Introduction to forcing for the working mathematician

lecture 1: Independence of the Continuum Hypothesis

David Chodounský

Institute of Mathematics CAS

Set Theory

In this talk, "Set Theory" means ZFC.

Objects are sets. Language is $\langle =, \in \rangle$.

Other relations and operations are derived; $\subset, \cap, \cup, \mathcal{P}(\cdot), \emptyset, \ldots$

Set Theory

In this talk, “Set Theory” means ZFC.

Objects are sets. Language is $\langle =, \in \rangle$.

Other relations and operations are derived; $\subset, \cap, \cup, \mathcal{P}(\cdot), \emptyset, \ldots$

Axioms:

1. Extensionality
   $\forall x \forall y \, (\forall z \, (z \in x \Leftrightarrow z \in y) \;\Leftrightarrow\; x = y)$
2. ...

Infinitely many axioms, recursive system.

Set Theory

In this talk, "Set Theory" means ZFC.
Objects are sets. Language is $\langle =, \in \rangle$.
Other relations and operations are derived; $\subset, \cap, \cup, \mathcal{P}(\cdot), \emptyset, \ldots$

Axioms:

1. Extensionality
   $\forall x \forall y \, (\forall z \, (z \in x \Leftrightarrow z \in y) \; \Leftrightarrow \; x = y)$
2. ...

Infinitely many axioms, recursive system.

Things provable in ZFC:

- If $2^{\aleph_n} < \aleph_\omega$ for all $n \in \omega$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$ (Shelah 94),
- There exists an L-space (Moore 05),
- $\mathfrak{t} = \mathfrak{p}$ (Malliaris–Shelah 13),
- ...

Set Theory

In this talk, “Set Theory” means ZFC.
Objects are sets. Language is $\langle =, \in \rangle$.
Other relations and operations are derived; $\subset, \cap, \cup, \mathcal{P}(\cdot), \emptyset, \ldots$

Axioms:

1. Extensionality
   $\forall x \forall y \, (\forall z \, (z \in x \Leftrightarrow z \in y) \;\Leftrightarrow\; x = y)$
2. …

Infinitely many axioms, recursive system.

Things provable in ZFC:

- If $2^{\aleph_n} < \aleph_\omega$ for all $n \in \omega$, then $2^{\aleph_\omega} < \aleph_{\omega_4}$ (Shelah 94),
- There exists an L-space (Moore 05),
- $\mathfrak{t} = \mathfrak{p}$ (Malliaris–Shelah 13),
- …

Gödel’s incompleteness implies there are undecidable sentences.

Continuum Hypothesis – CH

$$(|\mathbb{R}| =)\ \ |2^{\omega}| = \aleph_1$$

Continuum Hypothesis – CH

$$(|\mathbb{R}| =)\ \ |2^\omega| = \aleph_1$$

Defined by Cantor in 1878.
Cantor believed it is true.
One of Hilbert's 23 problems (1900).
Gödel (1940) shows it cannot be disproved (if ZFC consistent).
Cohen (1963) shows it is independent of ZFC, invented *forcing*.

Independence

How to show given sentence φ is independent of ZFC?

Independence

How to show given sentence φ is independent of ZFC?

Assume ZFC is consistent, demonstrate that
ZFC $+ \varphi$ is also consistent.

Given a model (universe of sets) V of ZFC,
modify V so that it satisfies ZFC $+ \varphi$.

Independence

How to show given sentence φ is independent of ZFC?

Assume ZFC is consistent, demonstrate that
ZFC $+ \varphi$ is also consistent.

Given a model (universe of sets) V of ZFC,
modify V so that it satisfies ZFC $+ \varphi$.

- Idea 1: Remove some sets from V.
  Inner models, e.g. Gödel's constructible universe L.

Independence

How to show given sentence φ is independent of ZFC?

Assume ZFC is consistent, demonstrate that
ZFC + φ is also consistent.

Given a model (universe of sets) V of ZFC,
modify V so that it satisfies ZFC + φ.

- Idea 1: Remove some sets from V.
  Inner models, e.g. Gödel's constructible universe L.
- Idea 2: Add some new sets.
  The method of *forcing*.

