangle \in A \times A : q(a) = q(a')\}$ induced by $q$.

(ii) $\implies$ (iii) is trivial if $\emptyset \notin \mathcal{C}$. Otherwise we consider a single-point extension $B \supseteq A$ such that $A$ is clopen in $B$ and use the same $E$. Also see Remark 2.13.

(iii) $\implies$ (i). We consider the induced rectangular composition $\mathcal{A}_F(q\colon F \to B)$ (see Construction 2.5). In the compact case with $B$ compact the proof is finished. If $B = \bigcup_{n\in\omega} B_n$ for some compacta $B_n$, then each $F \cap (A \times B_n)$ induces a compact composition of $\{F^b : b \in B_n\}$, and $\mathcal{C}$ is equivalent to a countable union of compactifiable classes, which is compactifiable by Observation 2.14. In the Polish case, there is a Polish space $B'$ and a continuous surjection $f\colon B' \twoheadrightarrow B$. Let $\mathcal{A}'$ be the pullback of $\mathcal{A}_F$ along $f$ (Construction 2.6). As in Corollary 2.7, $\mathcal{A}'$ is a composition of $\{F^b : b \in B\} \cong \mathcal{C}$, and it is Polish by Lemma 2.9.

(i) $\implies$ (iv), (v). Let $\mathcal{A}(q\colon A \to B)$ be a compact (resp. Polish) composition of $\mathcal{C}$. We may suppose that $B$ is nonempty. Otherwise, $\mathcal{C}$ is empty as well. Recall that every nonempty metrizable compactum is a continuous image of the Cantor space $2^\omega$ and that every nonempty Polish space is a continuous image of the Baire space $\omega^\omega$, so we may suppose that $B = 2^\omega$ (resp. $\omega^\omega$) by Construction 2.6. Recall that every separable metrizable space may be embedded into the Hilbert cube $[0,1]^\omega$, so we may suppose that $A \subseteq [0,1]^\omega$. Let $F$ be the graph of $q$. By the second part of Construction 2.5, $\{F^b : b \in B\} \cong \mathcal{C}$ and $F$ is closed in $A \times B$. Since $A$ is compact (resp. Polish), $A \times B$ and so $F$ is closed (resp. $G_\delta$) in $[0,1]^\omega$. This proves (iv). The proof of (v) is analogous and uses the fact that every Polish space may be embedded into $(0,1)^\omega$ as a closed subspace [8, 4.17].

The implications (iv), (v) $\implies$ (iii) are trivial. □

**Observation 2.12.** A class $\mathcal{C}$ of nonempty metrizable continua is compactifiable if and only if there exists a metrizable compactum $A$ whose set of components is equivalent to $\mathcal{C}$.

*Proof.* Let $\mathcal{A}(q\colon A \to B)$ be a compact composition of $\mathcal{C}$. By Theorem 2.10 the indexing space $B$ may be taken zero-dimensional (e.g. the Cantor space), and hence the spaces $q^{-1}(b)$ are precisely the components of $A$.

On the other hand, let $A$ be a metrizable compactum whose set of components is equivalent to $\mathcal{C}$. Let $q\colon A \to B$ be the quotient map induced by the decomposition of $A$ into its components. Since $A$ is a metrizable compactum, the components are equal to the quasi-components, and hence $B$ is totally separated (i.e. points can be separated by clopen sets), in particular Hausdorff. Therefore, $B$ is a metrizable compactum and $q$ induces the desired compact composition. □

Let us conclude this section with basic observations about (non)existence of compactifiable or Polishable classes.

**Remark 2.13.** If a class $\mathcal{C}$ is compactifiable (resp. Polishable), then so are the classes $\mathcal{C} \setminus \{\emptyset\}$ and $\mathcal{C} \cup \{\emptyset\}$. This is because if a map $q\colon A \to B$ induces a compact composition, then the maps $q\colon A \to q[A]$ and $q\colon A \to B \oplus \{\infty\}$ induces compact compositions as well. For Polishable $\mathcal{C}$ the case "$\mathcal{C} \cup \{\emptyset\}$" is the same, but the case "$\mathcal{C} \setminus \{\emptyset\}$" needs a comment. The map $q\colon A \to q[A]$ may not directly induce a Polish composition since $q[A]$ may not be $G_\delta$ in $B$. Nevertheless, it is analytic, so we use Corollary 2.7. In fact, this gives us the composition $\mathcal{A}_E$ for $E = \{\langle a, a'\rangle \in A \times A : q(a) = q(a')\}$.

**Observation 2.14.** Every countable union of compactifiable (resp. Polishable) classes is compactifiable (resp. Polishable).

*Proof.* Let $I$ be a set and for every $i \in I$ let $\mathcal{A}_i(q_i\colon A_i \to B_i)$ be a composition of a class $\mathcal{C}_i$. We consider the *sum composition* $\mathcal{A}(q\colon A \to B) := \sum_{i\in I} \mathcal{A}_i$, i.e. $A := \sum_{i\in I} A_i$, $B := \sum_{i\in I} B_i$, and $q := \sum_{i\in I} q_i\colon A \to B$. Clearly, $\mathcal{A}$ is a composition of $\bigcup_{i\in I} \mathcal{C}$. If $I$ is finite (resp. countable) and the compositions $\mathcal{A}_i$ are compact (resp. Polish), then $\mathcal{A}$ is also compact (resp. Polish).

It remains to consider a countable sum of compact compositions that is not compact. Without loss of generality, $\emptyset \notin \mathcal{C}_i \neq \emptyset$ for every $i \in I$ (Remark 2.13), and so $A$ and $B$ are separable metrizable locally compact non-compact spaces. We consider their one-point compactifications $A^+$ and $B^+$, which are metrizable, and the corresponding extension $q^+\colon A^+ \to B^+$ of the map $q$. The map $q^+$ is continuous since $q$ is perfect (i.e. closed with compact fibers), and it induces a composition of $\bigcup_{i\in I} \mathcal{C}_i \cup \{\{\infty\}\}$, so if the given classes contain a one-point space, we are done. Otherwise, we take any space $C \in \bigcup_{i\in I} \mathcal{C}_i$, attach it to the point $\infty \in A^+$, and modify the definition of $q^+$ accordingly. □

**Corollary 2.15.** Every countable family of metrizable compacta is compactifiable. Every countable family of Polish spaces is Polishable.

**Remark 2.16.** We require metrizability (or equivalently existence of a countable base) in the definition of compact composition not only to obtain a notion stronger than Polish composition, but because otherwise the corresponding compactifiability would be trivial. Using the one-point compactification as in the previous proof, we may easily construct a composition with compact composition space and compact indexing space for any family of compacta.

**Observation 2.17.** By Theorem 2.11 there are at most $\mathfrak{c}$-many nonequivalent Polishable classes since there are only $\mathfrak{c}$-many $G_\delta$ subsets of $[0,1]^\omega \times \omega^\omega$. On the other hand, there are $\mathfrak{c}$-many non-homeomorphic metrizable compact spaces – even in the real line. Hence, there are exactly $2^{\mathfrak{c}}$-many nonequivalent classes of metrizable compacta and also exactly $2^{\mathfrak{c}}$-many nonequivalent classes of Polish spaces. This cardinal argument gives us that many classes of metrizable compacta are not Polishable.

# 3 Compactifiability and hyperspaces

A class of topological spaces is often equivalent to a family of subspaces of some fixed ambient space. Therefore, it is natural to consider how compactifiability of such family is related to its properties when viewed as a subset of a hyperspace.

For a topological space $X$ we shall consider the hyperspaces of all subsets $\mathcal{P}(X)$, of all closed subsets $\mathcal{C}l(X)$, of all compact subsets $\mathcal{K}(X)$, and of all subcontinua $\mathcal{C}(X)$ endowed with the *Vietoris topology.* We include the empty set in the families. Recall that the *lower Vietoris topology* $\tau_V^-$ is generated by the sets $U^- = \{A : A \cap U \neq \emptyset\}$ for $U \subseteq X$ open, and the *upper Vietoris topology* $\tau_V^+$ is generated by the sets $U^+ = \{A : A \subseteq U\}$ for $U \subseteq X$ open. The Vietoris topology $\tau_V$ is their join.

Also recall that if $X$ is metrizable by a metric $d$, the corresponding *Hausdorff metric* $d_H$ on $\mathcal{C}l(X)$ is defined by $d_H(A, B) = \max(\delta(A, B), \delta(B, A))$ where

$\delta(A, B) = \sup_{x \in A} d(x, B) = \sup_{x \in A} \inf_{y \in B} d(x, y) = \inf\{\varepsilon : A \subseteq N_\varepsilon(B)\}$. We have $\delta(\emptyset, B) = 0$ for every $B$, and $\delta(A, \emptyset) = \infty$ for every $A \neq \emptyset$, and also $\delta(A, B) = \infty$ for every $A$ unbounded and $B$ bounded. Hence, strictly speaking, $d_H$ is an extended metric, but we may always cap it at 1 or suppose that $d \leq 1$ and interpret the infima in $[0, 1]$, so $\inf \emptyset = 1$. In any case, the singleton $\{\emptyset\}$ is clopen in $\mathcal{C}l(X)$ with both Vietoris topology and Hausdorff metric topology.

The Vietoris topology and the topology induced by the Hausdorff metric are not comparable on $\mathcal{C}l(X)$ in general, but they coincide on $\mathcal{K}(X)$. If $X$ is compact or Polish, so is $\mathcal{K}(X)$. Also, $\mathcal{C}(X)$ is a closed subspace of $\mathcal{K}(X)$ if $X$ is Hausdorff. For reference on the mentioned properties see [8, 4.F].

**Construction 3.1** (from hyperspace to composition)**.** Let $X$ be a topological space and let $\mathcal{F} \subseteq \mathcal{P}(X)$. We consider the set $A_\mathcal{F} := \{\langle x, F\rangle : x \in F \in \mathcal{F}\} \subseteq X \times \mathcal{F}$. Let us denote the corresponding composition (Construction 2.5) by $\mathcal{A}_\mathcal{F}$. Since $(A_\mathcal{F})^F = F$ for every $F \in \mathcal{F}$, we have that $\mathcal{A}_\mathcal{F}$ is a composition of the family $\mathcal{F}$ with composition space $A_\mathcal{F}$ and indexing space $\mathcal{F}$. The composition map is just the projection $\pi_\mathcal{F}\restriction_{A_\mathcal{F}}$. Also, $A_\mathcal{F} = \mathcal{R}_\in \cap (X \times \mathcal{F})$ where $\mathcal{R}_\in := \{\langle x, F\rangle \in X \times \mathcal{P}(X) : x \in F\}$ is the membership relation.

**Observation 3.2.** If $X$ is a regular space, then the membership relation of closed sets is closed, i.e. $\mathcal{R}_\in \cap (X \times \mathcal{C}l(X))$ is closed in $X \times \mathcal{C}l(X)$ (even with respect to $\tau_V^+$).

*Proof.* If $F \in \mathcal{C}l(X)$ and $x \in X \setminus F$, then there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $F \subseteq V$. We have that $U \times V^+$ is a neighborhood of $\langle x, F\rangle$ disjoint with $\mathcal{R}_\in$. □

**Proposition 3.3.**

(i) If $X$ is a metrizable compactum and $\mathcal{F}$ is an $F_\sigma$ subset of $\mathcal{K}(X)$ (resp. $\mathcal{C}(X)$), then $\mathcal{F}$ is a compactifiable class of compacta (resp. continua).

(ii) If $X$ is a Polish space and $\mathcal{F}$ is an analytic subset of $\mathcal{K}(X)$ (resp. $\mathcal{C}(X)$), then $\mathcal{F}$ is a Polishable class of compacta (resp. continua).

*Proof.* It is enough to use the set $A_\mathcal{F} \subseteq X \times \mathcal{F}$ from Construction 3.1 and Theorem 2.10 and 2.11. □

Next, we shall introduce a construction in the opposite direction, i.e. turning a composition into a subset of a hyperspace. But first, let us recall some further properties of hyperspaces and their induced maps.

**Observation 3.4.** If a space $X$ is identified with the family of its singletons $[X]^1$, then it becomes a subspace of $\mathcal{P}(X)$ with respect to all $\tau_V^-$, $\tau_V^+$, and $\tau_V$ since for every open $U \subseteq X$ we have $U^- \cap [X]^1 = U^+ \cap [X]^1 = [U]^1$.

**Notation 3.5.** Let $f\colon X \to Y$ be a map between sets. We shall use the notation for induced maps from [11, 5.9]:

- $f^*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ is the *image map* defined by $f^*(A) = f[A]$,

- $f^{-1*}\colon Y \to \mathcal{P}(X)$ is the *fiber map* defined by $f^{-1*}(y) = f^{-1}(y)$,
- $f^{-1**}\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ is the *preimage map* defined by $f^{-1**}(B) = f^{-1}[B]$.

The following proposition summarizes properties of the induced maps defined above. Some of the equivalences were proved by Michael [11, 5.10]. Note that our map $f$ does not have to be onto, we include the empty set in the hyperspace, and we also formulate the equivalences separately for $\tau_V^-$ and $\tau_V^+$.

**Proposition 3.6.** Let $f\colon X \to Y$ be a map between topological spaces.

(i) $f$ is continuous $\iff$ $f^*$ is $\tau_V^-$-continuous $\iff$ $f^*$ is $\tau_V^+$-continuous $\iff$ $f^*$ is $\tau_V$-continuous.

(ii) $f$ is an embedding $\iff$ $f^*$ is a $\tau_V^-$-embedding $\iff$ $f^*$ is a $\tau_V^+$-embedding $\iff$ $f^*$ is a $\tau_V$-embedding.

(iii) $f$ is an open embedding $\iff$ $f^*$ is a $\tau_V^+$-open embedding $\iff$ $f^*$ is a $\tau_V$-open embedding.

(iv) $f$ is a closed embedding $\iff$ $f^*$ is a $\tau_V^-$-closed embedding $\iff$ $f^*$ is a $\tau_V$-closed embedding.

(v) $f$ is open $\iff$ $f^{-1*}$ is $\tau_V^-$-continuous $\iff$ $f^{-1**}$ is $\tau_V^-$-continuous.

(vi) $f$ is closed $\iff$ $f^{-1*}$ is $\tau_V^+$-continuous $\iff$ $f^{-1**}$ is $\tau_V^+$-continuous.

(vii) $f$ is closed and open $\iff$ $f^{-1*}$ is $\tau_V$-continuous $\iff$ $f^{-1**}$ is $\tau_V$-continuous.

(viii) $f$ is continuous $\implies$ $f^{-1*}$ is $\tau_V$-(closed and open) onto its image.

*Proof sketch.* We use the following equalities.

$$
\begin{aligned}
(f^*)^{-1}[B^-] &= f^{-1}[B]^- & (f^*)^{-1}[B^+] &= f^{-1}[B]^+ \\
(f^*)[A^-] &= f[A]^- \cap \operatorname{rng}(f^*) & (f^*)[A^+] &= f[A]^+ \subseteq \operatorname{rng}(f^*) \quad f \text{ injective} \\
(f^{-1*})^{-1}[A^-] &= f[A] & (f^{-1*})^{-1}[A^+] &= f^{\forall}[A] := Y \setminus f[X \setminus A] \\
(f^{-1**})^{-1}[A^-] &= f[A]^- & (f^{-1**})^{-1}[A^+] &= f^{\forall}[A]^+
\end{aligned}
$$

$$
(f^{-1*})[B] = \begin{cases} f^{-1}[B]^- \cap \operatorname{rng}(f^{-1*}) & \text{if } B \subseteq \operatorname{rng}(f) \\ f^{-1}[B]^+ \cap \operatorname{rng}(f^{-1*}) & \text{if } B \nsubseteq \operatorname{rng}(f) \text{ or } f \text{ is onto} \end{cases}
$$

Regarding the embeddings, if $f$ is an embedding and $U \subseteq X$ is open, then $f[U] = V \cap \operatorname{rng}(f)$ for some open $V \subseteq Y$. Therefore, we have

$$
\begin{aligned}
&f^*[U^-] = f[U]^- \cap \operatorname{rng}(f) = (V \cap \operatorname{rng}(f))^- \cap \operatorname{rng}(f^*) = V^- \cap \operatorname{rng}(f^*), \\
&f^*[U^+] = f[U]^+ = (V \cap \operatorname{rng}(f))^+ = V^+ \cap \operatorname{rng}(f^*),
\end{aligned}
$$

and so $f^*$ is a $\tau_V^-$-, $\tau_V^+$- and hence a $\tau_V$-embedding. Regarding the closedness and openness, observe that $\operatorname{rng}(f^*) = \operatorname{rng}(f)^+$, so if $\operatorname{rng}(f)$ is open, then $\operatorname{rng}(f^*)$ is $\tau_V^+$-open, and if $\operatorname{rng}(f)$ is closed, then $\operatorname{rng}(f^*)$ is $\tau_V^-$-closed. For the backward implications we may use Observation 3.4 since $f$ may be viewed as a restriction $[X]^1 \to [Y]^1$ of $f^*$, and $\operatorname{rng}(f)$ is essentially $\operatorname{rng}(f^*) \cap [Y]^1$. $\square$

**Definition 3.7.** A composition $\mathcal{A}(q\colon A \to B)$ is called a *strong composition* if the composition map $q$ is closed and open and $|B \setminus \operatorname{rng}(q)| \leq 1$. A class $\mathcal{C}$ of topological spaces is called *strongly compactifiable* (resp. *strongly Polishable*) if there is a strong compact (resp. strong Polish) composition of $\mathcal{C}$.

The strongness of a composition means that the corresponding decomposition of $A$ is continuous (closedness correspond to upper semi-continuity and openness to lower semi-continuity). Note that the rather technical condition $|B \setminus \operatorname{rng}(q)| \leq 1$ and also clopenness of $\operatorname{rng}(q)$ can be obtained for every composition by removing $B \setminus \operatorname{rng}(q)$ and then eventually adding a clopen point (Remark 2.13). Also, the closedness of $q$ is trivial for compact compositions.

**Construction 3.8** (from composition to hyperspace)**.** To every composition $\mathcal{A}(q\colon A \to B)$ we assign the disjoint family $\mathcal{F}_{\mathcal{A}} := \{q^{-1}(b) : b \in B\} \subseteq \mathcal{P}(A)$.

We have $q^{-1*}\colon B \twoheadrightarrow \mathcal{F}_{\mathcal{A}} \subseteq \mathcal{P}(A)$, so we have two natural topologies on $\mathcal{F}_{\mathcal{A}}$ – the quotient topology induced by $q^{-1*}$ from $B$, and the subspace topology induced from the hyperspace $\mathcal{P}(A)$. By Proposition 3.6 the Vietoris topology is finer than the quotient topology. The converse holds if and only if $q$ is both closed and open. The map $q^{-1*}$ is a homeomorphism with respect to the quotient topology if and only if it is a bijection, which happens if and only if $|B \setminus \operatorname{rng}(q)| \leq 1$. Therefore, $\mathcal{F}_{\mathcal{A}}$ is homeomorphic to $B$ via $q^{-1*}$ if and only if the composition $\mathcal{A}$ is strong.

In this case, if $\mathcal{A}$ is a compact (resp. Polish) composition of compacta, then $\mathcal{F}_{\mathcal{A}}$ is compact (resp. Polish), and so it is a closed (resp. $G_\delta$) subset of the compact (resp. Polish) hyperspace $\mathcal{K}(A)$.

**Observation 3.9.** If $\mathcal{A}(q\colon A \to B)$ is a strong composition, then the family $\mathcal{F}_{\mathcal{A}}$ is closed in every Hausdorff space $\mathcal{H} \subseteq \mathcal{P}(A)$ containing it.

*Proof.* Let us consider the family $\mathcal{F}^{\bigcup} := \{F \in \mathcal{H} : q^{-1}[q[F]] = F\}$, which is closed since $q^{-1**} \circ q^*$ is continuous and $\mathcal{H}$ is Hausdorff, and the family $\mathcal{F}^{\downarrow} := (q^*)^{-1}[[B]^{\leq 1}]$ (where $[B]^{\leq 1}$ denotes the family of all subsets of $B$ with at most one element), which is also closed since $B \cong \mathcal{F}_{\mathcal{A}}$ is Hausdorff, and so $[B]^{\leq 1}$ is closed in $\mathcal{P}(B)$. To conclude, it is enough to observe that $\mathcal{F}_{\mathcal{A}} \subseteq \mathcal{F}^{\bigcup} \cap \mathcal{F}^{\downarrow} \subseteq \mathcal{F}_{\mathcal{A}} \cup \{\emptyset\}$. □

**Lemma 3.10.** Let $X$, $Y$ be topological spaces, and let $R \subseteq X \times Y$. Let us consider the map $\rho\colon Y \to \mathcal{P}(X)$ defined by $\rho(y) := R^y$.

(i) The map $\pi_Y{\restriction}_R\colon R \to Y$ is open if and only if the map $\rho$ is $\tau_V^-$-continuous.

(ii) The map $\pi_Y{\restriction}_R\colon R \to Y$ is closed if and only if the map $\rho$ is $\tau_V^+$-continuous and every set $R^y \times \{y\}$ has a rectangular neighborhood basis (r.n.b.), i.e. every its neighborhood in $R$ contains a neighborhood of form $R \cap (U \times V)$ for some open sets $U$ and $V$. The r.n.b. condition is satisfied if $\operatorname{rng}(\rho) \subseteq \mathcal{K}(X)$.

*Proof.* The necessity of $\tau_V^{+/-}$-continuity follows from equality $\rho = \pi_X^* \circ (\pi_Y{\restriction}_R)^{-1*}$ and from Proposition 3.6. The open case follows from equality $\pi_Y[R \cap (U \times V)] = \{y \in V : R^y \cap U \neq \emptyset\} = \rho^{-1}[U^-] \cap V$. The map $\pi_Y{\restriction}_R$ is closed if and only if for every closed $F \subseteq R$ and every $y \in Y \setminus \pi_Y[F]$ there is an open neighborhood $W$ of $y$ disjoint with $\pi_Y[F]$. Considering $R \cap (X \times W)$ gives us necessity of the r.n.b. condition. On the other hand, if $U \times V$ is an open neighborhood of $R^y \times \{y\} \neq \emptyset$

disjoint with $F$, then we put $W := \rho^{-1}[U^+] \cap V$. Note that $z \in \rho^{-1}[U^+]$ if and only if $R^z \subseteq U$. Hence, if $\langle x, z\rangle \in R$ and $z \in W$, then $\langle x, z\rangle \in U \times V$ and so it cannot be in $F$. If $R^y = \emptyset$, then we put $W := Y \setminus \pi_Y[R]$, which is open since $\pi_Y[R] = \rho^{-1}[X^-]$ and $X^-$ is $\tau_V^+$-closed. The r.n.b. condition holds if every $R^y \times \{y\}$ is compact by the tube lemma [6, 3.1.15]. □

**Corollary 3.11.** Let $\mathcal{A}_\mathcal{F}(q\colon A_\mathcal{F} \to \mathcal{F})$ be the composition obtained by Construction 3.1 from a family $\mathcal{F} \subseteq \mathcal{P}(X)$. We have that the map $q$ is open and $|\mathcal{F} \setminus \mathrm{rng}(q)| \leq 1$. If $\mathcal{F} \subseteq \mathcal{K}(X)$, then $q$ is also closed, and hence the composition is strong.