Forcing

Given a universe of sets V, choose a set $P \in V$, pick a 'suitable' new $G \subset P$.

Forcing

Given a universe of sets V, choose a set $P \in V$,
pick a 'suitable' new $G \subset P$.
Try to build $V[G]$, an extension of V containing G, satisfying ZFC.
$V[G]$ is the collection of all sets definable from G over V.

Forcing

Given a universe of sets V, choose a set $P \in V$,
pick a 'suitable' new $G \subset P$.
Try to build $V[G]$, an extension of V containing G, satisfying ZFC.
$V[G]$ is the collection of all sets definable from G over V.

Issues:

- How to choose G so that $V[G]$ satisfies ZFC?
- Control that a given sentence φ holds in $V[G]$.

Forcing

Given a universe of sets V, choose a set $P \in V$,
pick a 'suitable' new $G \subset P$.
Try to build $V[G]$, an extension of V containing G, satisfying ZFC.
$V[G]$ is the collection of all sets definable from G over V.

Issues:

- How to choose G so that $V[G]$ satisfies ZFC?
- Control that a given sentence φ holds in $V[G]$.

Theorem (Cohen)

If P is a poset and G is a V-generic filter, then $V[G] \models \mathrm{ZFC}$.

Forcing

Given a universe of sets V, choose a set $P \in V$,
pick a 'suitable' new $G \subset P$.
Try to build $V[G]$, an extension of V containing G, satisfying ZFC.
$V[G]$ is the collection of all sets definable from G over V.

Issues:

- How to choose G so that $V[G]$ satisfies ZFC?
- Control that a given sentence φ holds in $V[G]$.

Theorem (Cohen)

If P is a poset and G is a V-generic filter, then $V[G] \models \mathrm{ZFC}$.

Theorem (Balcar–Vopěnka)

If $V[G] \models \mathrm{ZFC}$, then (essentially) P is a poset and G a V-generic filter.

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \ \wedge \ b \leq c \ \Rightarrow a \leq c \quad$ and $\quad a \leq a$.

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \ \wedge \ b \leq c \ \Rightarrow a \leq c \quad$ and $\quad a \leq a$.
- $p, q \in P$ are compatible ($p \parallel q$) if there exists $r \in P$ such that $r \leq p, q$.

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \ \wedge \ b \leq c \ \Rightarrow a \leq c$ and $a \leq a$.
- $p, q \in P$ are compatible ($p \parallel q$) if there exists $r \in P$ such that $r \leq p, q$.
- If p, q not compatible, then p, q orthogonal ($p \perp q$).

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \ \wedge \ b \leq c \ \Rightarrow a \leq c \quad$ and $\quad a \leq a$.
- $p, q \in P$ are compatible ($p \parallel q$) if there exists $r \in P$ such that $r \leq p, q$.
- If p, q not compatible, then p, q orthogonal ($p \perp q$).
- We will assume that posets are nice; i.e.
  - antisymmetric
    $a \leq b \leq a \ \Rightarrow a = b$
  - separative
    $a \not\leq b \Rightarrow \exists c \leq a, c \perp b$
  - contains largest element $\mathbf{1}$

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \ \wedge \ b \leq c \ \Rightarrow a \leq c \quad$ and $\quad a \leq a$.
- $p, q \in P$ are compatible ($p \parallel q$) if there exists $r \in P$ such that $r \leq p, q$.
- If p, q not compatible, then p, q orthogonal ($p \perp q$).
- $a \in P$ is an atom if a is minimal.

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \wedge b \leq c \Rightarrow a \leq c$ and $a \leq a$.
- $p, q \in P$ are compatible ($p \parallel q$) if there exists $r \in P$ such that $r \leq p, q$.
- If p, q not compatible, then p, q orthogonal ($p \perp q$).
- $a \in P$ is an atom if a is minimal.
- $F \subset P$ is upwards closed if $p \in F \wedge p \leq q$ implies $q \in F$.