*Proof.* The map $q$ is the projection $\mathcal{R}_\in \cap (X \times \mathcal{F}) \to \mathcal{F}$, so we may use Lemma 3.10. The corresponding map $\rho$ is $\mathrm{id}\colon \mathcal{F} \to \mathcal{P}(X)$, which is both $\tau_V^-$- and $\tau_V^+$-continuous. The fact that $|\mathcal{F} \setminus \mathrm{rng}(q)| \leq 1$ is clear since there is only one empty set. □

**Corollary 3.12.** Let $\mathcal{A}(q\colon A \to B)$ be a composition of spaces $\langle A_b\rangle_{b \in B}$, let $f\colon B' \to B$ be a continuous map, and let $\mathcal{A}'(q'\colon A' \to B')$ be the pullback of $\mathcal{A}$ along $f$ (Construction 2.6). If $q$ is open, so is $q'$. If $q$ is closed and every space $A_b$ is compact, then $q'$ is also closed. It follows that strong compositions of compact spaces are preserved by pullbacks (such that $|f^{-1}[B \setminus \mathrm{rng}(f)]| \leq 1$).

*Proof.* We apply Lemma 3.10 to $A' \subseteq A \times B'$. The corresponding map $\rho$ is $q^{-1*} \circ f$, which is $\tau_V^-$- (resp. $\tau_V^+$-)continuous if $q$ is open (resp. closed) by Proposition 3.6. □

By putting all the previous claims and propositions together, we obtain the following characterizations – compare with Theorem 2.10 and 2.11.

**Theorem 3.13.** The following conditions are equivalent for a class $\mathcal{C}$ of topological spaces.

(i) $\mathcal{C}$ is strongly compactifiable.
(ii) There is a metrizable compactum $X$ and a closed family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.
(iii) There is a closed zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

**Theorem 3.14.** The following conditions are equivalent for a class $\mathcal{C}$ of topological spaces.

(i) $\mathcal{C}$ is a strongly Polishable class of compacta.
(ii) There is a Polish space $X$ and an analytic family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.
(iii) There is a $G_\delta$ zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.
(iv) There is a closed zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}((0,1)^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

*Proof.* Let $\mathcal{F} \subseteq \mathcal{K}(X)$. Construction 3.1 gives us the corresponding composition $\mathcal{A}_{\mathcal{F}}$, which is strong by Corollary 3.11. If $X$ is a metrizable compactum and $\mathcal{F}$ is closed, then the composition $\mathcal{A}_{\mathcal{F}}$ is compact. If $X$ is Polish and $\mathcal{F}$ is analytic, then there is a continuous surjection $f\colon Y \twoheadrightarrow \mathcal{F}$ from a Polish space $Y$ such that $|f^{-1}(\emptyset)| \leq 1$. The pullback of $\mathcal{A}_{\mathcal{F}}$ along $f$ (Construction 2.6) is a composition of $\mathcal{F}$ that is Polish by Lemma 2.9 and strong by Corollary 3.12.

On the other hand, let $\mathcal{A}(q\colon A \to B)$ be a strong compact (resp. strong Polish) composition of $\mathcal{C}$. Without loss of generality, $B$ is zero-dimensional (we use Construction 2.6 as in Theorem 2.10 and 2.11 together with Corollary 3.12). Construction 3.8 gives us the corresponding zero-dimensional disjoint family $\mathcal{F}_{\mathcal{A}} \subseteq \mathcal{K}(A)$, which is closed by Observation 3.9. There is an embedding $e\colon A \hookrightarrow [0,1]^\omega$, and so $e^*\colon \mathcal{K}(A) \hookrightarrow \mathcal{K}([0,1]^\omega)$ is an embedding by Proposition 3.6. In the compact case, $e^*[\mathcal{F}_{\mathcal{A}}]$ is compact and so closed in $\mathcal{K}([0,1]^\omega)$. In the Polish case, $e^*[\mathcal{F}_{\mathcal{A}}]$ is Polish and so $G_\delta$ in $\mathcal{K}([0,1]^\omega)$. Moreover, there is a closed embedding $i\colon A \hookrightarrow (0,1)^\omega$ by [8, 4.17], and so $i^*\colon \mathcal{K}(A) \hookrightarrow \mathcal{K}((0,1)^\omega)$ is a closed embedding by Proposition 3.6. Hence, $i^*[\mathcal{F}_{\mathcal{A}}]$ is a closed subset of $\mathcal{K}((0,1)^\omega)$.

The remaining implications are trivial. □

**Lemma 3.15.** Let $X$ be a metric space and let $\mathcal{R}$ denote the family $\{\langle A, B\rangle \in \mathcal{K}(X)^2 : A \subseteq B\}$ viewed as a subspace of $\langle \mathcal{K}(X), \tau_V\rangle \times \langle \mathcal{K}(X), \tau_V^+\rangle$. The Hausdorff metric $d_H\colon \mathcal{R} \to [0,\infty)$ is upper semi-continuous.

*Proof.* Let $\langle A, B\rangle \in \mathcal{R}$ and $r > d_H(A, B)$. We want to find $\mathcal{U}$ a $\tau_V$-neighborhood of $A$ and $\mathcal{V}$ a $\tau_V^+$-neighborhood of $B$ such that $d_H(A', B') < r$ for every $A' \in \mathcal{U}$ and $B' \in \mathcal{V}$.

For every $\langle A', B'\rangle \in \mathcal{R}$ we have $d_H(A', B') = \delta(B', A') = \inf\{\varepsilon > 0 : B' \subseteq N_\varepsilon(A')\}$. Hence, $d_H(A', B') = \delta(B', A') \leq \delta(B', B) + \delta(B, A) + \delta(A, A') \leq \delta(B', B) + d_H(A, B) + d_H(A, A')$.

Let $\varepsilon > 0$ such that $d_H(A, B) + 2\varepsilon < r$. We put $\mathcal{U} := \{A' : d_H(A, A') < \varepsilon\}$ and $\mathcal{V} := N_\varepsilon(B)^+$. The set $\mathcal{U}$ is $\tau_V$-open since the Hausdorff metric topology coincides with the Vietoris topology on $\mathcal{K}(X)$, and $\mathcal{V}$ is clearly $\tau_V^+$-open. Moreover, for every $B' \in \mathcal{V}$ we have $\delta(B', B) \leq \varepsilon$. Therefore, for every $\langle A', B'\rangle \in \mathcal{U} \times \mathcal{V}$ we have $d_H(A', B') < d_H(A, B) + 2\varepsilon < r$. □

**Proposition 3.16.** Let $\mathcal{A}(q\colon A \to B)$ be a Polish composition of compacta such that the composition map $q$ is closed. The family $\mathcal{F}_{\mathcal{A}} \subseteq \mathcal{K}(A)$ obtained via Construction 3.8 is $G_\delta$.

*Proof.* As in the proof of Observation 3.9 we have $\mathcal{F}_{\mathcal{A}} \subseteq \mathcal{F}^{\bigcup} \cap \mathcal{F}^{\downarrow} \subseteq \mathcal{F}_{\mathcal{A}} \cup \{\emptyset\}$, and the family $\mathcal{F}^{\downarrow}$ is closed. But the family $\mathcal{F}^{\bigcup} = \{F \in \mathcal{K}(A) : \widehat{F} := q^{-1}[q[F]] = F\}$ is now not necessarily closed since the map $q^{-1**} \circ q^*$ is not necessarily continuous. It is only $\tau_V^+$-continuous since $q$ is closed.

Let $d$ be a compatible metric on $A$ and let $\mathcal{G}_n := \{F \in \mathcal{K}(A) : d_H(F, \widehat{F}) < \frac{1}{n}\}$. Clearly, $\mathcal{F}^{\bigcup} = \bigcap_{n\in\mathbb{N}} \mathcal{G}_n$, so it is enough to show that each $\mathcal{G}_n$ is open. Let $\mathcal{R}$ be the space from Lemma 3.15 for the base space $A$. The map $\mathrm{id}_{\mathcal{F}_{\mathcal{A}}} \triangle (q^{-1**} \circ q^*)\colon \mathcal{F}_{\mathcal{A}} \to \mathcal{R}$ that maps $F \mapsto \langle F, \widehat{F}\rangle$ is continuous since $q^{-1**} \circ q^*$ is $\tau_V^+$-continuous. By Lemma 3.15 the map $d_H\colon \mathcal{R} \to [0,\infty)$ is upper semi-continuous. Together, the map $F \mapsto d_H(F, \widehat{F})$ is upper semi-continuous, and the families $\mathcal{G}_n$ are open. □

**Corollary 3.17.** Every compactifiable class is strongly Polishable. Also, in the definition of strong Polishability it is enough that the witnessing composition map is closed.

We have shown that every compactifiable class is strongly Polishable. On the other hand, strongly Polishable classes of compacta are sometimes close to being compactifiable. Compare the following characterization with Theorem 2.10 and 2.11.

**Theorem 3.18.** The following conditions are equivalent for a class $\mathcal{C}$ of topological spaces.

(i) $\mathcal{C}$ is a strongly Polishable class of compacta.

(ii) There is a metrizable compactum $A$, an analytic space $B$, and a closed set $F \subseteq A \times B$ such that $\{F^b : b \in B\} \cong \mathcal{C}$.

(iii) There is a closed set $F \subseteq [0,1]^\omega \times \omega^\omega$ such that $\{F^b : b \in \omega^\omega\} \cong \mathcal{C}$, or $\mathcal{C} = \emptyset$.

(iv) There is a closed set $F \subseteq [0,1]^\omega \times 2^\omega$ and a $G_\delta$ set $G \subseteq 2^\omega$ such that $\{F^b : b \in G\} \cong \mathcal{C}$ and $\{F^b : b \in 2^\omega\} = \overline{\{F^b : b \in G\}}$ in $\mathcal{K}([0,1]^\omega)$, or $\mathcal{C} = \emptyset$.

*Proof.* (ii) $\implies$ (i): Let $f\colon B' \twoheadrightarrow B$ be a continuous surjection from a Polish space $B'$. Let $F'$ denote the set $\{\langle a, b'\rangle \in A \times B' : \langle a, f(b')\rangle \in F\}$, which is closed as a continuous preimage of $F$. The induced rectangular composition $\mathcal{A}_{F'}(q\colon F' \to B')$ is a Polish composition of $\mathcal{C}$ (cf. Lemma 2.9). The map $q = \pi_{B'}\restriction_{F'}$ is closed by the Kuratowski theorem [6, Theorem 3.1.16] since $A$ is compact. Therefore, $\mathcal{C}$ is strongly Polishable by Proposition 3.16.

(i) $\implies$ (iii) and (i) $\implies$ (iv): By Theorem 3.14 there is a $G_\delta$ family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ equivalent to $\mathcal{C}$. We may suppose that $\mathcal{F}$ is nonempty. For (iii) we consider the composition $\mathcal{A}_\mathcal{F}$ (Construction 3.1) and its pullback (Construction 2.6) along a continuous surjection $f\colon \omega^\omega \twoheadrightarrow \mathcal{F}$, i.e. $F := \{\langle x, y\rangle \in [0,1]^\omega \times \omega^\omega : x \in f(y)\}$. For (iv) we do the same, but with $\overline{\mathcal{F}}$ and a continuous surjection $f\colon 2^\omega \twoheadrightarrow \overline{\mathcal{F}}$, i.e. $F := \{\langle x, y\rangle \in [0,1]^\omega \times 2^\omega : x \in f(y)\}$, and we put $G := f^{-1}[\mathcal{F}]$. Clearly, we have $\{F^b : b \in G\} = \mathcal{F}$ and $\{F^b : b \in 2^\omega\} = \overline{\mathcal{F}}$.

The remaining implications are trivial. □

The last condition condition of the previous theorem is quite close to compactifiability. It is enough to modify the fibers $F^b$ for $b \in 2^\omega \setminus G$, so they become spaces from the given class $\mathcal{C}$, while keeping the modified set $F' \subseteq [0,1]^\omega \times 2^\omega$ closed.

**Theorem 3.19.** Let $\langle X, d\rangle$ be a metric compactum and for every $n \in \omega$ let $\mathcal{A}_n$ be a finite covering of $X$ by closed sets of diameter $< 2^{-n}$. For every $F \in \mathcal{K}(X)$ let $\mathcal{A}_n(F)$ denote the space $\bigcup\{A \in \mathcal{A}_n : A \cap F \neq \emptyset\}$. Every $G_\delta$ family $\mathcal{F} \subseteq \mathcal{K}(X)$ containing a copy of every space from $\{\mathcal{A}_n(F) : F \in \overline{\mathcal{F}}, n \in \omega\}$ is compactifiable.

*Proof.* If $\mathcal{F} = \emptyset$, the theorem holds. Otherwise, there is a continuous surjection $f\colon 2^\omega \twoheadrightarrow \overline{\mathcal{F}}$. The set $G := f^{-1}[\mathcal{F}]$ is $G_\delta$ in $2^\omega$, and so its complement can be written as a disjoint union $\bigcup_{n\in\omega} K_n$ of compact sets. As before, we consider the pullback of the induced composition of $\overline{\mathcal{F}}$, i.e. the closed set $F := \{\langle x, b\rangle \in X \times 2^\omega : x \in f(b)\}$. For every $n \in \omega$ let $H_n := \bigcup\{\mathcal{A}_n(F^b) \times \{b\} : b \in K_n\} \subseteq X \times K_n$, and let $F' := F \cup \bigcup_{n\in\omega} H_n$.

We need to prove that $F'$ is closed. Then it is clear that $F'$ induces a compact composition of $\mathcal{F}$ since $\{(F')^b : b \in G\} = \mathcal{F}$ and $(F')^b = \mathcal{A}_n(F^b)$ for $b \in K_n$. Every set $H_n$ is closed since it is equal to $\bigcup_{A\in\mathcal{A}_n} A \times (f^{-1}[A^- \cap \overline{\mathcal{F}}] \cap K_n)$. Moreover, we have $H_n \subseteq N_{2^{-n}}(F)$ for a suitable metric on $X \times 2^\omega$. Altogether, $\overline{F'} = F \cup \bigcup_{n\in\omega} H_n \cup \bigcap_{k\in\omega} \overline{\bigcup_{n\geq k} H_n}$, and the last term is below $\bigcap_{k\in\omega} \overline{N_{2^{-k}}(F)} = F$. □

**Lemma 3.20.** Let $X$ be a Polish space such that $X \times \omega^\omega$ embeds into $X$. Every analytic family $\mathcal{F} \subseteq \mathcal{K}(X)$ is equivalent to a $G_\delta$ family $\mathcal{G} \subseteq \mathcal{K}(X)$.

*Proof.* There is a Polish space $Y \in \{\emptyset, \omega^\omega, \omega^\omega \oplus 1\}$ and a continuous surjection $f\colon Y \twoheadrightarrow \mathcal{F}$ such that $|f^{-1}(\emptyset)| \leq 1$. As in the proof of Theorem 3.14, the pullback $\mathcal{A}'(q\colon A' \to Y)$ of the composition $\mathcal{A}_{\mathcal{F}}$ along $f$ is a strong Polish composition of $\mathcal{F}$. Hence, the corresponding family of fibers $\mathcal{F}_{\mathcal{A}'} \subseteq \mathcal{K}(A')$ is $G_\delta$ (Construction 3.8). Since the composition space $A'$ is a subspace of $X \times Y$, it embeds into $X \times \omega^\omega$, and so into $X$. For an embedding $e\colon A' \hookrightarrow X$, the induced map $e^*\colon \mathcal{K}(A') \hookrightarrow \mathcal{K}(X)$ is also an embedding by Proposition 3.6. Since $A'$ and $\mathcal{K}(A')$ are Polish spaces, $e^*[\mathcal{F}_{\mathcal{A}'}] \subseteq \mathcal{K}(X)$ is the desired $G_\delta$ family equivalent to $\mathcal{F}$. □

**Corollary 3.21.** We have the following applications of the previous theorem.

(i) Every analytic subset of $\mathcal{C}([0,1]^\omega)$ containing a copy of every Peano continuum is compactifiable. In particular, the class of all Peano continua is compactifiable.

(ii) Every analytic subset of $\mathcal{K}(2^\omega)$ containing a copy of $2^\omega$ is compactifiable.

(iii) Every $G_\delta$ subset of $\mathcal{C}(D_\omega)$ containing a copy of $D_\omega$ is compactifiable ($D_\omega$ denotes the Ważewski's universal dendrite [14, 10.37]).

*Proof.* Let $X$ denote $[0,1]^\omega$ or $2^\omega$ or $D_\omega$, respectively. By Lemma 3.20 there is a $G_\delta$ family $\mathcal{F} \subseteq \mathcal{K}(X)$ equivalent to the original family. By Theorem 3.19 it is enough to find suitable coverings $\mathcal{A}_n$ of $X$ such that every space from $\mathcal{G} := \{\mathcal{A}_n(F) : F \in \overline{\mathcal{F}}, n \in \omega\}$ is homeomorphic to a space from $\mathcal{F}$. We cover $X$ by its copies of sufficiently small diameters. In (i) every space from $\mathcal{G}$ is a connected finite union Hilbert cubes, and so a Peano continuum. In (ii) every space from $\mathcal{G}$ is a finite union of Cantor spaces, and so a Cantor space. In (iii) every space from $\mathcal{G}$ is a connected finite union of copies of $D_\omega$ in $D_\omega$, and so a copy of $D_\omega$ if we choose the coverings so that for every $A \in \bigcup_{n\in\omega} \mathcal{A}_n$ all branching points of $D_\omega$ in $A$ are in the interior of $A$. □

In Theorem 3.18 we have characterized strong Polishability in the language of rectangular compositions to make a connection with compactifiability. Now we characterize compactifiability using families in hyperspaces to make a connection with strong compactifiability.

**Theorem 3.22.** The following conditions are equivalent for a class $\mathcal{C}$ of topological spaces.

(i) $\mathcal{C}$ is compactifiable.

(ii) There is a metrizable compactum $X$ and a family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$ and $\langle \mathcal{F}, \tau \rangle$ is a metrizable compactum for a topology $\tau \supseteq \tau_V^+$.

(iii) There is a $G_\delta$ disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$ and $\langle \mathcal{F}, \tau_V^+ \rangle$ is a zero-dimensional metrizable compactum.

*Proof.* For (ii) $\implies$ (i) we use Construction 3.1 on $\langle \mathcal{F}, \tau \rangle$. We obtain a composition of $\mathcal{C}$ that is compact by Observation 3.2.

(i) $\implies$ (iii). Let $\mathcal{A}(q\colon A \to B)$ be a compact composition of $\mathcal{C}$. We may suppose that $B$ is zero-dimensional by Theorem 2.10, that $|B \setminus \operatorname{rng}(q)| \leq 1$ by Observation 2.13, and that $A \subseteq [0,1]^\omega$. The family $\mathcal{F}_\mathcal{A}$ obtained by Construction 3.8 is disjoint and by Proposition 3.16 $G_\delta$ in $\mathcal{K}(A) \subseteq \mathcal{K}([0,1]^\omega)$. Since $|B \setminus \operatorname{rng}(q)| \leq 1$ and the map $q$ is closed, $q^{-1*}\colon B \to \langle \mathcal{F}, \tau_V^+ \rangle$ is a homeomorphism.

(iii) $\implies$ (ii) is trivial. □

**Question 3.23.** Is there a similar characterization for Polishable classes?

Figure 1 summarizes the implications between composition-related properties and descriptive complexity of the corresponding subsets of the space of all metrizable compacta $\mathcal{K}([0,1]^\omega)$. The left part and the right part follow from the characterization theorems: 2.10, 2.11, 3.13, 3.14. The implication "compactifiable $\implies$ $G_\delta$" follows from Proposition 3.16. As a byproduct, we obtain the dashed implications.

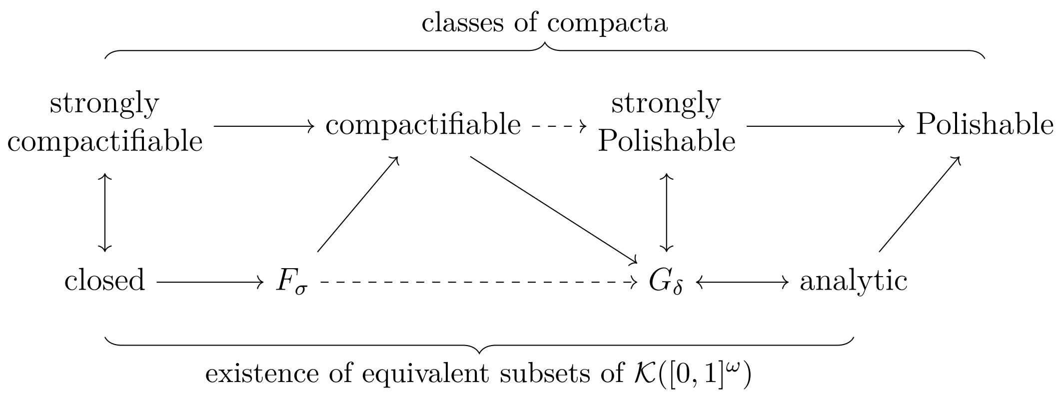


Figure 1: Implications between the considered classes.

**Question 3.24.** We do not know which implications can be reversed. Namely, we have the following questions.

(i) Is there a compactifiable class that is not strongly compactifiable?

(ii) Is there a strongly Polishable class that is not compactifiable?

(iii) Is there a Polishable class that is not strongly Polishable?

To summarize, this chapter relates (strong) compactifiability or Polishability of a class of metrizable compacta to the lowest complexity of its realizations in the hyperspace $\mathcal{K}([0,1]^\omega)$. This complexity in $\mathcal{K}([0,1]^\omega)$ up to the equivalence is studied in [1] by the first author.

## 3.1 The Wijsman hypertopologies

So far we have considered mostly the hyperspace of all compact subsets $\mathcal{K}(X)$ endowed with the Vietoris topology (or equivalently Hausdorff metric topology for metrizable $X$). There we have the one-to-one correspondence between subsets of the hyperspace and strong compositions (Construction 3.1 and 3.8). On the other hand, we are limited to Polishable classes of compact rather than Polish spaces.

For a Polish space $X$ we would like $\mathcal{C}l(X)$ to be Polish as well, but the Vietoris topology on $\mathcal{C}l(X)$ is not metrizable unless $X$ is compact, and the Hausdorff metric topology is not separable unless $X$ is compact. That is why we will also consider so-called *Wijsman topology*. The Wijsman topology induced by the metric $d$ is the one projectively generated by the family $\{d(x, \cdot)\colon \mathcal{C}l(X) \to \mathbb{R}\}_{x \in X}$. It was shown in [3] that $\mathcal{C}l(X)$ with the Wijsman topology induced by a complete metric is a Polish space for a Polish space $X$. Usually the Wijsman topology is defined only on $\mathcal{C}l(X) \setminus \{\emptyset\}$, and is then extended to $\mathcal{C}l(X)$ in a way related to the one-point compactification. For our purposes we may use the projectively generating definition directly to $\mathcal{C}l(X)$, which results in $\{\emptyset\}$ being clopen.

The Wijsman topology is coarser than both Vietoris and Hausdorff metric topology, and in general they are not equal even on $\mathcal{K}(X)$. In general, $\mathcal{K}(X)$ is an $F_{\sigma\delta}$-subspace of $\mathcal{C}l(X)$ with respect to the Wijsman topology, but it is not necessarily $G_\delta$ [3]. Given a metric $d$ on $X$ we may identify a set $A \in \mathcal{C}l(X)$ with the function $d(\cdot, A)\colon X \to \mathbb{R}$. Therefore, the Wijsman topology is inherited from the space of all continuous functions $C(X, \mathbb{R})$ with the topology of pointwise convergence. On the other hand, $d_H(A, B) = \sup_{x \in X} |d(x, A) - d(x, B)|$, so the Hausdorff metric topology is inherited from $C(X, \mathbb{R})$ with the topology of uniform convergence.

The Observation 3.2 holds also for the Wijsman topologies.

**Observation 3.25.** If $X$ is a metrizable space and $\mathcal{C}l(X)$ is endowed with a Wijsman topology, then $\mathcal{R}_\in \cap (X \times \mathcal{C}l(X))$ is closed in $X \times \mathcal{C}l(X)$.

*Proof.* If $F \in \mathcal{C}l(X)$ and $x \in X \setminus F$, then $r := d(x, F) > 0$. We put $U = \{y \in X : d(x, y) < \frac{r}{2}\}$ and $\mathcal{V} = \{H \in \mathcal{C}l(X) : d(x, H) > \frac{r}{2}\}$, so $U \times \mathcal{V}$ is a neighborhood of $\langle x, F \rangle$ disjoint with $\mathcal{R}_\in$. □

It follows that we may use Construction 3.1 also for Wijsman hyperspaces to obtain Polish compositions. The following proposition extends Proposition 3.3.

**Proposition 3.26.** If $X$ is a Polish space and $\mathcal{C}l(X)$ is endowed with the Wijsman topology induced by a complete metric, then every analytic subset of $\mathcal{C}l(X)$ is a Polishable class of Polish spaces.

**Remark 3.27.** Since every Polish space can be embedded as a closed subspace to $(0, 1)^\omega$, the hyperspace $\mathcal{C}l((0, 1)^\omega)$ endowed with the Wijsman topology induced by a complete metric may be viewed as a Polish space of all Polish spaces.

**Question 3.28.** Let $\mathcal{C}$ be a Polishable class and let $\mathcal{C}l((0, 1)^\omega)$ be endowed with a Wijsman topology induced by a complete metric. Does there exist an analytic (or even $G_\delta$ or closed) family $\mathcal{F} \subseteq \mathcal{C}l((0, 1)^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$?

# 4 Induced classes

In this section we shall analyze how the properties of being compactifiable and Polishable are preserved under various modifications and constructions of induced classes.

**Proposition 4.1.** Strongly compactifiable, compactifiable, strongly Polishable, and Polishable classes are stable under countable unions.

*Proof.* For compactifiable and Polishable classes this is Observation 2.14. Let $\mathcal{C}_n$, $n \in \omega$, be strongly Polishable classes. By Theorem 3.14 each of them is equivalent to an analytic family $\mathcal{F}_n \subseteq \mathcal{K}([0,1]^\omega)$. We have $\bigcup_{n\in\omega} \mathcal{C}_n \cong \bigcup_{n\in\omega} \mathcal{F}_n$, which is also analytic and hence strongly Polishable. In the strongly compactifiable case we proceed analogously, but end up with an $F_\sigma$ family $\bigcup_{n\in\omega} \mathcal{F}_n$. The conclusion follows from the non-trivial fact, that every $F_\sigma$ family in $\mathcal{K}([0,1]^\omega)$ is equivalent to a closed family, and hence is strongly compactifiable [1, Theorem 3.6]. □

**Remark 4.2.** In the previous proof we have used the fact that $\bigcup_{i\in I} \mathcal{C}_i \cong \bigcup_{i\in I} \mathcal{D}_i$ for every collection of equivalent classes $\mathcal{C}_i \cong \mathcal{D}_i$, $i \in I$. However, it is not necessary that even $\mathcal{C}_i \cap \mathcal{C}_j \cong \mathcal{D}_i \cap \mathcal{D}_j$, so we cannot use the same argument for proving preservation under intersections – compare with Proposition 4.32.

**Observation 4.3.** Let $X$ be a metric space. The map $\operatorname{diam}\colon \mathcal{P}(X) \to [0,\infty)$ is both $\langle \tau_V^+, \tau_U \rangle$- and $\langle \tau_V^-, \tau_L \rangle$-continuous, where $\tau_U$ and $\tau_L$ are the upper and lower semi-continuous topologies on $[0,\infty)$. It follows that diam is continuous.

*Proof.* If $\operatorname{diam}(A) < r$, then there is $\varepsilon > 0$ such that $\operatorname{diam}(N_\varepsilon(A)) < r$. Hence, $\operatorname{diam}(A') < r$ for every $A' \in N_\varepsilon(A)^+$. If $\operatorname{diam}(A) > r$, then there are points $x, y \in A$ and $\varepsilon > 0$ such that $d(x,y) \geq r + 2\varepsilon$. Hence, $\operatorname{diam}(A') > r$ for every $A' \in B(x,\varepsilon)^- \cap B(y,\varepsilon)^-$. □

**Corollary 4.4.** Let $\mathcal{A}(q\colon A \to B)$ be a compact composition of a family $\langle A_b \rangle_{b\in B}$. For every $\varepsilon > 0$ the set $B_\varepsilon := \{b \in B : \operatorname{diam}(A_b) \geq \varepsilon\}$ is closed, and the set $B_0 := \{b \in B : \operatorname{diam}(A_b) > 0\}$ is $F_\sigma$. It follows that the corresponding families of spaces are also compactifiable.

*Proof.* The map $(\operatorname{diam} \circ q^{-1*})\colon B \to [0,\infty)$ is upper semi-continuous since $q^{-1*}$ is $\tau_V^+$-continuous and diam is $\langle \tau_V^+, \tau_U \rangle$-continuous by Observation 4.3. Note that the intervals $[\varepsilon,\infty)$ are $\tau_U$-closed, and so the interval $(0,\infty)$ is $\tau_U$-$F_\sigma$. □

In definitions of many natural classes of compacta, degenerate spaces are occasionally included, resp. excluded. The following proposition shows that with respect to compactifiability, it does not matter.

**Proposition 4.5.** If a class $\mathcal{C}$ of metrizable compacta is strongly compactifiable, compactifiable, strongly Polishable, or Polishable, then so are the classes $\mathcal{C} \cup \{\emptyset\}$, $\mathcal{C} \setminus \{\emptyset\}$, $\mathcal{C} \cup \{1\}$, and $\mathcal{C}_{>1}$, where 1 denotes a one-point space and $\mathcal{C}_{>1}$ denotes the class of all nondegenerate members of $\mathcal{C}$.

*Proof.* The additive cases $\mathcal{C}\cup\{\emptyset\}$ and $\mathcal{C}\cup\{1\}$ follow directly from Proposition 4.1. The case $\mathcal{C}\setminus\{\emptyset\}$ for compactifiable and Polishable classes is covered by Observation 2.13. For strongly compactifiable and Polishable classes, it is easy since $\{\emptyset\}$ is clopen in $\mathcal{K}([0,1]^\omega)$, and so removing it from a realization of $\mathcal{C}$ does not change its complexity. Similarly, we obtain the $\mathcal{C}_{>1}$ case since the degenerate sets form a closed subset of the hyperspace. Hence, removing degenerate spaces from a realization of $\mathcal{C}$ preserves the $G_\delta$ complexity and turns a closed family to an $F_\sigma$ family (since the hyperspace is metrizable), which is enough for $\mathcal{C}_{>1}$ to be strongly compactifiable by Proposition 4.1.

It remains to cover the $\mathcal{C}_{>1}$ case for compactifiable and Polishable $\mathcal{C}$. Let $\mathcal{A}(q\colon A\to B)$ be a composition of $\mathcal{C}$ and let $C:=\{b\in B : |q^{-1}(b)|>1\}$. On one hand, if $A$ is a metric space, then $C$ is the preimage $(q^{-1*})^{-1}[\mathcal{G}]$ of the family $\mathcal{G}:=\{K\in\mathcal{K}(A) : \operatorname{diam}(K)>0\}$, which is $\tau_V^+$-open by Observation 4.3. Hence, if $\mathcal{A}$ is a compact composition, then $q$ is closed, $q^{-1*}$ is $\tau_V^+$-continuous, and $C$ is open and, in particular, $F_\sigma$, and so $\mathcal{C}_{>1}$ is compactifiable. On the other hand, $C$ is the projection of the set $\{\langle a,a',b\rangle\in A\times A\times B : q(a)=b=q(a'),\ a\neq a'\}$, which is the intersection of a closed set and an open set. Hence, if $\mathcal{A}$ is a Polish composition, then $C$ is analytic, and so $\mathcal{C}_{>1}$ is Polishable. □

**Notation 4.6.** Let $\mathcal{C}$ be a class of topological spaces.

- $\mathcal{C}^{\downarrow}$ denotes the class of all subspaces of members of $\mathcal{C}$.
- $\mathcal{C}^{\uparrow}$ denotes the class of all superspaces of members of $\mathcal{C}$.
- $\mathcal{C}^{\cong}$ denotes the class of all homeomorphic copies of members of $\mathcal{C}$.
- $\mathcal{C}^{\twoheadrightarrow}$ denotes the class of all continuous images of members of $\mathcal{C}$.
- $\mathcal{C}^{\twoheadleftarrow}$ denotes the class of all continuous preimages of members of $\mathcal{C}$, i.e. the class of all spaces than can be continuously mapped onto a member of $\mathcal{C}$.

We also denote the classes of all metrizable compacta and all continua by $\mathbf{K}$ and $\mathbf{C}$, respectively, so we can denote e.g. the class of all subcontinua of members of $\mathcal{C}$ by $\mathcal{C}^{\downarrow}\cap\mathbf{C}$. For a topological space $X$ and a family $\mathcal{F}\subseteq\mathcal{P}(X)$, the notation $\mathcal{F}^{\uparrow}\cap\mathcal{P}(X)$ means "all supersets of members of $\mathcal{F}$ that are subsets of $X$, all endowed with the subspace topology". This is consistent with the definition of $\mathcal{C}^{\uparrow}$ above when $\mathcal{P}(X)$ is viewed as a set of topological spaces.

**Observation 4.7.** If $\mathcal{C}$ is a strongly compactifiable or strongly Polishable class of compacta, then so is the class $\mathcal{C}\cap\mathbf{C}$ of all continua from $\mathcal{C}$ and the class $\mathcal{C}\setminus\mathbf{C}$ of all disconnected compacta from $\mathcal{C}$. If $\mathcal{C}$ is a strongly Polishable class of Polish spaces, then so is the class $\mathcal{C}\cap\mathbf{K}$ of all compacta from $\mathcal{C}$.

*Proof.* In the first case, there is a closed (resp. $G_\delta$) family $\mathcal{F}\subseteq\mathcal{K}([0,1]^\omega)$ such that $\mathcal{F}\cong\mathcal{C}$. We have $\mathcal{C}\cap\mathbf{C}\cong\mathcal{F}\cap\mathcal{C}([0,1]^\omega)$, which is closed (resp. $G_\delta$) not only in $\mathcal{C}([0,1]^\omega)$, but also in $\mathcal{K}([0,1]^\omega)$ since $\mathcal{C}(X)$ is closed in $\mathcal{K}(X)$ for every Hausdorff space $X$. Similarly, $\mathcal{C}\setminus\mathbf{C}\cong\mathcal{F}\setminus\mathcal{C}([0,1]^\omega)$, which is $F_\sigma$ (resp. $G_\delta$) in $\mathcal{K}([0,1]^\omega)$.

If $\mathcal{C}$ is a strongly Polishable class of Polish spaces, then by Construction 3.8 there is a Polish space $X$ and a Polish family $\mathcal{F}\subseteq\mathcal{C}l(X)$ equivalent to $\mathcal{C}$ which is closed by Observation 3.9 since the hyperspace $\mathcal{C}l(X)$ is Hausdorff. It follows that $\mathcal{C} \cap \mathbf{K}$ is equivalent to the family $\mathcal{F} \cap \mathcal{K}(X)$, which is closed in the Polish space $\mathcal{K}(X)$. □

**Question 4.8.** Is the previous observation true also for compactifiable and Polishable classes?

**Proposition 4.9.** If $\mathcal{C}$ is a compactifiable (resp. Polishable) class, then $\mathcal{C}^{\downarrow} \cap \mathbf{K}$ is a strongly compactifiable (resp. strongly Polishable) class.

*Proof.* Let $\mathcal{A}(q\colon A \to B)$ be a witnessing composition. It is enough to observe that $\mathcal{C}^{\downarrow} \cap \mathbf{K} \cong (q^*)^{-1}[[B]^{\leq 1}] \cap \mathcal{K}(A)$, which is a closed subset of $\mathcal{K}(A)$ since the family of all degenerate subspaces of $B$, $[B]^{\leq 1}$, is $\tau_V^-$-closed in $\mathcal{P}(B)$. □

**Corollary 4.10.** Every hereditary class of metrizable compacta or continua with a universal element (i.e. $\mathcal{C} \cong \{X\}^{\downarrow} \cap \mathbf{K}$ or $\mathcal{C} \cong \{X\}^{\downarrow} \cap \mathbf{C}$) is strongly compactifiable. This includes the classes of all compacta, totally disconnected compacta, continua, continua with dimension at most $n$, chainable continua, tree-like continua, and dendrites (in the realm of metrizable compacta).

In order to obtain a similar result for the induced class $\mathcal{C}^{\uparrow} \cap \mathbf{K}$, we shall analyze the set $\mathcal{F}^{\uparrow} \cap \mathcal{K}(X)$ for a family $\mathcal{F} \subseteq \mathcal{K}(X)$. First, we shall need the following refinement of Observation 3.2.

**Observation 4.11.** If $X$ is a Hausdorff space, then the inclusion relation of compacts sets is closed, i.e. $\mathcal{R}_{\subseteq} \cap \mathcal{K}(X)^2$ is closed in $\mathcal{K}(X)^2$ where $\mathcal{R}_{\subseteq} := \{\langle A, B\rangle \in \mathcal{P}(X)^2 : A \subseteq B\}$.

*Proof.* If $x \in A \setminus B$ for some $A, B \in \mathcal{K}(X)$, then there are disjoint open sets $U, V \subseteq X$ such that $x \in U$ and $B \subseteq V$, and hence $U^- \times V^+$ is an open neighborhood of $\langle A, B\rangle$ disjoint with $\mathcal{R}_{\subseteq}$. □

**Lemma 4.12.** Let $X$ be a topological space.

(i) The map $\mathcal{K}\colon \mathcal{K}(X) \to \mathcal{K}(\mathcal{K}(X))$ that maps every compact set $A \subseteq X$ to its compact hyperspace $\mathcal{K}(A)$ is continuous.

(ii) The projection $\pi_2\colon \mathcal{R}_{\subseteq} \cap \mathcal{K}(X)^2 \to \mathcal{K}(X)$ is closed and open.

*Proof.* Let $R$ denote the relation $\mathcal{R}_{\subseteq} \cap \mathcal{K}(X)^2$. Observe that for every $A \in \mathcal{K}(X)$ we have $R^A = A^+ \cap \mathcal{K}(X) = \mathcal{K}(A)$, which is compact. Hence, (i) $\iff$ (ii) by Lemma 3.10 since $\mathcal{K}$ is the map $\rho$ for $R$. We shall prove (i). In fact, $\mathcal{K}$ is both $\langle \tau_V^-, \tau_V^-(\tau_V)\rangle$-continuous and $\langle \tau_V^+, \tau_V^+(\tau_V)\rangle$-continuous. (The notation $\tau_V^{+/-}(\tau_V)$ means $\tau_V^{+/-}$ on $\mathcal{K}(Y)$ where $Y = \mathcal{K}(X)$ is endowed with $\tau_V$.)

Let $A \in \mathcal{K}(X)$ and let $\mathcal{V} \subseteq \mathcal{K}(X)$ be open such that $\mathcal{K}(A) \in \mathcal{V}^-$ (resp. $\mathcal{V}^+$). To prove that $\mathcal{K}$ is $\tau_V^-$-continuous (resp. $\tau_V^+$-continuous) it is enough to find $\mathcal{U}$ a $\tau_V^-$-open (resp. $\tau_V^+$-open) neighborhood of $A$ in $\mathcal{K}(X)$ such that $\mathcal{K}[\mathcal{U}] \subseteq \mathcal{V}^-$ (resp. $\mathcal{V}^+$). The set $\mathcal{V}$ is of the from $\bigcup_{i\in I} \bigcap_{j \in J_i} \mathcal{V}_{i,j}$ where $J_i$ are finite sets and every $\mathcal{V}_{i,j}$ is $V^-$ or $V^+$ for some open set $V \subseteq X$.

Let us start with the $\tau_V^-$-continuity. By $(\bigcup_{i\in I} \bigcap_{j\in J_i} \mathcal{V}_{i,j})^- = \bigcup_{i \in I}(\bigcap_{j \in J_i} \mathcal{V}_{i,j})^-$, we may suppose without loss of generality that $\mathcal{V} = \bigcap_{j<m} U_j^+ \cap \bigcap_{i<n} V_i^-$ for

some open sets $U_j, V_i \subseteq X$. Also, $\bigcap_{j<m} U_j^+ = (\bigcap_{j<m} U_j)^+ =: U^+$. We put $\mathcal{U} := \bigcap_{i<n}(U \cap V_i)^-$. Since $\mathcal{K}(A) \in \mathcal{V}^-$, there is $B \in \mathcal{K}(A) \cap U^+ \cap \bigcap_{i<n} V_i^-$, so $B \cap (U \cap V_i) \neq \emptyset$ for every $i < n$, and since $A \supseteq B$, we have $A \in \mathcal{U}$. On the other hand, for every $B \in \mathcal{U}$ we may choose points $x_i \in B \cap U \cap V_i$ for $i < n$, and hence $\{x_i : i < n\} \in \mathcal{K}(B) \cap \mathcal{V}$, so $\mathcal{K}(B) \in \mathcal{V}^-$.

Now let us prove the $\tau_V^+$-continuity. We have

$$\mathcal{K}(A) \subseteq \mathcal{V} = \bigcup_{i\in I} \bigcap_{j\in J_i} \mathcal{V}_{i,j} = \bigcap_{f\in\prod_{i\in I} J_i} \bigcup_{i\in I} \mathcal{V}_{i,f(i)} = \bigcap_{f\in F} \mathcal{V}_f$$

where $F := \prod_{i\in I} J_i$ and $\mathcal{V}_f := \bigcup_{i\in I} \mathcal{V}_{i,f(i)}$ for $f \in F$. Since $\mathcal{K}(A)$ is compact, we may suppose the sets $I$ and $F$ are finite. Since $(\bigcap_{f\in F} \mathcal{V}_f)^+ = \bigcap_{f\in F} \mathcal{V}_f^+$, it is enough to find for every $f \in F$ an open neighborhood $\mathcal{U}_f$ of $A$ such that $\mathcal{K}[\mathcal{U}_f] \subseteq \mathcal{V}_f^+$. Therefore, we may suppose without loss of generality that $\mathcal{V} = \bigcup_{i<n} U_i^+ \cup \bigcup_{j<m} V_j^-$ for some open sets $U_i, V_j \subseteq X$. Also, $\bigcup_{j<m} V_j^- = (\bigcup_{j<m} V_j)^- =: V^-$.

We have $A \setminus V \in \mathcal{K}(A) \subseteq \mathcal{V} = \bigcup_{i<n} U_i^+ \cup V^-$, and $n > 0$ since $\emptyset \in \mathcal{K}(A) \setminus V^-$. Hence, there is some $i < n$ such that $A \setminus V \subseteq U_i$. We put $\mathcal{U} := (U_i \cup V)^+$. We have $A = (A \setminus V) \cup (A \cap V) \subseteq U_i \cup V$, so $A \in \mathcal{U}$. Let $B \in \mathcal{U}$. For every $C \in \mathcal{K}(B)$ we have $C \subseteq B \subseteq U_i \cup V$. Therefore, $\mathcal{K}(B) \subseteq (U_i \cup V)^+ \subseteq U_i^+ \cup V^- \subseteq \mathcal{V}$, and so $\mathcal{K}(B) \in \mathcal{V}^+$. □

**Corollary 4.13.** Let $X$ be a topological space and $\mathcal{F} \subseteq \mathcal{K}(X)$.

(i) If $\mathcal{F}$ is closed, then $\mathcal{F}^\uparrow \cap \mathcal{K}(X)$ is closed.