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \ \wedge \ b \leq c \ \Rightarrow a \leq c \quad$ and $\quad a \leq a$.
- $p, q \in P$ are compatible ($p \parallel q$) if there exists $r \in P$ such that $r \leq p, q$.
- If p, q not compatible, then p, q orthogonal ($p \perp q$).
- $a \in P$ is an atom if a is minimal.
- $F \subset P$ is upwards closed if $p \in F \ \wedge \ p \leq q$ implies $q \in F$.
- $D \subset P$ is downwards closed if … (also called open)

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \wedge b \leq c \Rightarrow a \leq c$ and $a \leq a$.
- $p, q \in P$ are compatible ($p \parallel q$) if there exists $r \in P$ such that $r \leq p, q$.
- If p, q not compatible, then p, q orthogonal ($p \perp q$).
- $a \in P$ is an atom if a is minimal.
- $F \subset P$ is upwards closed if $p \in F \wedge p \leq q$ implies $q \in F$.
- $D \subset P$ is downwards closed if ... (also called open)
- $D \subset P$ is dense if $\forall p \in P \, \exists q \leq p, q \in D$.

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \ \wedge \ b \leq c \ \Rightarrow a \leq c \quad$ and $\quad a \leq a$.
- $p, q \in P$ are compatible ($p \parallel q$) if there exists $r \in P$ such that $r \leq p, q$.
- If p, q not compatible, then p, q orthogonal ($p \perp q$).
- $a \in P$ is an atom if a is minimal.
- $F \subset P$ is upwards closed if $p \in F \ \wedge \ p \leq q$ implies $q \in F$.
- $D \subset P$ is downwards closed if … (also called open)
- $D \subset P$ is dense if $\forall p \in P \, \exists q \leq p, q \in D$.
- $F \subset P$ is a filter if F is upwards closed and $\forall p, q \in F \, \exists r \in F$ such that $r \leq p, q$.

Posets and Filters

- $(P, \leq)$ is a poset if $\leq$ is a transitive, reflexive relation on P.
  $a \leq b \ \wedge \ b \leq c \ \Rightarrow a \leq c \quad$ and $\quad a \leq a$.
- $p, q \in P$ are compatible ($p \parallel q$) if there exists $r \in P$ such that $r \leq p, q$.
- If p, q not compatible, then p, q orthogonal ($p \perp q$).
- $a \in P$ is an atom if a is minimal.
- $F \subset P$ is upwards closed if $p \in F \ \wedge \ p \leq q$ implies $q \in F$.
- $D \subset P$ is downwards closed if … (also called open)
- $D \subset P$ is dense if $\forall p \in P \, \exists q \leq p, q \in D$.
- $F \subset P$ is a filter if F is upwards closed and $\forall p, q \in F \, \exists r \in F$ such that $r \leq p, q$.
- $A \subset P$ is an antichain if $a, b \in A, a \neq b$ implies $a \perp b$.

Generic Filters

Suppose $(P, \leq) \in V$ is a poset.

A filter $G \subset P$ is V-generic if $G \cap D \neq \emptyset$ for each dense $D \subset P$, $D \in V$.

Generic Filters

Suppose $(P, \leq) \in V$ is a poset.
A filter $G \subset P$ is V-generic if $G \cap D \neq \emptyset$ for each dense $D \subset P$, $D \in V$.

If $a \in P$ is an atom, then the associated principal filter $F_a = \{ p \in P : p \geq a \}$ is V-generic.

Generic Filters

Suppose $(P, \leq) \in V$ is a poset.
A filter $G \subset P$ is V-generic if $G \cap D \neq \emptyset$ for each dense $D \subset P$, $D \in V$.

If $a \in P$ is an atom, then the associated principal filter $F_a = \{ p \in P : p \geq a \}$ is V-generic.

Claim

If P atomless, then there is no V-generic filter $F \in V$.

Generic Filters

Suppose $(P, \leq) \in V$ is a poset.
A filter $G \subset P$ is V-generic if $G \cap D \neq \emptyset$ for each dense $D \subset P$, $D \in V$.

If $a \in P$ is an atom, then the associated principal filter $F_a = \{ p \in P : p \geq a \}$ is V-generic.