(ii) If $X$ is Polish and $\mathcal{F}$ is analytic, then $\mathcal{F}^\uparrow \cap \mathcal{K}(X)$ is analytic.

*Proof.* Observe that $\mathcal{F}^\uparrow \cap \mathcal{K}(X)$ is the $\pi_2$-image of the set $\mathcal{H} := \mathcal{R}_\subseteq \cap (\mathcal{F} \times \mathcal{K}(X))$. If $\mathcal{F}$ is closed, then $\mathcal{H}$ is closed in $\mathcal{R}_\subseteq \cap \mathcal{K}(X)^2$, and the claim follows since the map $\pi_2{\restriction}_{\mathcal{R}_\subseteq \cap \mathcal{K}(X)^2}$ is closed by Lemma 4.12. If $\mathcal{F}$ is analytic, then $\mathcal{H}$ is analytic since $\mathcal{K}(X)$ is Polish and $\mathcal{R}_\subseteq$ is closed in $\mathcal{K}(X)^2$ by Observation 4.11. The claim follows since the map $\pi_2$ is continuous. □

**Proposition 4.14.** If $\mathcal{C}$ is a strongly compactifiable or a strongly Polishable class of compacta, then so is the corresponding class of all metrizable compact superspaces $\mathcal{C}^\uparrow \cap \mathbf{K}$.

*Proof.* Let us denote the Hilbert cube by $Q$ and let $Z$ be a Z-set in $Q$ that is homeomorphic to $Q$ (it exists by [12, Lemma 5.1.3]). Our class $\mathcal{C}$ is equivalent to a closed or an analytic family $\mathcal{F} \subseteq \mathcal{K}(Z)$. We show that $\mathcal{C}^\uparrow \cap \mathbf{K}$ is equivalent to $\mathcal{F}^\uparrow \cap \mathcal{K}(Q)$, which is closed or analytic by Corollary 4.13. Clearly, every member of $\mathcal{F}^\uparrow \cap \mathcal{K}(Q)$ is homeomorphic to a member of $\mathcal{C}^\uparrow \cap \mathbf{K}$. On the other hand, let $K \in \mathcal{C}^\uparrow \cap \mathbf{K}$. We may suppose that $K \in \mathcal{K}(Z)$. Since $K$ has a subspace $C \in \mathcal{C}$, there is a homeomorphism $h\colon C \to F \in \mathcal{F}$. By [12, Theorem 5.3.7] $h$ can be extended to a homeomorphism $\bar{h}\colon Q \to Q$. We have $K \cong \bar{h}[K] \in \mathcal{F}^\uparrow \cap \mathcal{K}(Q)$. □

**Example 4.15.** The class of all uncountable metrizable compacta is strongly compactifiable. Since every uncountable metrizable compactum contains a copy of the Cantor space, the class is equivalent to $\{2^\omega\}^\uparrow \cap \mathbf{K}$.

**Proposition 4.16.** If $\mathcal{C}$ is a strongly compactifiable or a strongly Polishable class of compacta, then so is the corresponding class of all metrizable compact continuous preimages $\mathcal{C}^{\twoheadleftarrow} \cap \mathbf{K}$.

*Proof.* Let $Q$ denote the Hilbert cube $[0,1]^\omega$ and let $\mathcal{F} \subseteq \mathcal{K}(Q)$ be equivalent to $\mathcal{C}$. We will show that $\mathcal{C}^{\twoheadleftarrow} \cap \mathbf{K} \cong \mathcal{H} := \{K \in \mathcal{K}(Q \times Q) : \pi_2[K] \in \mathcal{F}\}$. Clearly, $\mathcal{H} \subseteq \mathcal{F}^{\twoheadleftarrow} \cap \mathbf{K}$. On the other hand, let $K \in \mathcal{F}^{\twoheadleftarrow} \cap \mathbf{K}$. There is an embedding $e\colon K \hookrightarrow Q$, and there is a continuous map $f\colon K \twoheadrightarrow Y \subseteq Q$ for some $Y \in \mathcal{F}$. The map $(e \mathbin{\triangle} f)\colon K \to Q \times Q$ defined by $x \mapsto \langle e(x), f(x)\rangle$ is an embedding because of the embedding $e$, so $K \cong \operatorname{rng}(e \mathbin{\triangle} f) \subseteq Q \times Q$. At the same time $\pi_2[\operatorname{rng}(e \mathbin{\triangle} f)] = \operatorname{rng}(f) = Y$, and so $\operatorname{rng}(e \mathbin{\triangle} f) \in \mathcal{H}$. Altogether, we have $\mathcal{C}^{\twoheadleftarrow} \cap \mathbf{K} \cong \mathcal{F}^{\twoheadleftarrow} \cap \mathbf{K} \cong \mathcal{H}$. Since $\mathcal{H} = (\pi_2^*)^{-1}[\mathcal{F}]$, if $\mathcal{F}$ is closed or analytic, so is $\mathcal{H}$. $\square$

**Example 4.17.** We have another way to see that the class of all disconnected metrizable compacta $\mathbf{K} \setminus \mathbf{C}$ is strongly compactifiable (besides Observation 4.7) since it is exactly $\{2\}^{\twoheadleftarrow} \cap \mathbf{K}$, where 2 denotes the two-point discrete space.

**Example 4.18.** The class of all metrizable compact spaces with infinitely many components is strongly compactifiable since it is exactly $\{\omega+1\}^{\twoheadleftarrow} \cap \mathbf{K}$, where $\omega+1$ denotes the convergent sequence.

*Proof.* For every metrizable compactum $X$ we consider the equivalence $\sim$ induced by its components. $X/{\sim}$ may be viewed as a subspace of the Cantor space $2^\omega$. If $X$ has infinitely many components, then $X/{\sim}$ contains a nontrivial converging sequence. The conclusion follows from the fact that every closed subspace of $2^\omega$ is its retract. $\square$

**Example 4.19.** Let $\mathcal{N}$ denote the class of all topological spaces that are *not* locally connected. The class of all non-Peano metrizable continua $\mathcal{N} \cap \mathbf{C}$ is strongly compactifiable since it is exactly $\{H\}^{\twoheadleftarrow} \cap \mathbf{C}$, where $H$ denotes the harmonic fan. The class of all non-locally connected metrizable compacta $\mathcal{N} \cap \mathbf{K}$ is strongly compactifiable since it is exactly $\{\omega+1, H\}^{\twoheadleftarrow} \cap \mathbf{K}$.

*Proof.* Since Peano continua are exactly continuous images of the unit interval, every continuum that maps continuously onto $H$ (which is clearly not locally connected) is not Peano, so $\{H\}^{\twoheadleftarrow} \cap \mathbf{C} \subseteq \mathcal{N} \cap \mathbf{C}$. On the other hand, it is known that each member of $\mathcal{N} \cap \mathbf{C}$ maps continuously onto $H$ [4].

Let $K \in \mathbf{K}$. By Example 4.18, $K$ has infinitely many components if and only if $K$ continuously maps onto $\omega+1$, and in this case $K$ is not locally connected. This is because $K$ contains a convergent sequence such that each its member and the limit are in different components. So we may suppose that $K$ has finitely many components. If $K \in \mathcal{N}$, then one of the components is a non-Peano continuum, and so $K \in \{H\}^{\twoheadleftarrow}$ as before. On the other hand if $K \in \{H\}^{\twoheadleftarrow}$, then one of its components maps onto a subfan $H' \subseteq H$ that contains infinitely many endpoints of $H$. It follows that $H' \in \mathcal{N}$, and so $K \in \mathcal{N}$. $\square$

**Question 4.20.** Is the class of all Peano continua strongly compactifiable? We will show in Corollary 4.25 that it is compactifiable.

**Example 4.21.** Let $\mathcal{D}$ denote the class of all dendrites, $\mathcal{N}$ the class of all non-locally connected spaces, and $S^1$ the unit circle. Both $\mathcal{D}$ and $\mathbf{C} \setminus \mathcal{D}$ are strongly compactifiable classes – $\mathcal{D}$ by Corollary 4.10, and $\mathbf{C} \setminus \mathcal{D}$ since dendrites are exactly Peano continua not containing a simple closed curve, so $\mathbf{C} \setminus \mathcal{D} \cong (\{S^1\}^{\uparrow} \cup \mathcal{N}) \cap \mathbf{C}$, which is strongly compactifiable by Proposition 4.14 and Example 4.19.

In the following paragraphs we shall prove a preservation theorem for $\mathcal{C}^{\twoheadrightarrow} \cap \mathbf{K}$ and a necessary condition for being a strongly Polishable class.

**Lemma 4.22.** Let $X$, $Y$ be metrizable. The following sets are $G_\delta$.

- $\mathcal{G}_{\cong} := \{G \in \mathcal{K}(X \times Y) : G \text{ is a graph of a partial homeomorphism}\}$,
- $\mathcal{G}_{\twoheadrightarrow} := \{G \in \mathcal{K}(X \times Y) : G \text{ is a graph of a partial continuous surjection}\}$.

*Proof.* A set $G \in \mathcal{K}(X \times Y)$ is a member of $\mathcal{G}_{\cong}$ if and only if the maps $\pi_X{\restriction}_G$ and $\pi_Y{\restriction}_G$ are injective. The necessity is clear. On the other hand, if they are injective, then they are homeomorphisms onto their images since $G$ is compact. It follows that $G$ is the graph of the homeomorphism $\pi_Y{\restriction}_G \circ (\pi_X{\restriction}_G)^{-1} \colon \pi_X[G] \to \pi_Y[G]$. Analogously, $G \in \mathcal{G}_{\twoheadrightarrow}$ if and only if $\pi_X{\restriction}_G$ is injective.

For every $n \in \mathbb{N}$ let $\mathcal{F}_n := \{F \in \mathcal{K}(X \times Y) : |\pi_Y[F]| = 1 \text{ and } \operatorname{diam}(F) \geq \frac{1}{n}\}$, which is a closed set since $\pi_Y^*$ is continuous, $[Y]^1$ is closed in $\mathcal{K}(Y)$, and $\operatorname{diam} \colon \mathcal{K}(X \times Y) \to [0, \infty)$ is continuous. The map $\pi_Y{\restriction}_G$ is *not* injective if and only if there are $x_1 \neq x_2 \in X$ and $y \in Y$ such that $\{\langle x_1, y\rangle, \langle x_2, y\rangle\} \subseteq G$ if and only if there is $n \in \mathbb{N}$ and a set $F \in \mathcal{F}_n$ such that $F \subseteq G$, i.e. if and only if $G \in \bigcup_{n \in \mathbb{N}} \mathcal{F}_n^{\uparrow} \cap \mathcal{K}(X \times Y)$, which is an $F_\sigma$ set by Corollary 4.13. Analogously for $\pi_X{\restriction}_G$. □

It is known that the homeomorphic classification for compact metric spaces is analytic [7, Proposition 14.4.3]. We shall use the following formulation of the result.

**Corollary 4.23.** Let $X$, $Y$ be Polish spaces. The following relations are analytic.

- $\mathcal{R}_{\cong} := \{\langle A, B\rangle \in \mathcal{K}(X) \times \mathcal{K}(Y) : B \text{ is homeomorphic to } A\}$,
- $\mathcal{R}_{\twoheadrightarrow} := \{\langle A, B\rangle \in \mathcal{K}(X) \times \mathcal{K}(Y) : B \text{ is a continuous image of } A\}$.

*Proof.* We have $\mathcal{R}_{\cong} = \{\langle \pi_X[G], \pi_Y[G]\rangle : G \in \mathcal{G}_{\cong}\} = (\pi_X^* \mathbin{\triangle} \pi_Y^*)[\mathcal{G}_{\cong}]$, which is a continuous image of a $G_\delta$ set by Lemma 4.22. Analogously for $\mathcal{R}_{\twoheadrightarrow}$. □

**Proposition 4.24.** If $\mathcal{C}$ is a strongly Polishable class of compacta, then the corresponding class of all metrizable compact continuous images $\mathcal{C}^{\twoheadrightarrow} \cap \mathbf{K}$ is also strongly Polishable. Moreover, the class $\mathcal{C}^{\twoheadrightarrow} \cap \mathbf{C}$ is compactifiable.

*Proof.* There is an analytic family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{C} \cong \mathcal{F}$. We have $\mathcal{C}^{\twoheadrightarrow} \cap \mathbf{K} \cong \mathcal{F}^{\twoheadrightarrow} \cap \mathcal{K}([0,1]^\omega) = \mathcal{R}_{\twoheadrightarrow}[\mathcal{F}] = \pi_2[\mathcal{H}]$ where $\mathcal{H} = \mathcal{R}_{\twoheadrightarrow} \cap (\mathcal{F} \times \mathcal{K}([0,1]^\omega))$, which is an analytic set by Corollary 4.23. Moreover, either $\mathcal{C}^{\twoheadrightarrow} \cap \mathbf{C}$ contains $[0,1]$ and so every Peano continuum, and hence $\mathcal{F}^{\twoheadrightarrow} \cap \mathcal{C}([0,1]^\omega)$ is compactifiable by Corollary 3.21, or it consists only of degenerate spaces. In both cases, $\mathcal{C}^{\twoheadrightarrow} \cap \mathbf{C}$ is compactifiable. □

We obtain a corollary dual to Corollary 4.10.

**Corollary 4.25.** Every class of metrizable compacta (resp. continua) closed under continuous images with a common model in the class is strongly Polishable (resp. compactifiable). This includes the class of all Peano continua (images of $[0,1]$) and the class of all weakly chainable continua (images of the pseudoarc).

We finally give the necessary condition.

**Theorem 4.26.** If $\mathcal{C}$ is a strongly Polishable class of compacta, then $\mathcal{C}^{\cong} \cap \mathcal{K}(X)$ is analytic for every Polish space $X$.

*Proof.* There is an analytic set $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$. We have $\mathcal{C}^{\cong} \cap \mathcal{K}(X) = \mathcal{R}_{\cong}[\mathcal{F}] = \pi_2[\mathcal{H}]$ where $\mathcal{R}_{\cong}$ is the relation of being homeomorphic on $\mathcal{K}([0,1]^\omega) \times \mathcal{K}(X)$ and $\mathcal{H} = \mathcal{R}_{\cong} \cap (\mathcal{F} \times \mathcal{K}(X))$, which is an analytic set by Corollary 4.23. □

**Corollary 4.27.** If $\mathcal{C}$ is a class of metrizable compacta embeddable into a Polish space $X$, then it is equivalent to $\mathcal{C}^{\cong} \cap \mathcal{K}(X)$. Hence, $\mathcal{C}$ is strongly Polishable if and only if $\mathcal{C}^{\cong} \cap \mathcal{K}(X)$ is analytic.

**Example 4.28.** Every strongly Polishable class of zero-dimensional compacta is equivalent to an analytic family in $\mathcal{K}(2^\omega)$ by Corollary 4.27, and if it contains a copy of $2^\omega$, then it is compactifable by Corollary 3.21.

**Remark 4.29.** For a strongly compactifiable class $\mathcal{C}$, the family $\mathcal{C}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ is almost never closed. In fact, this happens if and only if $\mathcal{C}^{\cong}$ is one of the countably many classes listed in [1, Observation 4.3].

**Example 4.30.** By [8, Theorem 27.5] the class of all uncountable compacta in $\mathcal{K}([0,1]^\omega)$ is analytically complete. Together with Example 4.15 this shows that there is a strongly compactifiable class $\mathcal{C}$ such that $\mathcal{C}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ is not Borel. It also follows that the class of all countable metrizable compacta is coanalytically complete, and hence is not strongly Polishable. Note that by a classical result of Mazurkiewicz and Sierpiński [10], countable metrizable compacta are exactly countable successor ordinals and zero.

**Example 4.31.** By [9] the following classes are also coanalytically complete, and hence not strongly Polishable: hereditarily decomposable continua, dendroids, $\lambda$-dendroids, arcwise connected continua, uniquely arcwise connected continua, hereditarily locally connected continua.

Let us conclude with a result on preservation under intersections.

**Proposition 4.32.** Let $\{\mathcal{C}_n : n \in \omega\}$, $\mathcal{C}$, $\mathcal{D}$ be classes of metrizable compacta.

(i) If the classes $\mathcal{C}_n$ are strongly Polishable (resp. Polishable), then so is the class $\bigcap_{n\in\omega} \mathcal{C}_n^{\cong}$.

(ii) If the classes $\mathcal{C}$ and $\mathcal{D}$ are strongly Polishable (resp. Polishable), then so is the class $\mathcal{C} \cap \mathcal{D}^{\cong}$.

*Proof.* In the strongly Polishable case we have $\bigcap_{n\in\omega} \mathcal{C}_n^{\cong} \cong \bigcap_{n\in\omega} \mathcal{C}_n^{\cong} \cap \mathcal{K}([0,1]^\omega)$, which is an analytic set by Theorem 4.26.

In the Polishable case, by Theorem 2.11 for every $n \in \omega$ there is a $G_\delta$ subset $F_n \subseteq [0,1]^\omega \times \omega^\omega$ such that $\{F_n^x : x \in \omega^\omega\} \cong \mathcal{C}_n$. By [8, Theorem 28.8] the maps $\rho_n\colon \omega^\omega \to \mathcal{K}([0,1]^\omega)$ defined by $x \mapsto F_n^x$ are Borel. Let $i, j \in \omega$. We put $A_{i,j} := \{\langle x, y\rangle \in \omega^\omega \times \omega^\omega : F_i^x \cong F_j^y\} = (\rho_i \times \rho_j)^{-1}[\mathcal{R}_{\cong}]$. Since the relation $\mathcal{R}_{\cong}$ is analytic and the map $\rho_i \times \rho_j$ is Borel, the set $A_{i,j}$ is analytic. Hence, also the set $A := \{\langle x_n\rangle_{n\in\omega} \in (\omega^\omega)^\omega : \langle x_i, x_j\rangle \in A_{i,j}$ for every $i, j \in \omega\}$ and its projection $\pi_0[A] \subseteq \omega^\omega$ are analytic. Observe that $\bigcap_{n\in\omega} \mathcal{C}_n^{\cong} \cong \{F_0^x : x \in \pi_0[A]\}$, and so the intersection is Polishable by Corollary 2.7.

Unlike $\mathcal{C} \cap \mathcal{D}$, the class $\mathcal{C} \cap \mathcal{D}^{\cong}$ is equivalent to $\mathcal{C}^{\cong} \cap \mathcal{D}^{\cong}$, which is (strongly) Polishable by the previous claim. □

**Remark 4.33.** A similar argument would give us that if $\mathcal{C}$ is strongly compactifiable and $\mathcal{D}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ is closed, then $\mathcal{C} \cap \mathcal{D}^{\cong}$ is strongly compactifiable, but by Remark 4.29, $\mathcal{D}^{\cong}$ would have to be one of countably many special classes. One of these classes is the class of all metrizable continua $\mathbf{C}$, so Observation 4.7 is a special case.

**Example 4.34.** We shall extend Example 4.21. Let $\mathcal{P}$ be the class of all Peano continua. The class $\mathcal{P} \setminus \mathcal{D}$ is strongly Polishable by Corollary 4.25 and Proposition 4.32 since it is equivalent to $\mathcal{P} \cap \{S^1\}^{\uparrow}$.

# 5 Compactifiability and inverse limits

In the last section we give a construction of compact or Polish compositions of classes of spaces expressible as inverse limits of sequences of spaces and bonding maps from suitable families.

First, we shall recall some standard notions and the related notation. An *inverse sequence* is a pair $\langle X_*, f_*\rangle$ where $X_* = \langle X_n\rangle_{n\in\omega}$ is a sequence of topological spaces and $f_* = \langle f_n\colon X_n \leftarrow X_{n+1}\rangle_{n\in\omega}$ is a sequence of continuous maps. For every $n \le m \in \omega$ we denote by $f_{n,m}$ the composition $(f_n \circ f_{n+1} \circ \cdots \circ f_{m-1})\colon X_n \leftarrow X_m$. In particular, $f_{n,n} = \mathrm{id}_{X_n}$ and $f_{n,n+1} = f_n$ for every $n$.

The *limit* of $\langle X_*, f_*\rangle$ is the pair $\langle X_\infty, \langle f_{n,\infty}\rangle_{n\in\omega}\rangle$ where the *limit space* $X_\infty$ is the subspace of $\prod_{n\in\omega} X_n$ consisting of all sequences $x_* = \langle x_n\rangle_{n\in\omega}$ such that $x_n = f_n(x_{n+1})$ for every $n$, and the *limit maps* $f_{n,\infty}\colon X_n \leftarrow X_\infty$ are just the coordinate projections restricted to $X_\infty$. Abstractly, the limit is defined by its universal property: the limit maps satisfy $f_{n,\infty} = f_n \circ f_{n+1,\infty}$ for every $n$, and for every other family of continuous maps $g_n\colon X_n \leftarrow Y$ satisfying $g_n = f_n \circ g_{n+1}$ for every $n$, there is a unique continuous map $g_\infty\colon X_\infty \leftarrow Y$ such that $g_n = f_{n,\infty} \circ g_\infty$ for every $n$.

Recall that a *tree* is a partially ordered set $T$ with the smallest element such that for every node $t \in T$ the set $\{s \in T : s < t\}$ is well-ordered. A *lower subset* of $T$ is a subset $S \subseteq T$ such that for every $t \le s \in T$ with $s \in S$ we have also $t \in S$. A *subtree* of a tree $T$ is a lower subset $S \subseteq T$ endowed with the induced ordering. We will be interested in trees of countable height. These can be always represented as subtrees of $A^{<\omega} = \bigcup_{n\in\omega} A^n$ for a sufficiently large set $A$. The members of $A^{<\omega}$ are $A$-valued tuples $t$ of finite length $|t|$, and they are ordered by extension, i.e. $t \leq s$ if and only if $s{\restriction}_{|t|} = t$. For $T$ a subtree of $A^{<\omega}$ and $n \in \omega$, the *level* $n$ of $T$, denoted by $T_n$, is the set $\{t \in T : |t| = n\} = T \cap A^n$.

Let $T$ be a tree. A node $s \in T$ is a *successor* of a node $t \in T$ if $s > t$ and there is no other node $s > s' > t$. We denote this by $s \succ t$. A tree is *countably* (resp. *finitely*) *splitting* if every node has only countably (resp. finitely) many successors. Every countably splitting tree of countable height may be realized as a subtree of $\omega^{<\omega}$.

Let $t, s \in A^{<\omega}$ for some $A$. We denote the *concatenation* of the tuples $t$ and $s$ by $t^\frown s$. That means, $t^\frown s \in A^{<\omega}$, $(t^\frown s)(n) = t(n)$ for $n < |t|$ and $(t^\frown s)(|t|+n) = s(n)$ for $n < |s|$. For $a \in A$, the notation $t^\frown a$ is a shortcut for $t^\frown\langle a\rangle$. Note that for a subtree $T \subseteq A^{<\omega}$, every successor of a node $t \in T$ is of the form $t^\frown a$ for some $a \in A$.

Let $T$ be a tree. Recall that a *branch* of $T$ is any maximal chain $\alpha \subseteq T$, i.e. a subset of $T$ whose elements are pairwise comparable and which is maximal with respect to inclusion. Suppose that $T$ is a subtree of some $A^{<\omega}$. In that case, for every infinite branch of $T$ there is a unique sequence $\alpha \in A^\omega$ such that the infinite branch as a set is $\{\alpha{\restriction}_n : n \in \omega\}$. For this reason it is common to identify infinite branches of $A^{<\omega}$ with $A^\omega$. By $T_\infty$ we denote the *body* of $T$, i.e. the set of all infinite branches of $T \subseteq A^{<\omega}$ viewed as a subspace of $A^\omega$ with the product topology with $A$ being discrete. The standard basic open subsets of $T_\infty$ are of the form $N_t := \{\alpha \in T_\infty : \alpha{\restriction}_{|t|} = t\}$ for $t \in T$. It is easy to see that $T_\infty$ is always a closed subspace of the space $A^\omega$, and so is Polish if $A$ is countable. For more details on trees see for example [8, Section I.2].

**Definition 5.1.** Let $T$ be a subtree of $A^{<\omega}$ for some $A$. By a *$T$-inverse system* we mean a pair $\langle X_*, f_*\rangle$ where $X_* = \langle X_t\rangle_{t\in T}$ is a family of topological spaces and $f_* = \langle f_{t,s}\colon X_t \leftarrow X_s\rangle_{t\leq s\in T}$ is a family of continuous maps such that $f_{t,t} = \mathrm{id}_{X_t}$ for every $t$ and $f_{t,s} \circ f_{s,r} = f_{t,r}$ for every $t \leq s \leq r$. Of course, the system is determined by the successor maps $f_{t,t^\frown\langle a\rangle}$ where $t^\frown\langle a\rangle \in T$. Note that an inverse sequence may be viewed as a $1^{<\omega}$-inverse system.

**Construction 5.2.** Let $T$ be a subtree of $\omega^{<\omega}$ and let $\langle X_*, f_*\rangle$ be a $T$-inverse system. The following construction produces a composition of the limit spaces along the infinite branches of $T$.

We consider the inverse sequence $\langle X^\oplus_*, f^\oplus_*\rangle$ obtained by summing $\langle X_*, f_*\rangle$ along each level of $T$, i.e. for each $n \in \omega$ we put $X^\oplus_n := \sum_{t\in T_n} X_t$ and $f^\oplus_n := (\sum_{t\in T_n} f^\oplus_t)\colon X^\oplus_n \leftarrow X^\oplus_{n+1}$ where the maps $f^\oplus_t := (\bigtriangledown_{s\succ t} f_{t,s})\colon X_t \leftarrow \sum_{s\succ t} X_s$ are preliminary codiagonal sums of all maps going to $X_t$. (We denote codiagonal sums by $\bigtriangledown$ and diagonal products by $\bigtriangleup$. The notation is inspired by [6, 2.1.11 and 2.3.20].)

Moreover, for each branch $\alpha \in T_\infty$ we consider the inverse sequence $\langle X^\alpha_*, f^\alpha_*\rangle$ defined as the restriction of $\langle X_*, f_*\rangle$ to $\alpha$, i.e. $X^\alpha_n = X_{\alpha{\restriction}_n}$ and $f^\alpha_n = f_{\alpha{\restriction}_n,\alpha{\restriction}_{n+1}}\colon X^\alpha_n \leftarrow X^\alpha_{n+1}$. For every $n \in \omega$ we denote the embedding $X^\alpha_n \hookrightarrow X^\oplus_n$ by $e^\alpha_n$. This yields a natural transformation $e^\alpha_*\colon \langle X^\alpha_*, f^\alpha_*\rangle \hookrightarrow \langle X^\oplus_*, f^\oplus_*\rangle$ and the limit embedding $e^\alpha_\infty\colon X^\alpha_\infty \hookrightarrow X^\oplus_\infty$.

*Claim.* The family of subspaces $\langle \mathrm{rng}(e^\alpha_\infty)\rangle_{\alpha\in T_\infty}$ is a decomposition of $X^\oplus_\infty$, and the induced map $q\colon X^\oplus_\infty \to T_\infty$ (where $q^{-1}(\alpha) = \mathrm{rng}(e^\alpha_\infty)$) is continuous. Hence, we have a composition $\mathcal{A}(q\colon X^\oplus_\infty \to T_\infty)$ of the family of embeddings $\langle e^\alpha_\infty\rangle_{\alpha\in T_\infty}$. If all spaces $X_t$ for $t \in T$ are Polish, then the composition is Polish. If all spaces $X_t$ for $t \in T$ are metrizable compacta and $T$ is finitely splitting, then the composition is compact.

*Proof.* Without loss of generality, we may suppose that $X_t \subseteq X^\oplus_n$ for every $n \in \omega$ and $t \in T_n$, and that $X^\alpha_\infty \subseteq X^\oplus_\infty$ for every $\alpha \in T_\infty$.

First, $\langle X^\alpha_\infty\rangle_{\alpha\in T_\infty}$ is a decomposition of $X^\oplus_\infty$. Clearly, for every $x_* \in X^\oplus_\infty \subseteq \prod_{n\in\omega} X^\oplus_n$ and every $n \in \omega$ there is a unique node $t_n \in T_n$ such that $x_n \in X_{t_n}$, and since $x_n = f^\oplus_n(x_{n+1})$, we have that $t_{n+1}$ is a successor of $t_n$ and $x_n = f_{t_n,t_{n+1}}(x_{n+1})$. Hence, $\alpha := \{t_n : n \in \omega\}$ is the unique infinite branch such that $x_n = f^\alpha_n(x_{n+1})$ for every $n \in \omega$, i.e. such that $x_* \in X^\alpha_\infty$.

Let $n \in \omega$ and $t \in T_n$. For every $x_* \in X^\alpha_\infty \subseteq X^\oplus_\infty$ we have $\alpha(n) = t$ if and only if $x_n \in X_t$. Hence, we have $q^{-1}[N_t] = \{x_* \in X^\oplus_\infty : x_n \in X_t\} = (f^\oplus_{n,\infty})^{-1}[X_t]$, and $X_t$ is clopen in $X^\oplus_n$. Therefore, $q\colon X^\oplus_\infty \to T_\infty$ is continuous.

If $T$ is countably (resp. finitely) splitting, then every level $T_n$ is countable (resp. finite), and so every space $X^\oplus_n$ is Polish (resp. metrizable compact) if all spaces $X_t$ are. So is their limit $X^\oplus_\infty$ as a closed subspace of their product. The indexing space $T_\infty$ is a closed subset of $\omega^\omega$, and therefore is Polish. Moreover, if $T$ is finitely splitting, $T_\infty$ is a closed subset of $\prod_{n\in\omega} F_n$ for some finite sets $F_n \subseteq \omega$ since every level $T_n$ is finite, and so it is a metrizable compactum. □

**Remark 5.3.** Construction 5.2 gives a way of proving that some class of spaces is compactifiable or Polishable. On the other hand, note that every compact composition $\mathcal{A}(q\colon A \to 2^\omega)$ gives us a $2^{<\omega}$-inverse system of inclusions. Namely, for every $t \in T := 2^{<\omega}$ we put $X_t := q^{-1}[N_t]$, and for very $s \geq t$ we define $f_{t,s}$ by the inclusion $X_s \subseteq X_t$. We obtain a $T$-inverse system $\langle X_*, f_*\rangle$ and for every $\alpha \in T_\infty = 2^\omega$ we have $X^\alpha_\infty = \bigcap_{n\in\omega} q^{-1}[N_{\alpha\restriction_n}] = q^{-1}[\bigcap_{n\in\omega} N_{\alpha\restriction_n}] = q^{-1}(\alpha)$. Moreover, $X^\oplus_n = A$ for every $n$, so by applying Construction 5.2 to $\langle X_*, f_*\rangle$, we obtain the composition $\mathcal{A}$ we started with.

**Definition 5.4.** For a class $\mathcal{F}$ of continuous maps, we call a topological space *$\mathcal{F}$-like* if it is the limit of an inverse sequence with bonding maps in $\mathcal{F}$. By $\mathrm{Obj}(\mathcal{F})$ we denote the class of all domains and codomains of the maps from $\mathcal{F}$.

For a class $\mathcal{P}$ of topological spaces, we call a topological space *$\mathcal{P}$-like* if it is $\mathcal{F}$-like for $\mathcal{F}$ being the class of all continuous surjections between spaces from $\mathcal{P}$. Classically, $\{[0,1]\}$-like spaces are called *arc-like*, and $\{S^1\}$-like spaces are called *circle-like.*

**Proposition 5.5.** Let $\mathcal{F}$ be a countable family of continuous maps. There is a subtree $T \subseteq \omega^{<\omega}$ and a $T$-inverse system $\langle X_*, f_*\rangle$ such that $\{X^\alpha_\infty : \alpha \in T_\infty\}$ is equivalent to the class of all $\mathcal{F}$-like spaces. Moreover, we may have $T \subseteq 2^{<\omega}$ if every space $X$ that is the codomain of infinitely maps from $\mathcal{F}$ is $\mathcal{F}$-like (in particular, if $\mathrm{id}_X \in \mathcal{F}$).

*Proof.* If every $\mathcal{F}$-like space is empty, then the empty tree or a single-branch tree with empty maps works. Otherwise, let us fix a nonempty $\mathcal{F}$-like space $X_\emptyset$ formally distinct from each member of $\mathrm{Obj}(\mathcal{F})$. Moreover, for every $X \in \mathrm{Obj}(\mathcal{F})$, let us fix a constant map $c_X\colon X \to X_\emptyset$. We put $\mathcal{F}' := \mathcal{F} \cup \{c_X : X \in \mathrm{Obj}(\mathcal{F})\}$. A space is $\mathcal{F}'$-like if and only if it is $\mathcal{F}$-like since every inverse sequence with bonding maps from $\mathcal{F}'$ either has all bonding maps in $\mathcal{F}$ or starts with some $c_X$ and continues with maps from $\mathcal{F}$. We have extended $\mathcal{F}$ to $\mathcal{F}'$ just to have a common codomain to serve as the root of our tree.

Let $A := |\mathcal{F}'| \leq \omega$ and let $\langle f_n \rangle_{n \in A}$ be an enumeration of $\mathcal{F}'$. We associate every $t \in A^{<\omega}$ with the composition $f_{t(0)} \circ f_{t(1)} \circ \cdots \circ f_{t(|t|-1)}$ if the composition is possible and if the codomain is $X_\emptyset$. Namely, let $T$ be the subtree of $A^{<\omega} \subseteq \omega^{<\omega}$ consisting of all tuples $t$ such that $\mathrm{dom}(f_{t(n)}) = \mathrm{cod}(f_{t(n+1)})$ for every $n+1 < |t|$ and $\mathrm{cod}(f_{t(0)}) = X_\emptyset$ or $t = \emptyset$. We put $X_t := \mathrm{dom}(f_{t(|t|-1)})$ for $t \in T \setminus \{\emptyset\}$. Note that $X_\emptyset$ is already defined. For every $t^\frown n \in T$ we put $f_{t,t^\frown n} := f_n$. This defines the desired $T$-inverse system $\langle X_*, f_* \rangle$. The first level consists exactly of the added maps $c_X$, i.e. $\{f_{\emptyset,\langle n \rangle} : \langle n \rangle \in T\} = \{c_X : X \in \mathrm{Obj}(\mathcal{F})\}$. Moreover, the restrictions $\langle X^\alpha_*, f^\alpha_* \rangle$ along infinite branches $\alpha \in T_\infty$ are exactly inverse sequences with bonding maps in $\mathcal{F}'$ and starting at $X_\emptyset$, which are exactly all inverse sequences with bonding maps from $\mathcal{F}$ prepended with the corresponding map $c_X$.

Now let us turn the tree $T \subseteq \omega^{<\omega}$ into a tree $S \subseteq 2^{<\omega}$, and define the corresponding $S$-inverse system $\langle Y_*, g_* \rangle$. First, we define canonical transformations between $\omega^{<\omega}$ and $2^{<\omega}$. For every $n \in \omega$ let $[n]$ be the sequence of $n$ ones followed by zero, and for every $t \in \omega^{<\omega}$ let $\varphi(t)$ be the concatenation $[t(0)]^\frown[t(1)]^\frown \cdots {}^\frown[t(|t|-1)]$. This defines an injective map $\varphi\colon \omega^{<\omega} \to 2^{<\omega}$. Essentially, each branching $t^\frown 0, t^\frown 1, t^\frown 2, \ldots$ is replaced by $t^\frown 0, t^\frown\langle 1,0\rangle, t^\frown\langle 1,1,0\rangle, \ldots$ The image $\varphi[\omega^{<\omega}]$ consists of all sequences ending with 0 and the empty sequence. Let $\psi\colon 2^{<\omega} \to \omega^{<\omega}$ be the extension of $\varphi^{-1}$ by $\psi(s^\frown 1) := \psi(s)$ for $s \in 2^{<\omega}$.

Let $S := \psi^{-1}[T]$, which is the tree generated by $\varphi[T]$. For each $s \in S$ let $Y_s := X_{\psi(s)}$, $g_{s,s^\frown 1} := \mathrm{id}_{X_{\psi(s)}}$, and $g_{s,s^\frown 0} := f_{\psi(s),\psi(s^\frown 0)}$. This defines the desired $S$-inverse system $\langle Y_*, g_* \rangle$. Infinite branches $\alpha \in T_\infty$ are in a one-to-one correspondence with infinite branches $\beta \in S_\infty$ with infinitely many zeroes, and the limits of the corresponding inverse sequences $\langle X^\alpha_*, f^\alpha_* \rangle$ and $\langle Y^\beta_*, g^\beta_* \rangle$ are the same – the maps $g^\beta_n$ with $\beta(n) = 0$ are exactly the maps $f^\alpha_n$, while the maps $g^\beta_n$ with $\beta(n) = 1$ are identities. Note that $S_\infty$ may contain also branches with only finitely many zeroes, but the corresponding inverse sequence is eventually constant $\mathrm{id}_X$ for some $X \in \mathrm{Obj}(\mathcal{F}')$, and so its limit is $X$. By the construction, $X = X_t$ for some $t \in T$ with infinitely many successors. Hence, $X$ is the codomain of infinitely many maps from $\mathcal{F}'$, so it is either the codomain of infinitely many maps from $\mathcal{F}$, or $X = X_\emptyset$. In both cases, $X$ is $\mathcal{F}$-like. □

**Proposition 5.6.** Let $\mathcal{F}$ be a family of continuous maps such that $\mathrm{Obj}(\mathcal{F})$ is a countable family of metrizable compacta. There is a countable family $\mathcal{G} \subseteq \mathcal{F}$ such that a space is $\mathcal{F}$-like if and only if it is $\mathcal{G}$-like.

*Proof.* For every $X, Y \in \mathrm{Obj}(\mathcal{F})$ let $\mathcal{F}(X, Y)$ denote the family of all maps $f \in \mathcal{F}$ such that $f\colon X \to Y$. Every $\mathcal{F}(X, Y)$ is a subspace of the space of all continuous maps $X \to Y$ with the topology of uniform convergence, which is separable and metrizable since $X$ and $Y$ are metrizable compacta, and hence $\mathcal{F}(X,Y)$ is also separable. Let $\mathcal{G}(X,Y) \subseteq \mathcal{F}(X,Y)$ be a countable dense subset and let $\mathcal{G}$ be the countable family $\bigcup_{X,Y\in\mathrm{Obj}(\mathcal{F})} \mathcal{G}(X,Y)$.

Clearly, every $\mathcal{G}$-like space is $\mathcal{F}$-like. On the other hand, by Brown's approximation theorem [5, Theorem 3], for every inverse sequence $\langle X_*, f_* \rangle$ with bonding maps from $\mathcal{F}$ and fixed metrics on the spaces $X_n$, there is a sequence of numbers $\varepsilon_n > 0$ and a sequence of maps $g_n \in \mathcal{G}(X_{n+1}, X_n)$ such that $d(f_n, g_n) < \varepsilon_n$ for every $n$ and such that the limit space of $\langle X_*, g_* \rangle$ is homeomorphic to the limit space of $\langle X_*, f_* \rangle$. Therefore, every $\mathcal{F}$-like space is $\mathcal{G}$-like. □

Now we combine the previous propositions into the following theorem.

**Theorem 5.7.** Let $\mathcal{F}$ be a family of continuous maps.

(i) If $\mathcal{F}$ is countable and $\mathrm{Obj}(\mathcal{F})$ is a class of Polish spaces, then the class of all $\mathcal{F}$-like spaces is Polishable.

(ii) If $\mathrm{Obj}(\mathcal{F})$ is a countable family of metrizable compacta such that every $X \in \mathrm{Obj}(\mathcal{F})$ is $\mathcal{F}$-like (in particular if $\mathrm{id}_X \in \mathcal{F}$), then the class of all $\mathcal{F}$-like spaces is compactifiable.

*Proof.* By Proposition 5.6 we may suppose that $\mathcal{F}$ is countable also in the compact case. Using Proposition 5.5 we build a tree $T \subseteq \omega^{<\omega}$ and a $T$-inverse system such that $\mathcal{F}$-like spaces are exactly the limit spaces along the branches. Moreover, in the compact space our tree can be made finitely splitting. Finally, we build a Polish (resp. compact) composition of the class of all $\mathcal{F}$-like spaces using Construction 5.2. □

**Corollary 5.8.** For a countable family $\mathcal{P}$ of metrizable compacta, the class of all $\mathcal{P}$-like spaces is compactifiable.

**Remark 5.9.** The class of all arc-like continua is strongly compactifiable by Corollary 4.10 since there is a universal arc-like continuum. Theorem 5.7 gives another way to prove that the class of all arc-like continua is compactifiable. In fact, Construction 5.2 is based on [14, Theorem 12.22], where a universal arc-like continuum is constructed. The difference is that in [14, Theorem 12.22] all spaces $X_t$ are copies of the unit interval, the spaces $X_n^{\oplus}$ are extended to bigger arcs $A_n$, and the surjections $f_n^{\oplus}\colon X_n^{\oplus} \leftarrow X_{n+1}^{\oplus}$ are continuously extended to surjections $g\colon A_n \leftarrow A_{n+1}$, so we get an arc-like continuum $A_\infty \supseteq X_\infty^{\oplus}$ as limit. However, such extension cannot be done with circles. In fact, there is no universal circle-like continuum (Observation 5.10). Yet, by Theorem 5.7 the class of all circle-like continua is compactifiable. Because of this and Corollary 4.10, a compact composition may be viewed as a weaker form of a universal element.

**Observation 5.10.** There is no universal circle-like continuum.

*Proof.* Let $\langle X_*, f_* \rangle$ be an inverse sequence of circles and continuous surjections. We will show that if $S^1 \subseteq X_\infty$, then already $S^1 = X_\infty$, so $X_\infty$ cannot be universal.

Let us divide $S^1$ into four quarter-arcs $A_k := \{e^{ix} : x \in [k\frac{\pi}{2}, (k+1)\frac{\pi}{2}]\}$, $k \in \{0,1,2,3\}$. There is $n$ such that $f_{n,\infty}[A_0] \cap f_{n,\infty}[A_2] = \emptyset = f_{n,\infty}[A_1] \cap f_{n,\infty}[A_3]$.

Necessarily, the same condition holds for every $f_{m,\infty}$ where $m \geq n$. We have that $f_{n,\infty}{\restriction}_{S^1}$ is onto. Otherwise, $A := f_{n,\infty}[S^1]$ is an arc, $f_{n,\infty}[A_0]$ and $f_{n,\infty}[A_2]$ are its disjoint subcontinua, and no two subarcs of $A$ meeting both $f_{n,\infty}[A_0]$ and $f_{n,\infty}[A_2]$ are disjoint, which is a contradiction with disjointness of $f_{n,\infty}[A_1]$ and $f_{n,\infty}[A_3]$.

We have shown that $f_{m,\infty}{\restriction}_{S^1}$ is onto for every $m \geq n$. But for $x \in X_\infty \setminus S^1$ there is $m \geq n$ such that $f_{m,\infty}(x) \notin f_{m,\infty}[S^1] = X_m$, which is a contradiction. □

We wonder if the constructions from this chapter may be modified to obtain strong compact or strong Polish compositions. In particular, we have the following question.

**Question 5.11.** Is the class of all circle-like continua strongly compactifiable?

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# IV. Borel complexity up to the equivalence

Adam Bartoš[*,1]

**Abstract**

We say that two classes of topological spaces are equivalent if each member of one class has a homeomorphic copy in the other class and vice versa. Usually when the Borel complexity of a class of metrizable compacta is considered, the class is realized as the subset of the hyperspace $\mathcal{K}([0,1]^\omega)$ containing all homeomorphic copies of members of the given class. We are rather interested in the lowest possible complexity among all equivalent realizations of the given class in the hyperspace.

We recall that to every analytic subset of $\mathcal{K}([0,1]^\omega)$ there exist an equivalent $G_\delta$ subset. Then we show that up to the equivalence open subsets of the hyperspace $\mathcal{K}([0,1]^\omega)$ correspond to countably many classes of metrizable compacta. Finally we use the structure of open subsets up to equivalence to prove that to every $F_\sigma$ subset of $\mathcal{K}([0,1]^\omega)$ there exists an equivalent closed subset.

*Classification:* 54H05, 54B20, 54E45, 54F15.

*Keywords:* Borel hierarchy, complexity, homeomorphism equivalence, metrizable compactum, Polish space, hyperspace, Z-set, saturated family, compactifiable class, Polishable class.

## 1 Introduction

We denote that topological spaces $X$, $Y$ are *homeomorphic* by $X \cong Y$. This equivalence of topological spaces may be lifted to an equivalence of classes of topological spaces. We say that two classes $\mathcal{C}$ and $\mathcal{D}$ are *equivalent* (and we also write $\mathcal{C} \cong \mathcal{D}$) if every space in $\mathcal{C}$ is homeomorphic to a space in $\mathcal{D}$ and vice versa. This is the equivalence from the title. Given a class $\mathcal{C}$ we denote by $\mathcal{C}^{\cong}$ the class of all homeomorphic copies of members of $\mathcal{C}$. Clearly, this is the largest class equivalent to $\mathcal{C}$. We say that the class $\mathcal{C}$ is *saturated* if $\mathcal{C} \cong \mathcal{C}^{\cong}$.

We denote the classes of all metrizable compacta and all metrizable continua by $\mathbf{K}$ and $\mathbf{C}$, respectively. We are interested in the complexity of classes of metrizable compacta and continua, i.e. of subclasses of $\mathbf{K}$ and $\mathbf{C}$. To express the complexity of a given class $\mathcal{C}$ using the Borel hierarchy, we first have to view the class as a subset of a Polish space. For this we use *hyperspaces*.

Let us recall the notation and basic properties of standard hyperspaces. For a topological space $X$ we denote the families of all compacta and continua (i.e. connected compacta) in $X$ including the empty set by $\mathcal{K}(X)$ and $\mathcal{C}(X)$, respectively, and we endow the families with the *Vietoris topology*. This is the topology

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generated by the sets of form $U^+ := \{A : A \subseteq U\}$ and $U^- := \{A : A \cap U \neq \emptyset\}$ for $U$ open in $X$. Clearly, $\mathcal{C}(X)$ is a subspace of $\mathcal{K}(X)$. It is a closed subspace if $X$ is Hausdorff.

If the space $X$ is metrizable with a metric $d$, the hyperspaces are metrizable with the induced *Hausdorff metric* $d_H$. The distance $d_H(A, B)$ is the infimum of all values $\varepsilon > 0$ such that $A \subseteq N_\varepsilon(B)$ and $B \subseteq N_\varepsilon(A)$. Here, $N_\varepsilon(A)$ denotes the set $\{x \in X : d(x, A) < \varepsilon\} = \bigcup_{x \in A} B(x, \varepsilon)$ where $B(x, \varepsilon)$ is the open ball of radius $\varepsilon$. To incorporate the empty set it makes sense to consider a bounded metric $d$ and to define $d_H(A, \emptyset)$ as the bound.

Every continuous map $f\colon X \to Y$ between topological spaces induces the map $f^*\colon \mathcal{K}(X) \to \mathcal{K}(Y)$ defined by $f^*(A) := f[A]$. This map is also continuous. Moreover, if $f$ is an embedding or a homeomorphism, so is the map $f^*$. These properties are well known, and we summarize them in [1, Proposition 3.6].

The *Hilbert cube* $[0, 1]^\omega$ is a universal space for all separable metrizable spaces, in particular, every metrizable compactum has a homeomorphic copy in $\mathcal{K}([0, 1]^\omega)$, which is itself a metrizable compactum. It is standard to view this space as the hyperspace of all metrizable compacta. For a class $\mathcal{C} \subseteq \mathbf{K}$ we consider the collection of all families $\mathcal{F} \subseteq \mathcal{K}([0, 1]^\omega)$ equivalent to $\mathcal{C}$, and we denote this collection by $[\mathcal{C}]$. Note that this is the equivalence class of $\cong$ restricted to $\mathcal{P}(\mathcal{K}([0, 1]^\omega))$. Analogously to the saturated class we say that $\mathcal{F} \subseteq \mathcal{K}([0, 1]^\omega)$ is a *saturated family* if $\mathcal{F} = \mathcal{F}^{\cong} \cap \mathcal{K}([0, 1]^\omega)$. The collection $[\mathcal{C}]$ has the largest element, namely the saturated family $\mathcal{C}^{\cong} \cap \mathcal{K}([0, 1]^\omega)$. Also, if $\mathcal{F} \in [\mathcal{C}]$, then $\mathcal{H} \in [\mathcal{C}]$ whenever $\mathcal{F} \subseteq \mathcal{H} \subseteq \max([\mathcal{C}])$. In particular, $[\mathcal{C}]$ is stable under arbitrary unions. The minimal elements of $[\mathcal{C}]$ are those families $\mathcal{F} \in [\mathcal{C}]$ whose members are pairwise non-homeomorphic.

Usually, when considering the complexity of a class $\mathcal{C} \subseteq \mathbf{K}$, the class is identified with $\max([\mathcal{C}])$ and its complexity in $\mathcal{K}([0, 1]^\omega)$ is considered. There are many results on complexity of $\max([\mathcal{C}])$, see for example the survey [4]. We are rather interested in the lowest complexity among families in $[\mathcal{C}]$. This is rarely the complexity of the saturated family. For example, every singleton $\{K\} \subseteq \mathcal{K}([0, 1]^\omega)$ is closed, but the corresponding saturated family is not unless $K$ is degenerate (see Observation 4.5).

The reason we are interested in the lowest complexity among the members of $[\mathcal{C}]$ for a class of metrizable compacta are the following notions introduced in [1]. A class of topological spaces $\mathcal{C}$ is *compactifiable* (or *Polishable*) if there is a continuous map $q\colon A \to B$ between metrizable compacta (or Polish spaces) $A$, $B$ such that the family of fibers $\{q^{-1}(b) : b \in B\}$ is equivalent to the given class $\mathcal{C}$. The map $q$ encodes how some representants of the members of $\mathcal{C}$ are disjointly composed together in one metrizable compactum (or Polish space) $A$. We call the resulting structure $\mathcal{A}(q\colon A \to B)$ a *composition*. Clearly, if the class $\mathcal{C}$ is compactifiable (or Polishable), then it necessarily consists of metrizable compacta (or Polish spaces).

We also define *strongly compactifiable* and *strongly Polishable classes* where the composition map $q$ additionally has to be closed and open. By [1, Corollary 3.17] every compactifiable class is strongly Polishable. Therefore, we have the implications:

$$\begin{matrix}\text{strongly}\\ \text{compactifiable}\end{matrix} \implies \text{compactifiable} \implies \begin{matrix}\text{strongly}\\ \text{Polishable}\end{matrix} \implies \text{Polishable.}$$

The following theorems show that the strong notions are directly connected to hyperspaces, and since the definition of (strongly) compactifiable and Polishable classes is inherently up to the equivalence $\cong$, we face the question of Borel complexity up to the equivalence.

**Theorem 1.1** ([1, Theorem 3.13])**.** The following conditions are equivalent for a class of topological spaces $\mathcal{C}$.

(i) $\mathcal{C}$ is strongly compactifiable.

(ii) There is a metrizable compactum $X$ and a closed family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.

(iii) There is a closed zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

**Theorem 1.2** ([1, Theorem 3.14])**.** The following conditions are equivalent for a class of topological spaces $\mathcal{C}$.

(i) $\mathcal{C}$ is a strongly Polishable class of compacta.

(ii) There is a Polish space $X$ and an analytic family $\mathcal{F} \subseteq \mathcal{K}(X)$ such that $\mathcal{F} \cong \mathcal{C}$.

(iii) There is a $G_\delta$ zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

(iv) There is a closed zero-dimensional disjoint family $\mathcal{F} \subseteq \mathcal{K}((0,1)^\omega)$ such that $\mathcal{F} \cong \mathcal{C}$.

So strong compactifiability correspond to existence of a closed equivalent subfamily of $\mathcal{K}([0,1]^\omega)$, and strong Polishability correspond to existence of an analytic or equivalently $G_\delta$ equivalent subfamily of $\mathcal{K}([0,1]^\omega)$. The theorems are proved by translating back and forth between families in hyperspaces and compositions. As a byproduct, we obtain the following theorem. We include a sketch of a standalone proof that gathers all the translations needed together.

**Theorem 1.3.** To every analytic family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ there exists an equivalent $G_\delta$ family $\mathcal{G} \subseteq \mathcal{K}([0,1]^\omega)$.

*Proof.* Let $R := \{\langle x, F\rangle \in [0,1]^\omega \times \mathcal{F} : x \in F\}$ and let $\pi\colon R \to \mathcal{F}$ be the projection. Since the family $\mathcal{F}$ is analytic, there is a Polish space $B$ and a continuous surjection $f\colon B \to \mathcal{F}$. Let $A := \{\langle x, b\rangle \in [0,1]^\omega \times B : x \in f(b)\}$ and let $q\colon A \to B$ be the projection. The space $A$ is separable metrizable, and so there is an embedding $e\colon A \hookrightarrow [0,1]^\omega$. We put $\mathcal{G} := \{e[q^{-1}(b)] : b \in B\}$.

For every $b \in B$ we have $e[q^{-1}(b)] \cong q^{-1}(b) = f(b) \times \{b\} \cong f(b) \in \mathcal{F}$, so $\mathcal{G}$ is equivalent to $\mathcal{F}$. The map $\pi$ is closed and open, and $q$ may be regarded as a pullback of $q$ along $f$. It follows that $q$ is also closed and open. We may suppose that $f$ also satisfies $|f^{-1}(\emptyset)| \leq 1$. We obtain that $B$ is homeomorphic to $\{q^{-1}(b) : b \in B\} \subseteq \mathcal{K}(A)$, which is homeomorphic to $\mathcal{G}$ via $e^*$. Hence, $\mathcal{G}$ is Polish and so $G_\delta$ in $\mathcal{K}([0,1]^\omega)$. For details see [1]. □

Let us note that for $\sigma$-ideals the previous theorem holds in a much stronger way.

**Theorem 1.4** ([3, Theorem 11])**.** Let $X$ be a metrizable compactum. Every analytic $\sigma$-ideal $\mathcal{F} \subseteq \mathcal{K}(X)$ is in fact $G_\delta$.

In this paper we analyze the remaining complexities, namely clopen, open, and $F_\sigma$ subsets of $\mathcal{K}([0,1]^\omega)$. The situation with clopen subsets is quite simple. It is well-known that the hyperspaces $\mathcal{K}(X) \setminus \{\emptyset\}$ and $\mathcal{C}(X) \setminus \{\emptyset\}$ are connected for any connected space $X$ (see for example [6, Exercises 4.32 and 5.25]). Hence, we obtain the following proposition.

**Proposition 1.5.** There are exactly four clopen subsets of $\mathcal{K}([0,1]^\omega)$: $\emptyset$, $\{\emptyset\}$, $\mathcal{K}([0,1]^\omega) \setminus \{\emptyset\}$, $\mathcal{K}([0,1]^\omega)$. Hence, there are only four corresponding classes: $\emptyset$, $\{\emptyset\}$, $\mathbf{K} \setminus \{\emptyset\}$, $\mathbf{K}$. Similarly, there are exactly four clopen subsets of $\mathcal{C}([0,1]^\omega)$: $\emptyset$, $\{\emptyset\}$, $\mathcal{C}([0,1]^\omega) \setminus \{\emptyset\}$, $\mathcal{C}([0,1]^\omega)$, and four corresponding classes of continua: $\emptyset$, $\{\emptyset\}$, $\mathbf{C} \setminus \{\emptyset\}$, $\mathbf{C}$.

The situation with open and $F_\sigma$ families is more involved and is the subject of the next sections. In the second section we prove that every open subset of $\mathcal{K}([0,1]^\omega)$ is equivalent to one of countably many saturated open subfamilies of the hyperspace (Theorem 2.18). In the third section we show that every $F_\sigma$ subset of $\mathcal{K}([0,1]^\omega)$ is equivalent to a closed subset (Theorem 3.6). In the fourth section we gather some observations on saturated and so-called type-saturated classes and families.

# 2 Open classes

Now let us look at open subsets of $\mathcal{K}([0,1]^\omega)$ up to the equivalence. First, we shall consider the following rough classification of metrizable compacta.

**Definition 2.1.** Let $X$ be a metrizable compactum.

- By $m(X)$ we denote the number of all connected components. By $n(X)$ we denote the number of all nondegenerate connected components.
- Let $T$ denote the set of all *finite types* $\{\langle m, n\rangle : m \geq n \in \omega\}$, and let $T_+$ denote the set of all *positive finite types* $\{\langle m, n\rangle \in T : m > 0\}$.
- We define the *type function* $t\colon \mathbf{K} \to T \cup \{\infty\}$ by $t(X) := \langle m(X), n(X)\rangle$ if $m(X) < \omega$, $\infty$ otherwise. Clearly, the type function is onto.
- We define a partial order $\leq$ on $T \cup \{\infty\}$: $\langle 0, 0\rangle$ is not comparable with anything; $T_+$ is ordered by the product order, i.e. $\langle m_1, n_1\rangle \leq \langle m_2, n_2\rangle$ if and only if $m_1 \leq m_2$ and $n_1 \leq n_2$; and $\infty \geq t$ for every $t \in T_+$.

- We define the *principal upper class* $\mathcal{U}_t := \{X \in \mathbf{K} : t(X) \geq t\}$ for every $t \in T \cup \{\infty\}$. Since the type function is onto, we have $t = \min\{t(X) : X \in \mathcal{U}_t\}$ for every $t \in T \cup \{\infty\}$, and so $t_1 \leq t_2 \iff \mathcal{U}_{t_1} \supseteq \mathcal{U}_{t_2}$ for every $t_1, t_2 \in T \cup \{\infty\}$.

**Example 2.2.** We have the following examples of principal upper classes.

- $\mathcal{U}_{m,0}$ is the class of all metrizable compacta with at least $m$ components.
- $\mathcal{U}_{m,0} \cup \mathcal{U}_{1,1}$ is the class of all metrizable compacta with at least $m$ points.
- $\mathcal{U}_{2,0} \cup \mathcal{U}_{1,1}$ is the class of all nondegenerate metrizable compacta.
- $\mathcal{U}_{1,1}$ is the class of all infinite metrizable compacta.
- $\mathcal{U}_{1,0}$ is the class of all nonempty metrizable compacta, i.e. $\mathbf{K} \setminus \{\emptyset\}$.
- $\mathcal{U}_{0,0} = \{\emptyset\}$ and $\mathcal{U}_{0,0} \cup \mathcal{U}_{1,0} = \mathbf{K}$.

We will show that open subsets of $\mathcal{K}([0,1]^\omega)$ are equivalent to some unions of principal upper classes. Since the finite spaces are dense in $\mathcal{K}([0,1]^\omega)$, not every principal upper class is open. However, this is essentially the only obstacle. That is why we define *nice* sets of types.

**Definition 2.3.** Let $R \subseteq T \cup \{\infty\}$.

- We say that $R$ is *nice* if $\langle m, 0\rangle \in R$ for some $m > 0$ whenever $R \cap (T_+ \cup \{\infty\}) \neq \emptyset$. This holds if and only if $\bigcup_{t\in R} \mathcal{U}_t$ contains a nonempty finite space whenever it contains a nonempty space.
- We say that $R$ is an *antichain* if it is pairwise $\leq$-incomparable. Note that every antichain is finite, and that no nice antichain contains $\infty$.
- By $A(R)$ we denote the set of all $\leq$-minimal elements of $R$. Note that this is the only antichain $A$ such that $\bigcup_{t\in A} \mathcal{U}_t = \bigcup_{t\in R} \mathcal{U}_t$. It follows that $A(R)$ is nice if and only if $R$ is nice.

Eventually, we will show that open subsets of $\mathcal{K}([0,1]^\omega)$ correspond to nice antichains in $T$ (Theorem 2.18), but first we determine which unions of principal upper classes are open.

**Definition 2.4.** For every finite function $s\colon I \to \mathbb{N}_+$, where $\mathbb{N}_+$ denotes the set of all positive integers, we define the *special open class* $\mathcal{O}_s$ of all metrizable compacta $K$ having a clopen decomposition $\{K_i : i \in I\}$ such that $|K_i| \geq s(i)$ for every $i \in I$.

Moreover, let $X$ be a metrizable space, and let $\mathcal{U} \subseteq \mathcal{K}(X)$ be open. We say that $\mathcal{U}$ is *of the shape* $s$ if there are disjoint open sets $U_i \subseteq X$, $i \in I$, and for every $i \in I$ there are disjoint open sets $V_{i,j} \subseteq U_i$, $j < s(i)$, such that $\mathcal{U} = (\bigcup_{i\in I} U_i)^+ \cap \bigcap_{i\in I, j<s(i)} V_{i,j}^-$. We say that $\mathcal{U}$ is *exactly of the shape* $s$ if moreover the set $U_i^+ \cap \bigcap_{j<s(i)} V_{i,j}^-$ contains a connected space for every $i \in I$.

By $n(s)$ we denote $|\{i \in I : s(i) > 1\}|$. To every type $t \in T \cup \{\infty\}$ we associate a set of finite functions $S_t$. If $t = \langle m, n\rangle$, we put $S_t := \{s\colon m \to \mathbb{N}_+ : n(s) \leq n\}$, if $t = \infty$, we put $S_t := \{s\colon m \to \mathbb{N}_+ : m > 0\}$.

**Observation 2.5.** Let $s\colon I \to \mathbb{N}_+$ be a finite function, let $X$ be a metrizable space, and let $K \in \mathcal{K}(X)$. $K$ has a neighborhood of the shape $s$ in $\mathcal{K}(X)$ if and only if $K \in \mathcal{O}_s$. It follows that $\mathcal{O}_s \cap \mathcal{K}(X)$ is open.

**Observation 2.6.** Let $s\colon I \to \mathbb{N}_+$ be a nonempty finite function and let be $K$ a metrizable compactum. If there are pairwise disjoint sets $A_i \subseteq K$, $i \in I$, such that for every $i \in I$ either $A_i$ is a nondegenerate component of $K$ or $A_i$ is the union of $s(i)$-many components, then $K \in \mathcal{O}_s$.

This is because the components and the quasi-components are the same and we have used only finitely many components when building the sets $A_i$, and hence there is a clopen decomposition $\{K_i : i \in I\}$ of $K$ such that $A_i \subseteq K_i$ for every $i \in I$. Also, every nondegenerate component is infinite, so $|K_i| \geq |A_i| \geq s(i)$ for every $i \in I$.

Note that each antichain in $T_+$ is of the form $\{\langle m+\sum_{i<j} \Delta m_i,\ n-\sum_{i<j} \Delta n_i\rangle : j \leq k\}$ for some $\{\Delta m_i, \Delta n_i : i < k\} \subseteq \mathbb{N}_+$, and it is nice if and only if $\sum_{i<k} \Delta n_i = n$, so the last member is $\langle m + \sum_{i<k} \Delta m_i, 0\rangle$. The next proposition says that each special open class $\mathcal{O}_s$ corresponds to such nice antichain additionally satisfying that each $\Delta n_i$ is 1 and that the sequence $\langle \Delta m_i : i < k\rangle$ is increasing.

**Proposition 2.7.** Let $s\colon I \to \mathbb{N}_+$ be a finite function. We have $\mathcal{O}_s = \bigcup_{t\in R_s} \mathcal{U}_t$ where $R_s$ is a nice antichain in $T$ defined as follows.

Let $\langle i_k : k < |I|\rangle$ be an enumeration of $I$ such that the map $k \mapsto s(i_k)$ is increasing. For every $n \leq n(s)$ let us consider the type $t_{s,n} := \langle n+\sum_{k<|I|-n} s(i_k), n\rangle$. In particular, $t_{s,0} = \langle \sum_{i\in I} s(i), 0\rangle$ and $t_{s,n(s)} = \langle |I|, n(s)\rangle$. We put $R_s := \{t_{s,n} : n \leq n(s)\}$.

*Proof.* First, if $s = \emptyset$, we have $\mathcal{O}_s = \{\emptyset\} = \mathcal{U}_{0,0} = \mathcal{U}_{t_{s,0}}$, so we may suppose that $s \neq \emptyset$.

If $K \in \mathcal{O}_s$, then it has a clopen decomposition $\{K_i : i \in I\}$ such that for every $i \in I$ we have $|K_i| \geq s(i)$. Let $J := \{i \in I : s(i) > 1$ and $K_i$ contains a nondegenerate component$\}$ and $n := |J|$. Clearly, $n \leq n(s)$. We have $\sum_{k<|I|-n} s(i_k) \leq \sum_{i\in I\setminus J} s(i)$ since the map $k \mapsto s(i_k)$ is increasing. Therefore, $t(K) \geq \langle |J| + \sum_{i\in I\setminus J} s(i), |J|\rangle \geq \langle n + \sum_{k<|I|-n} s(i_k), n\rangle = t_{s,n}$. It follows that $\mathcal{O}_s \subseteq \bigcup_{t\in R_s} \mathcal{U}_t$.

On the other hand, if $K \in \mathcal{U}_{t_{s,n}}$ for some $n \leq n(s)$, then $K$ has at least $n + \sum_{k<|I|-n} s(i_k)$ components at least $n$ of which are nondegenerate. Hence, we may find disjoint sets $A_i \subseteq K$, $i \in I$, such that $A_{i_k}$ is a nondegenerate component if $k \geq |I| - n$ and $A_{i_k}$ is the union of $s(i_k)$ components if $k < |I| - n$. From Observation 2.6 it follows that $K \in \mathcal{O}_s$, and so $\bigcup_{t\in R_s} \mathcal{U}_t \subseteq \mathcal{O}_s$. □

**Example 2.8.** We have the following examples of special open classes.

- $\mathcal{O}_\emptyset = \mathcal{U}_{0,0} = \{\emptyset\}$ is the empty space class.
- $\mathcal{O}_{\langle 1\rangle} = \mathcal{U}_{1,0} = \mathbf{K} \setminus \{\emptyset\}$ is the class of all nonempty metrizable compacta.
- $\mathcal{O}_{\langle 2\rangle} = \mathcal{U}_{1,1} \cup \mathcal{U}_{2,0}$ is the class of all nondegenerate metrizable compacta.
- $\mathcal{O}_{\langle m\rangle} = \mathcal{U}_{1,1} \cup \mathcal{U}_{m,0}$ is the class of all metrizable compacta with at least $m$ points.

- $\mathcal{O}_{\langle 1:i<m\rangle} = \mathcal{U}_{m,0}$ is the class of all metrizable compacta with at least $m$ components.
- $\mathcal{O}_{\langle 1,1,1,2,3,4\rangle} = \mathcal{U}_{6,3} \cup \mathcal{U}_{7,2} \cup \mathcal{U}_{9,1} \cup \mathcal{U}_{12,0}$.

**Corollary 2.9.** For every $t \in T \cup \{\infty\}$ and every $m \in \mathbb{N}_+$ there is $s_{t,m} \in S_t$ such that $\mathcal{U}_t \subseteq \mathcal{O}_{s_{t,m}} \subseteq \mathcal{U}_t \cup \mathcal{U}_{m,0}$.

*Proof.* For $t = \infty$ we simply put $s_{t,m} := \langle 1 : i < m\rangle$ so $\mathcal{O}_{s_{t,m}} = \mathcal{U}_{m,0}$. For $t = \langle m', n'\rangle \in T$ we define $s_{t,m} = s$ as a function with domain $m'$ taking the value $m$ $n'$ times and the value $1$ $m' - n'$ times. By Proposition 2.7 we have $\mathcal{O}_{s_{t,m}} = \bigcup_{n \le n'} \mathcal{U}_{t_{s,n}}$ and $t_{s,n} = \langle n + (m' - n') + (n' - n) \cdot m,\ n\rangle$. Hence, for $n = n'$ we obtain $\mathcal{U}_{t_{s,n}} = \mathcal{U}_t$ and for $n' - n > 0$ the first item is at least $m$, so $\mathcal{U}_{t_{s,n}} \subseteq \mathcal{U}_{m,0}$. □

**Proposition 2.10.** For every $t \in T \cup \{\infty\}$ we have $\mathcal{U}_t = \bigcap_{s \in S_t} \mathcal{O}_s$. In particular, $\mathcal{U}_t \cap \mathcal{K}(X)$ is $G_\delta$ for every metrizable space $X$, so every principal upper class is strongly Polishable. It also follows that $\mathcal{U}_{t'} \subseteq \mathcal{O}_s$ for every $t' \ge t$ and $s \in S_t$.

*Proof.* First let us show that $\mathcal{U}_t \subseteq \bigcap_{s \in S_t} \mathcal{O}_s$, so let $K \in \mathcal{U}_t$ and $s \in S_t$. If $t = \langle m, n\rangle \in T$, then $K$ has a clopen decomposition $\{K_i : i < m\}$ into components. Since $n(s) \le n$, we may choose the enumeration such that $K_i$ is nondegenerate whenever $s(i) > 1$. Since nondegenerate components are infinite, we have $|K_i| \ge s(i)$ for every $i < m$. If $t = \infty$, then $K$ has infinitely many components, so we may find suitable sets $A_i$ and use Observation 2.6. In both cases we have $K \in \mathcal{O}_s$.

Now, $\mathcal{U}_t \supseteq \bigcap_{s \in S_t} \mathcal{O}_s$. If $t \le \infty$, then for every $m > 0$ we take $s_{t,m} \in S_t$ from Corollary 2.9, and we have $\mathcal{U}_t \subseteq \bigcap_{m \in \mathbb{N}_+} \mathcal{O}_{s_{t,m}} \subseteq \mathcal{U}_t \cup \bigcap_{m \in \mathbb{N}_+} \mathcal{U}_{m,0} = \mathcal{U}_t \cup \mathcal{U}_\infty = \mathcal{U}_t$. Otherwise, $t = \langle 0, 0\rangle$ and $\mathcal{U}_t = \{\emptyset\} = \mathcal{O}_\emptyset$. □

**Proposition 2.11.** Let $R \subseteq T \cup \{\infty\}$. The set $\bigcup_{t \in R} \mathcal{U}_t \cap \mathcal{K}([0,1]^\omega)$ is open if and only if $R$ is nice.

*Proof.* First, suppose that $R$ is nice. Let $t \in R$. If $t = \langle 0, 0\rangle$ we put $s_t := \emptyset$ and we have $\mathcal{U}_t = \mathcal{O}_{s_t}$. Otherwise, there is $m > 0$ such that $\langle m, 0\rangle \in R$, and we put $s_t := s_{t,m}$ from Corollary 2.9, so $\mathcal{U}_t \subseteq \mathcal{O}_{s_t} \subseteq \mathcal{U}_t \cup \mathcal{U}_{m,0}$. Altogether, we have $\bigcup_{t \in R} \mathcal{U}_t = \bigcup_{t \in R} \mathcal{O}_{s_t}$, which has open intersection with $\mathcal{K}([0,1]^\omega)$ by Observation 2.5.

On the other hand, if $\mathcal{U} := \bigcup_{t \in R} \mathcal{U}_t \cap \mathcal{K}([0,1]^\omega)$ is open and $R$ meets $T_+ \cup \{\infty\}$, we have $\mathcal{U} \setminus \{\emptyset\} \ne \emptyset$. Since finite sets are dense, there is a finite set $F \in \mathcal{U} \setminus \{\emptyset\}$, and there is some $t \in R$ such that $F \in \mathcal{U}_t$. Since $F$ is finite and nonempty, we have $t = \langle m, 0\rangle$ for some $m > 0$, so $R$ is nice. □

The previous propositions regarding the properties of principal upper classes and special open classes would hold as well in the realm of Hausdorff compacta instead of metrizable compacta. Hausdorffness is needed so that components and quasi-components are the same in compacta and that nondegenerate connected spaces are infinite.

We have shown that open unions of principal upper classes are exactly unions over nice antichains. Now we show that every open subset of $\mathcal{K}([0,1]^\omega)$ is equivalent to such union.

**Lemma 2.12.** The set of all homeomorphic copies of $[0,1]^\omega$ is dense in $\mathcal{C}([0,1]^\omega)\setminus\{\emptyset\}$.

*Proof.* Let $U^+\cap\bigcap_{i<n}V_i^-$ be a basic neighborhood of a nonempty continuum $C\subseteq[0,1]^\omega$. Since $C$ is connected and $[0,1]^\omega$ is locally path-connected, we may suppose that the set $U$ is path-connected. For $i<n$ let $y_i\in U\cap V_i$, and let $Y$ be the union of finitely many paths in $U$ connecting the points $y_i$. There is some $\varepsilon>0$ such that $N_\varepsilon(Y)\subseteq U$. Let $f\colon[0,1]^\omega\to Y$ be a continuous surjection, for every $i<n$ let $x_i\in[0,1]^\omega$ be such that $f(x_i)=y_i$, and let $A:=\{x_i:i<n\}$. By the Mapping Replacement Theorem [5, 5.3.11] there is a Z-embedding $g\colon[0,1]^\omega\to[0,1]^\omega$ such that $g{\restriction_A}=f{\restriction_A}$ and $d(g,f)<\varepsilon$. Therefore, $[0,1]^\omega\cong\operatorname{rng}(g)\in U^+\cap\bigcap_{i<n}V_i^-$. □

**Lemma 2.13.** Let $F\subseteq[0,1]^\omega$ be a finite set. For every separable metrizable space $X$ such that $|X|\geq|F|$ there exists an embedding $f\colon X\hookrightarrow[0,1]^\omega$ such that $F\subseteq f[X]$.

*Proof.* Since $X$ is separable metrizable, we may suppose that $X\subseteq[0,1]^\omega$. Since $|X|\geq|F|$, there is a bijection $h\colon H\to F$ for some $H\subseteq X$. The map $h$ is a homeomorphism of Z-sets in $[0,1]^\omega$, so by [5, Theorem 5.3.7] it can be extended to a homeomorphism $\bar{h}\colon[0,1]^\omega\to[0,1]^\omega$. The restriction $\bar{h}{\restriction_X}$ is the desired embedding. □

**Proposition 2.14.** Let $s\colon I\to\mathbb{N}_+$ be a finite function. For every compactum $X\in\mathcal{O}_s$ and every open set $\mathcal{U}\subseteq\mathcal{K}([0,1]^\omega)$ exactly of the shape $s$ there is a compactum $Y\in\mathcal{U}$ homeomorphic to $X$.

*Proof.* Let $\{U_i,V_{i,j}:i\in I,j<s(i)\}$ be the open subsets of $[0,1]^\omega$ witnessing that $\mathcal{U}$ is exactly of the shape $s$, and let $\{X_i:i\in I\}$ be a clopen decomposition of $X$ such that $|X_i|\geq s(i)$ for every $i\in I$. Let $i\in I$. Since $U_i^+\cap\bigcap_{j<s(i)}V_{i,j}^-$ contains a connected space, it also contains a space $Q_i\cong[0,1]^\omega$ by Lemma 2.12. Let $F_i\subseteq Q_i$ be such that $|F_i|=s(i)$ and $F_i\cap V_{i,j}\neq\emptyset$ for every $j<s(i)$. By Lemma 2.13 there is a copy $Y_i\cong X_i$ such that $F_i\subseteq Y_i\subseteq Q_i$. Hence, $Y_i\in U_i^+\cap\bigcap_{j<s(i)}V_{i,j}^-$. Altogether we have $X\cong Y:=\bigcup_{i\in I}Y_i\in\mathcal{U}$. □

**Lemma 2.15.** Let $t\in T\cup\{\infty\}$. Every $K\in\mathcal{U}_t\cap\mathcal{K}(X)$ for any metrizable space $X$ has a neighborhood basis such that for every basic set $\mathcal{U}$ there is $s\in S_t$ such that $\mathcal{U}$ is exactly of the shape $s$.

*Proof.* Let $\mathcal{V}\subseteq\mathcal{K}(X)$ be any neighborhood of $K$. Without loss of generality $\mathcal{V}$ is of the form $V^+\cap\bigcap\{W^-:W\in\mathcal{W}\}$ for some open set $V\subseteq X$ and a finite family of open sets $\mathcal{W}$.

If $t=\infty$, let $\{C_i:i\in I\}$ be a finite collection of distinct components of $K$ such that every $W\in\mathcal{W}$ meets some of them, and let $\{K_i:i\in I\}$ be a clopen decomposition of $K$ such that $C_i\subseteq K_i$ for every $i\in I$. Such sets $K_i$ exist since components of $K$ are the quasi-components. If $t=\langle m,n\rangle\in T$, let $\{C_i=K_i:i\in I=m\}$ be the enumeration of all components of $K$.

For every $i\in I$ let $F_i:=\{x_{i,j}:j<s(i)\}\subseteq C_i$ be a nonempty finite set of minimal size such that $F_i\cap W\neq\emptyset$ for every $W\in\mathcal{W}\cap C_i^-$. This defines a function

$s\colon I \to \mathbb{N}_+$. For every $i \in I$ we have $s(i) \le |C_i|$, and so $n(s) \le n$ if $t = \langle m, n\rangle$. Hence, $s \in S_t$.

Since the set $I$ is finite, there are disjoint open sets $U_i \subseteq V$, $i \in I$, such that $K_i \subseteq U_i$, and for every $i \in I$ there are disjoint open sets $U_{i,j} \subseteq U_i$, $j < s(i)$, such that $x_{i,j} \in U_{i,j} \subseteq \bigcap\{W \in \mathcal{W} : x_{i,j} \in W\}$. We put $\mathcal{U} := (\bigcup_{i\in I} U_i)^+ \cap \bigcap_{i\in I, j<s(i)} U_{i,j}^-$ and $\mathcal{U}_i := U_i^+ \cap \bigcap_{j<s(i)} U_{i,j}^-$ for every $i \in I$. Since $\bigcup_{i\in I} U_i \subseteq V$ and for every $W \in \mathcal{W}$ there is $i \in I$ and $j < s(i)$ such that $U_{i,j} \subseteq W$, we have $\mathcal{U} \subseteq \mathcal{V}$. Since $C_i, K_i \in \mathcal{U}_i$ for every $i \in I$, we have that $\mathcal{U}$ is exactly of the shape $s$ and $K \in \mathcal{U}$. □

**Proposition 2.16.** Let $X, Y \in \mathcal{K}([0,1]^\omega)$. A homeomorphic copy of $Y$ is contained in every neighborhood of $X$ if and only if $t(Y) \ge t(X)$.

*Proof.* "$\Longleftarrow$": Suppose that $t(Y) \ge t(X)$ and let $\mathcal{U}$ be a neighborhood of $X$. By Lemma 2.15 we may suppose that $\mathcal{U}$ is exactly of the shape $s$ for some $s \in S_{t(X)}$. By Proposition 2.10 we have $Y \in \mathcal{U}_{t(Y)} \subseteq \mathcal{U}_{t(X)} \subseteq \mathcal{O}_s$. Finally, by Proposition 2.14, there is a space $Y' \in \mathcal{U}$ homeomorphic to $Y$.

"$\Longrightarrow$": Suppose that $t(Y) \not\ge t(X)$. We have $Y \notin \mathcal{U}_{t(X)} = \bigcap_{s\in S_{t(X)}} \mathcal{O}_s$ by Proposition 2.10. Hence, there is some $s \in S_{t(X)}$ such that $Y \notin \mathcal{O}_s \cap \mathcal{K}([0,1]^\omega) \ni X$. Since $\mathcal{O}_s$ is closed under homeomorphic copies, we are done. □

**Definition 2.17.** By $\mathcal{R}$ we denote the countable set of all nice antichains of $T \cup \{\infty\}$. For every $R \in \mathcal{R}$ we define the *open class* $\mathcal{O}_R := \bigcup_{t\in R} \mathcal{U}_t$. Proposition 2.7 says that every special open class is an open class, namely $\mathcal{O}_s = \mathcal{O}_{R_s}$ for every finite $s\colon I \to \mathbb{N}_+$.

**Theorem 2.18.** For every open $\mathcal{U} \subseteq \mathcal{K}([0,1]^\omega)$ there exists exactly one $R \in \mathcal{R}$ such that $\mathcal{U} \cong \mathcal{O}_R$. On the other hand, for every $R \in \mathcal{R}$ we have $\mathcal{O}_R \cong \mathcal{O}_R \cap \mathcal{K}([0,1]^\omega)$, which is open.

*Proof.* By Proposition 2.16 and by universality of $\mathcal{K}([0,1]^\omega)$ we have the equivalence $\mathcal{U} \cong \bigcup_{X\in\mathcal{U}} \mathcal{U}_{t(X)}$. We put $R := A(\{t(X) : X \in \mathcal{U}\})$. Since $\mathcal{U}$ is open, it contains a nonempty finite space whenever it contains a nonempty space. Therefore, $R$ is nice and $\mathcal{U} \cong \mathcal{O}_R$.

Clearly, if $R \ne R' \in \mathcal{R}$, there is a type $t \in T$ that is above some member of $R$ and above no member of $R'$ or the other way around. Any metrizable compactum $X$ of type $t$ satisfies $X \in (\mathcal{O}_R \setminus \mathcal{O}_{R'}) \cup (\mathcal{O}_{R'} \setminus \mathcal{O}_R)$, and hence $\mathcal{O}_R \not\cong \mathcal{O}_{R'}$.

On the other hand, let $R \in \mathcal{R}$. $\mathcal{O}_R \cap \mathcal{K}([0,1]^\omega)$ is open by Proposition 2.11, and $\mathcal{O}_R \cong \mathcal{O}_R \cap \mathcal{K}([0,1]^\omega)$ since $\mathcal{K}([0,1]^\omega)$ is universal for metrizable compacta. □

**Corollary 2.19.** There are exactly six nonequivalent classes corresponding to open subsets of $\mathcal{C}([0,1]^\omega)$. Besides the four clopen classes $\emptyset$, $\{\emptyset\}$, $\mathbf{C} \setminus \{\emptyset\}$, and $\mathbf{C}$, there is the class of all nondegenerate continua $\mathcal{U}_{1,1} \cap \mathbf{C}$ and the class $(\mathcal{U}_{1,1} \cup \mathcal{U}_{0,0}) \cap \mathbf{C} = (\mathcal{U}_{1,1} \cap \mathbf{C}) \cup \{\emptyset\}$.

*Proof.* Every open subset $\mathcal{V}$ of $\mathcal{C}([0,1]^\omega)$ is of form $\mathcal{U} \cap \mathbf{C}$ where $\mathcal{U}$ is open in $\mathcal{K}([0,1]^\omega)$. By Theorem 2.18 we have $\mathcal{U} \cong \mathcal{O}_R$ for some nice antichain $R$, and hence $\mathcal{V} \cong \mathcal{O}_R \cap \mathbf{C}$. Since $\mathcal{U}_{2,0} \cap \mathbf{C} = \emptyset$, open subsets of $\mathcal{C}([0,1]^\omega)$ are equivalent to classes $\bigcup_{t\in R} \mathcal{U}_t \cap \mathbf{C}$ where $R$ is any antichain in $\{\langle 0,0\rangle, \langle 1,0\rangle, \langle 1,1\rangle\}$. These are the six declared classes. □

# 3 Countable unions of strongly compactifiable classes

In this section we show that every $F_\sigma$ subset of $\mathcal{K}([0,1]^\omega)$ is equivalent to a closed subset, or equivalently, that strongly compactifiable classes are stable under countable unions. But first we have to improve several results from the previous section.

**Lemma 3.1.** Let $X$ be a metrizable space and let $\mathcal{F} \subseteq \mathcal{K}(X)$ be a compact family. For every open set $U \subseteq X$ such that $\mathcal{F} \subseteq U^-$ there exists a closed set $A \subseteq X$ such that $A \subseteq U$ and $\mathcal{F} \subseteq A^-$.

*Proof.* Let $d$ be a compatible metric on $X$. For every $F \in \mathcal{F}$ there is $x_F \in F$ and $\delta_F > 0$ such that $B(x_F, \delta_F) \subseteq U$. Since $\mathcal{F}$ is compact, there is a finite collection $\mathcal{H} \subseteq \mathcal{F}$ such that $\mathcal{F} \subseteq \bigcup_{H\in\mathcal{H}} B(x_H, \delta_H/2)^-$. Hence, for every $F \in \mathcal{F}$ there is $H_F \in \mathcal{H}$ and $y_F \in F \cap B(x_{H_F}, \delta_{H_F}/2)$. We put $Y := \{y_F : F \in \mathcal{F}\}$ and $\delta := \min\{\delta_H/2 : H \in \mathcal{H}\}$. For every $F \in \mathcal{F}$ we have that $B(y_F, \delta) \subseteq B(x_{H_F}, \delta_{H_F}) \subseteq U$. Therefore, $d(Y, X \setminus U) \geq \delta$ and $A := \overline{Y} \subseteq U$. □

**Lemma 3.2.** Let $X$ be a separable metrizable space, let $J$ be finite, and let $F_j \subseteq X$, $j \in J$, be disjoint compact sets. Let $V_j \subseteq [0,1]^\omega$, $j \in J$, be disjoint nonempty open sets. There is an embedding $f\colon X \hookrightarrow [0,1]^\omega$ such that $f[F_j] \subseteq V_j$ for every $j \in J$.

*Proof.* There exists a Z-set $Q \in \bigcap_{j\in J} V_j^-$ such that $Q \cong [0,1]^\omega$. This follows from [5, Lemma 5.1.3] since there is $n \in \omega$ such that every set $V_j$ contains a point $x_j$ such that $\pi_n(x_j) = 1$. Also, by Lemma 2.12 there are sets $Q_j \subseteq Q \cap V_j$, $j \in J$, such that $Q_j \cong [0,1]^\omega$ for every $j \in J$.

Since $X$ is separable metrizable, we may suppose that $X \subseteq Q$. There are homeomorphisms $h_j\colon F_j \to H_j \subseteq Q_j$ for $j \in J$. The map $h := \bigcup_{j\in J} h_j$ is a homeomorphism of Z-sets in $[0,1]^\omega$ since $\bigcup_{j\in J} F_j$ and $\bigcup_{j\in J} H_j$ are closed subsets of the Z-set $Q$. By [5, Theorem 5.3.7] the map $h$ can be extended to a homeomorphism $\bar{h}\colon [0,1]^\omega \to [0,1]^\omega$. The restriction $\bar{h}\restriction_X$ is the desired embedding. □

**Proposition 3.3.** Let $s\colon I \to \mathbb{N}_+$ be a finite function, let $\mathcal{U} \subseteq \mathcal{K}([0,1]^\omega)$ be an open set exactly of the shape $s$, and let $\mathcal{V} \subseteq \mathcal{K}(X)$ be an open set of the shape $s$ for some metrizable space $X$. For every compact family $\mathcal{H} \subseteq \mathcal{V}$ there is a compact family $\mathcal{F} \subseteq \mathcal{U}$ and a homeomorphism $\Phi\colon \mathcal{H} \to \mathcal{F}$ such that $\Phi(H) \cong H$ for every $H \in \mathcal{H}$.

*Proof.* Let $\{U_i, U_{i,j} : i \in I, j < s(i)\}$ be open subsets of $[0,1]^\omega$ witnessing that $\mathcal{U}$ is exactly of the shape $s$, let $\{V_i, V_{i,j} : i \in I, j < s(i)\}$ be open subsets of $X$ witnessing that $\mathcal{V}$ is of the shape $s$, and let $\mathcal{H} \subseteq \mathcal{V}$ be a compact family. We fix $i \in I$ and put $\mathcal{U}_i := U_i^+ \cap \bigcap_{j<s(i)} U_{i,j}^-$. Since $\mathcal{U}_i$ contains a connected space, it also contains a space $Q_i \cong [0,1]^\omega$ by Lemma 2.12. For every $j < s(i)$ there is a compact set $A_{i,j} \subseteq V_{i,j}$ such that $\mathcal{H} \subseteq A_{i,j}^-$ (Lemma 3.1). By Lemma 3.2 there is an embedding $e_i\colon [0,1]^\omega \to Q_i$ such that $e_i[A_{i,j}] \subseteq U_{i,j}$ for every $j < s(i)$.

For every $i \in I$ we have the homeomorphism $h_i := e_i \restriction_{V_i} : V_i \to \operatorname{rng}(e_i) \subseteq Q_i$. Since the families $\{V_i : i \in I\}$ and $\{\operatorname{rng}(e_i) : i \in I\}$ are separated, the map $h := \bigcup_{i \in I} h_i : \bigcup_{i \in I} V_i \to \bigcup_{i \in I} \operatorname{rng}(e_i)$ is also a homeomorphism. We put $\Phi := h^* \restriction_{\mathcal{H}}$ and $\mathcal{F} := \operatorname{rng}(\Phi)$. Clearly, $\Phi : \mathcal{H} \to \mathcal{F}$ is a homeomorphism and $\Phi(H) \cong H$ for every $H \in \mathcal{H}$.

For every $H \in \mathcal{H}$ and $i \in I$ we have $e_i[H \cap V_i] \in \mathcal{U}_i$. This is because $e_i[H \cap V_i] \subseteq Q_i \subseteq U_i$ and $H \cap V_i \in \bigcap_{j < s(i)} A_{i,j}^-$ so $e_i[H \cap V_i] \in \bigcap_{j < s(i)} U_{i,j}^-$. It follows that $\Phi(H) = \bigcup_{i \in I} e_i[H \cap V_i] \in \mathcal{U}$, and so $\mathcal{F} \subseteq \mathcal{U}$. □

Now we are ready to improve Proposition 2.14 from spaces to compact families of spaces.

**Proposition 3.4.** Let $s : I \to \mathbb{N}_+$ be a finite function. For every strongly compactifiable class $\mathcal{C} \subseteq \mathcal{O}_s$ and every open set $\mathcal{U} \subseteq \mathcal{K}([0,1]^\omega)$ exactly of the shape $s$ there is a compact zero-dimensional family $\mathcal{F} \subseteq \mathcal{U}$ equivalent to $\mathcal{C}$.

*Proof.* By Theorem 1.1 there is a closed zero-dimensional family $\mathcal{H} \subseteq \mathcal{K}([0,1]^\omega)$ equivalent to $\mathcal{C}$. For every $H \in \mathcal{H}$ let $\mathcal{V}_H \subseteq \mathcal{K}([0,1]^\omega)$ be a neighborhood of $F$ of the shape $s$ (Observation 2.5). The collection $\{\mathcal{V}_H : H \in \mathcal{H}\}$ is an open cover of $\mathcal{H}$. Since $\mathcal{H}$ is compact and zero-dimensional, there is a finite clopen decomposition $\{\mathcal{H}_k : k < n\}$ of $\mathcal{H}$ and a finite subcover $\{\mathcal{V}_k : k < n\} \subseteq \{\mathcal{V}_H : H \in \mathcal{H}\}$ such that $\mathcal{H}_k \subseteq \mathcal{V}_k$ for every $k < n$.

By Proposition 3.3 for every $k < n$ there is homeomorphism $\Phi_k : \mathcal{H}_k \to \mathcal{F}_k \subseteq \mathcal{U}$ such that $\mathcal{H}_k$ is equivalent to $\mathcal{F}_k$. Clearly, $\mathcal{F} := \bigcup_{k<n} \mathcal{F}_k \subseteq \mathcal{U}$ is a compact zero-dimensional family equivalent to $\mathcal{C}$. □

**Corollary 3.5.** For every strongly compactifiable class of infinite compacta $\mathcal{C}$ and $\varepsilon > 0$ there is a closed zero-dimensional family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ equivalent to $\mathcal{C}$ such that every space $F \in \mathcal{F}$ is $\varepsilon$-dense in $[0,1]^\omega$.

*Proof.* Let $A \subseteq [0,1]^\omega$ be a finite $2\varepsilon/3$-dense $2\varepsilon/3$-separated set and let $\mathcal{U} := \bigcap_{x \in A} B(x, \varepsilon/3)^-$. The balls $B(x, \varepsilon/3)$ are pairwise disjoint, and so the open set $\mathcal{U}$ is exactly of the shape $s := \langle |A| \rangle$. We have $\mathcal{C} \subseteq \mathcal{O}_s$ since all members of $\mathcal{C}$ are infinite and $\mathcal{O}_s$ is the class of all metrizable compacta with at least $|A|$ points. By Proposition 3.4 there is a closed zero-dimensional family $\mathcal{F} \subseteq \mathcal{U}$ equivalent to $\mathcal{C}$. For every $F \in \mathcal{F}$ and $x \in A$ we have $F \cap B(x, \varepsilon/3) \neq \emptyset$, and hence $F$ is $\varepsilon$-dense. □

**Theorem 3.6.** Every countable union of strongly compactifiable classes is strongly compactifiable, i.e. every $F_\sigma$ subset of $\mathcal{K}([0,1]^\omega)$ is strongly compactifiable and equivalent to a closed subset of $\mathcal{K}([0,1]^\omega)$.

*Proof.* Let $\mathcal{C}_n$, $n \in \omega$, be strongly compactifiable classes and let $\mathcal{C} = \bigcup_{n \in \omega} \mathcal{C}_n$. For every $n \in \omega$ there is a compact zero-dimensional family $\mathcal{H}_n \subseteq \mathcal{K}([0,1]^\omega)$ equivalent to $\mathcal{C}_n$ (Theorem 1.1). The set of minimal types $R := A(\{t(X) : X \in \mathcal{C}\})$ is finite as any antichain in $T \cup \{\infty\}$. For every $t \in R$ let us fix a space $F_{t,\infty} \in \mathcal{K}([0,1]^\omega)$ such that $F_{t,\infty} \in \mathcal{C}^{\cong}$ and $t(F_{t,\infty}) = t$. Every space $F_{t,\infty}$ has a countable decreasing neighborhood base $\{\mathcal{B}_{t,n} : n \in \omega\}$ such that every $\mathcal{B}_{t,n}$ is exactly of the shape $s_{t,n}$ for some $s_{t,n} \in S_t$ (Lemma 2.15).

For every $n \in \omega$ the family $\{\mathcal{O}_{s_{t,n}} : t \in R\}$ covers the compact zero-dimensional family $\mathcal{H}_n$ by Proposition 2.10, and so there is a clopen decomposition $\{\mathcal{H}_{t,n} : t \in R\}$ of $\mathcal{H}_n$ such that $\mathcal{H}_{t,n} \subseteq \mathcal{O}_{s_{t,n}}$ for every $t \in R$. By Proposition 3.4 there is a compact family $\mathcal{F}_{t,n} \subseteq \mathcal{B}_{t,n}$ equivalent to $\mathcal{H}_{t,n}$ for every $t \in R$. We put $\mathcal{F}_t := \bigcup_{n\in\omega} \mathcal{F}_{t,n} \cup \{F_{t,\infty}\}$ and $\mathcal{F} := \bigcup_{t\in R} \mathcal{F}_t$. Every family $\mathcal{F}_t$ is closed since the families $\mathcal{F}_{t,n}$ are closed and $\bigcap_{n\in\omega} \overline{\bigcup_{m\geq n} \mathcal{F}_{t,m}} \subseteq \bigcap_{n\in\omega} \overline{\mathcal{B}_{t,n}} = \{F_{t,\infty}\}$. The theorem follows since $\mathcal{C} = \bigcup_{n\in\omega} \mathcal{C}_n \cong \bigcup_{n\in\omega} \mathcal{H}_n = \bigcup_{t\in R, n\in\omega} \mathcal{H}_{t,n} \cong \bigcup_{t\in R} \mathcal{F}_t = \mathcal{F}$. □

**Corollary 3.7.** Every $F_\sigma$ subset of $\mathcal{C}([0,1]^\omega)$ is strongly compactifiable and equivalent to a closed subset of $\mathcal{C}([0,1]^\omega)$.

Theorem 3.6 together with Theorem 1.3 and 2.18 completes the picture of Borel complexity up to the equivalence – see Figure 1. The complexities reduce to four nontrivial groups of classes – clopen classes, open classes, strongly compactifiable classes, and strongly Polishable classes.

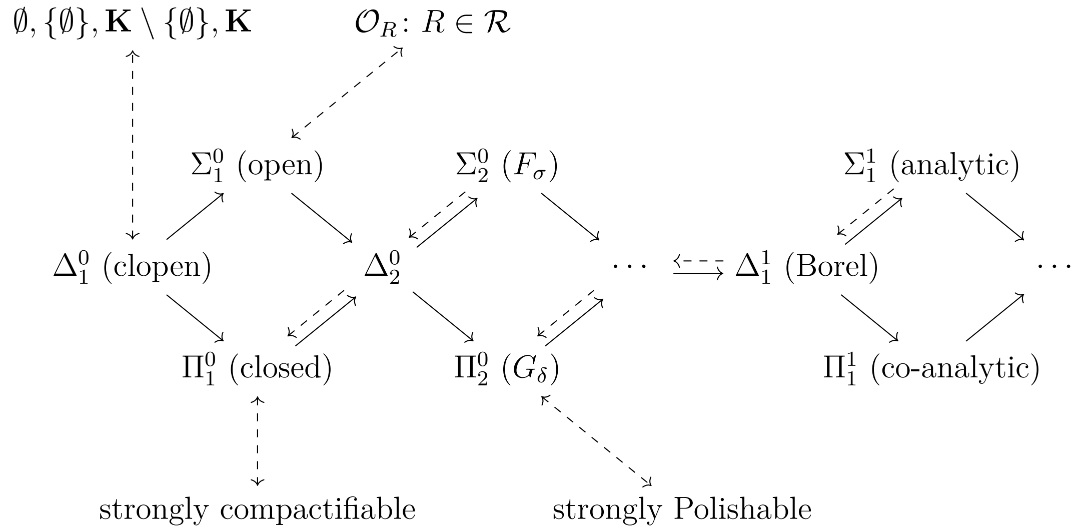


Figure 1: Complexities and corresponding classes. "⟶" denotes implication, "- - →" denotes implication up to the equivalence.

It is easy to see that there are open classes which are not clopen and that there are strongly compactifiable classes that are nor open. Also, there are classes which are not strongly Polishable. Nevertheless, the following remains open.

**Question 3.8.** Is there an analytic subset of $\mathcal{K}([0,1]^\omega)$ that is not equivalent to a closed subset? In other words, is there a class of metrizable compacta that is strongly Polishable, but not strongly compactifiable? One candidate is the class of all Peano continua [1, Question 4.25].

# 4 Saturated and type-saturated classes

We have defined saturated classes and saturated families. In general, on any set or class $X$ endowed with an equivalence we may consider its saturated subsets or subclasses – $A \subseteq X$ is saturated if it is the union of some equivalence classes, i.e. if it is closed under equivalent elements. So our saturated classes are saturated with respect to the equivalence of topological spaces where two spaces are equivalent if they are homeomorphic, and our saturated families are saturated with respect to the same equivalence but restricted to $\mathcal{K}([0,1]^\omega)$.

**Definition 4.1.** We say that a class of metrizable compacta $\mathcal{C}$ is *type-saturated* if is it saturated with respect to the equivalence induced by the type function $t\colon \mathbf{K} \to T \cup \{\infty\}$, i.e. $X, Y \in \mathbf{K}$ are equivalent if $t(X) = t(Y)$. That means type-saturated classes are the unions $\bigcup_{t\in R} \mathcal{T}_t$ for $R \subseteq T \cup \{\infty\}$ where $\mathcal{T}_t$ for $t \in T \cup \{\infty\}$ denotes the *principal type-saturated class* $\{K \in \mathbf{K} : t(K) = t\}$. For a set of types $R \subseteq T \cup \{\infty\}$ we denote the type-saturated class $\{K \in \mathbf{K} : t(K) \in R\} = \bigcup_{t\in R} \mathcal{T}_t$ by $\mathcal{T}_R$.

Clearly, every type-saturated class is saturated.

**Remark 4.2.** For every saturated class $\mathcal{C}$ of metrizable compacta we have $(\mathcal{C} \cap \mathcal{K}([0,1]^\omega))^{\cong} = \mathcal{C}$, and for every saturated family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ we have $\mathcal{F}^{\cong} \cap \mathcal{K}([0,1]^\omega) = \mathcal{F}$. This gives us a canonical identification between saturated classes and saturated families of metrizable compacta. Therefore, we may lift topological properties of saturated families to the corresponding saturated classes, e.g. we may say "closed class" or "open class" in the sense that the corresponding saturated family is closed or open. Note that this usage of "open class" is consistent with Definition 2.17. This also includes the type-saturated classes, so for example "$\mathcal{T}_\infty$ is $G_\delta$" means that the corresponding family $\mathcal{T}_\infty \cap \mathcal{K}([0,1]^\omega)$ is $G_\delta$ in $\mathcal{K}([0,1]^\omega)$. On the other hand, we have defined only type-saturated classes, but this correspondence allows us to talk about type-saturated families without an explicit definition.

**Observation 4.3.** By Theorem 2.18 every open family $\mathcal{U} \subseteq \mathcal{K}([0,1]^\omega)$ is equivalent to some open class $\mathcal{O}_R$, which is by definition type-saturated. Hence, $\mathcal{U}^{\cong} = \mathcal{O}_R$. It follows that the saturation of an open family is still an open family, and that every saturated open or closed family is type-saturated. In particular, for a class $\mathcal{C}$ of metrizable compacta, $\mathcal{C}^{\cong} \cap \mathcal{K}([0,1]^\omega)$ is closed if and only if $\mathcal{C}^{\cong} = \mathbf{K} \setminus \mathcal{O}_R$ for some $R \in \mathcal{R}$.

By Proposition 1.5 the situation with clopen families is even simpler – they just are type-saturated.

The following corollary summarizes which complexities are preserved by saturation.

**Corollary 4.4.** If a family $\mathcal{F} \subseteq \mathcal{K}([0,1]^\omega)$ is clopen, open, or analytic, then so is the corresponding saturated family $\mathcal{F}^{\cong} \cap \mathcal{K}([0,1]^\omega)$. On the other hand, there is a closed family $\mathcal{F}$ such that the corresponding saturated family is not Borel.

*Proof.* For clopen and open families, this follows Observation 4.3. The saturation of an analytic family is analytic by [1, Theorem 4.26] and Theorem 1.2.

The class of all uncountable metrizable compacta is analytically complete [2, Theorem 27.5], but yet strongly compactifiable [1, Example 4.15], and so equivalent to a closed family $\mathcal{F}$. □

Let us make some remarks on the complexity of the saturation of a singleton family. So let $X$ be a metrizable compactum and let $\mathcal{F}$ be the corresponding saturated family $\{X\}^{\cong} \cap \mathcal{K}([0,1]^\omega)$. $\mathcal{F}$ is always Borel [7, Theorem 2], but besides that it can be arbitrarily complex [4, Fact 3.12]. Section 3.5 of [4] also gives us some examples:

- If $X$ is a graph or a dendrite with finitely many branching points, then $\mathcal{F}$ is $F_{\sigma\delta}$-complete.
- If $X$ is the pseudo-arc, then $\mathcal{F}$ is $G_\delta$-complete.
- If $X$ is the Sierpiński universal curve or the Menger universal curve, then $\mathcal{F}$ is $F_{\sigma\delta}$-complete.

**Observation 4.5.** It follows from Proposition 2.16 that $\overline{\mathcal{F}} = \{K \in \mathcal{K}([0,1]^\omega) : t(K) \leq t(X)\}$. Therefore, $\mathcal{F}$ is closed if and only if $X$ is degenerate. $\mathcal{F}$ is dense in nonempty compacta if and only if $t(X) = \infty$, i.e. if $X$ has infinitely many components. $\mathcal{F}$ is dense in nonempty continua if and only if $X$ is a nondegenerate continuum.

In the last part we shall look at the type-saturated classes in more detail. We say that a type-saturated class $\mathcal{T}_R$ is *lower* or *upper* if the corresponding set $R$ is lower or upper in the ordered set $T \cup \{\infty\}$. Observe that every open type-saturated class is upper, and every closed type-saturated class is lower.

Also recall that a subset of a topological space is called *locally closed* if it is the intersection of an open set and a closed set.

**Observation 4.6.** The class $\mathcal{T}_{0,0}$ is clopen, $\mathcal{T}_{1,0}$ is closed, $\mathcal{T}_\infty$ is $G_\delta$, and $\mathcal{T}_t$ is locally closed for every other $t \in T \cup \{\infty\}$. No principal type-saturated class has a lower complexity than stated.

*Proof.* We already know that $\mathcal{T}_{0,0} = \mathcal{U}_{0,0} = \{\emptyset\}$ is (with its complement) the only nontrivial clopen class (Proposition 1.5). We have $\mathcal{T}_{1,0} = \mathbf{K} \setminus (\mathcal{O}_\emptyset \cup \mathcal{O}_{\langle 2 \rangle})$, so it is closed. We already know that $\mathcal{T}_\infty = \mathcal{U}_\infty$ is $G_\delta$ (Proposition 2.10) and dense (Observation 4.5), and so it is comeager. Since finite spaces are dense, $\mathcal{T}_\infty$ has empty interior. So if it was $F_\sigma$, it would be also meager. For $t = \langle m, n \rangle \in T_+$ we put $t' := \langle m, n+1 \rangle$ if $m > n$ and $\langle m+1, 0 \rangle$ otherwise. Let $\mathcal{V} := \mathcal{U}_t \cup \mathcal{U}_{\langle m+1, 0 \rangle}$ and $\mathcal{V}' := \mathcal{U}_{t'} \cup \mathcal{U}_{\langle m+1, 0 \rangle}$. Both classes $\mathcal{V}$ and $\mathcal{V}'$ are open and $\mathcal{T}_t = \mathcal{V} \setminus \mathcal{V}'$, so $\mathcal{T}_t$ is locally closed. $\mathcal{T}_t$ for $t \notin \{\langle 0,0 \rangle, \langle 1,0 \rangle, \infty\}$ is neither open nor closed since it is neither upper nor lower. □

**Corollary 4.7.** Let $R \subseteq T \cup \{\infty\}$. If $\infty \notin R$, then $\mathcal{T}_R$ is $F_\sigma$. Otherwise, $\mathcal{T}_R$ is $G_\delta$.

*Proof.* We have $\mathcal{T}_R = \bigcup_{t \in R} \mathcal{T}_t$, and if $\infty \notin R$, then each such $\mathcal{T}_t$ is $F_\sigma$. If $\infty \in R$, then the complementing type-saturated class is $F_\sigma$ by the previous claim. □

**Remark 4.8.** Even though the class $\mathcal{U}_\infty = \mathcal{T}_\infty$ of all metrizable compacta with infinitely many components is not $F_\sigma$, it is strongly compactifiable [1, Example 4.18]. It follows that every type-saturated class $\mathcal{T}_R$, $R \subseteq T \cup \{\infty\}$, is strongly compactifiable since it is either $\mathcal{T}_{R \setminus \{\infty\}}$ or $\mathcal{T}_{R \setminus \{\infty\}} \cup \mathcal{T}_\infty$, and $\mathcal{T}_{R \setminus \{\infty\}}$ is $F_\sigma$ by the previous corollary.

**Remark 4.9.** In the previous corollary we used the fact that every open saturated family is $F_\sigma$. But that does not mean it is the countable union of saturated closed families. Saturated closed families are type-saturated (Observation 4.3), so every union of them is also type-saturated. On the other hand, there are $F_\sigma$ or $G_\delta$ saturated families that are not type-saturated (see the examples before Observation 4.5).

**Observation 4.10.** Let us consider the quotient $q_{\cong}\colon \mathcal{K}([0,1]^\omega) \to \mathcal{K}([0,1]^\omega)/{\cong}$, so open subsets of $\mathcal{K}([0,1]^\omega)/{\cong}$ correspond to saturated open families. In general, subsets of $\mathcal{K}([0,1]^\omega)/{\cong}$ correspond to saturated families, and for example $F_\sigma$ subsets of $\mathcal{K}([0,1]^\omega)/{\cong}$ correspond to countable unions of saturated closed families. Since by the proof of Observation 4.6 every principal type-saturated class is obtained as a Borel combination of open type-saturated classes, we have that type-saturated classes correspond exactly to Borel subsets of $\mathcal{K}([0,1]^\omega)/{\cong}$.

It is not true that open subsets of $\mathcal{K}([0,1]^\omega)/{\cong}$ are $F_\sigma$. This space is not metrizable. In fact, it is not even $T_0$. Two points of $\mathcal{K}([0,1]^\omega)/{\cong}$ represented by spaces $X, Y \in \mathcal{K}([0,1]^\omega)$ are indistinguishable if and only if $t(X) = t(Y)$, so we may consider the Kolmogorov quotient $q_{T_0}\colon \mathcal{K}([0,1]^\omega)/{\cong} \to T \cup \{\infty\}$. In fact, the composition quotient map $q_{T_0} \circ q_{\cong}$ is just the type function $t\colon \mathcal{K}([0,1]^\omega) \to T \cup \{\infty\}$. This endows the set of all types $T \cup \{\infty\}$ with the topology where $R \subseteq T \cup \{\infty\}$ is open if and only if it is upper and nice.

It is also easy to directly see that these sets form a topology. Upper sets are stable under arbitrary unions and intersections, and nice sets are stable under arbitrary unions. Moreover, nice upper sets are stable under finite intersections: if $R_1 \cap R_2 \cap T_+ \neq \emptyset$, then since $R_1$ and $R_2$ are nice, there are some $m_1, m_2 > 0$ such that $\langle m_1, 0\rangle \in R_1$ and $\langle m_2, 0\rangle \in R_1$. Since $R_1$ and $R_2$ are upper, we have $\max\{\langle m_1, 0\rangle, \langle m_2, 0\rangle\} \in R_1 \cap R_2$.

**Observation 4.11.** The proof of Observation 4.6 in fact works in $T \cup \{\infty\}$, i.e. $\{\langle 0,0\rangle\}$ is clopen, $\{\langle 1,0\rangle\}$ is closed, $\{\infty\}$ is $G_\delta$, and $\{t\}$ is locally closed for every other $t \in T \cup \{\infty\}$. Also, no singleton has a lower complexity than stated.

Here we have to be more careful since open sets are not necessarily $F_\sigma$. Instead of $F_\sigma$ we should consider the complexity $\Sigma^0_2$ – the countable unions of members of $\Pi^0_1$. Instead of starting just with open sets and closed sets, we let $\Pi^0_1 = \Sigma^0_1$ be the algebra generated by open sets and closed sets. Members of the algebra are called *constructible sets*, and they are finite unions of locally closed sets.

So let us show that $\{\infty\}$ is not $\Sigma^0_2$. Since our set is a singleton, it would mean $\{\infty\}$ is locally closed. If $\{\infty\}$ was locally closed in $T \cup \{\infty\}$, we would have $\{\infty\} = \overline{\{\infty\}} \cap U = (T_+ \cup \{\infty\}) \cap U = U$ for some open set $U \subseteq T_+ \cup \{\infty\}$. So $\{\infty\}$ would be open, which it is not since it is not nice.

Also, for $t \neq \langle 0,0\rangle, \langle 1,0\rangle, \infty$ the singleton $\{t\}$ is neither in any class $F_\sigma$, $F_{\sigma\delta}$, $F_{\sigma\delta\sigma}$, … since they consist only of lower sets, nor in any class $G_\delta$, $G_{\delta\sigma}$, $G_{\delta\sigma\delta}$, … since they consist only of upper sets.

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unions and that this could be proved by approximating a limit object in the hyperspace.

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