Claim

If P atomless, then there is no V-generic filter $F \in V$.

Fact

We can assume that for every poset $P \in V$ and $p \in P$ there exists (not in V) a V-generic filter G such that $p \in G$.

Generic Filters

Suppose $(P, \leq) \in V$ is a poset.
A filter $G \subset P$ is V-generic if $G \cap D \neq \emptyset$ for each dense $D \subset P$, $D \in V$.

If $a \in P$ is an atom, then the associated principal filter $F_a = \{ p \in P : p \geq a \}$ is V-generic.

Claim

If P atomless, then there is no V-generic filter $F \in V$.

Fact

We can assume that for every poset $P \in V$ and $p \in P$ there exists (not in V) a V-generic filter G such that $p \in G$.

Theorem (Cohen)

If P is a poset and G is a V-generic filter, then $V[G] \models \mathrm{ZFC}$.

Choosing poset for φ

Given sentence φ, we want to find G such that $V[G] \models \varphi$.

Choosing poset for φ

Given sentence φ, we want to find G such that $V[G] \models \varphi$.

Cases:

1. φ is $\exists x \, \forall y \ldots$
2. φ is $\forall x \, \exists y \ldots$

Choosing poset for φ

Given sentence φ, we want to find G such that $V[G] \models \varphi$.

Cases:

1. φ is $\exists x\, \forall y \ldots$
2. φ is $\forall x\, \exists y \ldots$

Focus on case 1.

- CH $\Leftrightarrow$ there exists a surjective function $f : \omega_1 \to \mathbb{R}$.
- $\neg$CH $\Leftrightarrow$ there exists $C \subset \mathbb{R}\ (= 2^\omega),\ |C| = \aleph_2$.

Choosing poset for φ

Given sentence φ, we want to find G such that $V[G] \models \varphi$.

Cases:

1. φ is $\exists x \, \forall y \ldots$
2. φ is $\forall x \, \exists y \ldots$

Focus on case 1.

- CH $\Leftrightarrow$ there exists a surjective function $f : \omega_1 \to \mathbb{R}$.
- $\neg$CH $\Leftrightarrow$ there exists $C \subset \mathbb{R} \; (= 2^\omega)$, $|C| = \aleph_2$.

We want G to be the object witnessing $\exists x \ldots$ in φ.

Choosing poset for φ

Given sentence φ, we want to find G such that $V[G] \models \varphi$.

Cases:

1. φ is $\exists x \forall y \ldots$
2. φ is $\forall x \exists y \ldots$

Focus on case 1.

- CH $\Leftrightarrow$ there exists a surjective function $f : \omega_1 \to \mathbb{R}$.
- $\neg$CH $\Leftrightarrow$ there exists $C \subset \mathbb{R}\ (= 2^\omega)$, $|C| = \aleph_2$.

We want G to be the object witnessing $\exists x \ldots$ in φ.
Choose P to be a poset of approximations of the desired G.

CH

There exists a surjective function $f\colon \omega_1 \to \mathbb{R}$.

CH

There exists a surjective function $f\colon \omega_1 \to \mathbb{R}$.

$$P = \{\, f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \le \aleph_0 \,\}$$

$$f \le g \quad \text{iff} \quad g \subseteq f$$

CH

There exists a surjective function $f\colon \omega_1 \to \mathbb{R}$.

$$P = \{\, f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \,\}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_1 \to \mathbb{R}$.

CH

There exists a surjective function $f\colon \omega_1 \to \mathbb{R}$.

$$P = \{\, f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \,\}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_1 \to \mathbb{R}$.

For $r \in \mathbb{R}$ let $D_r = \{\, f \in P : r \in \operatorname{Rng}(f)\,\}$. D_r is dense.

CH

There exists a surjective function $f \colon \omega_1 \to \mathbb{R}$.

$$P = \{\, f \colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \,\}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_1 \to \mathbb{R}$.

For $r \in \mathbb{R}$ let $D_r = \{\, f \in P : r \in \operatorname{Rng}(f) \,\}$. D_r is dense.

If $F \cap D_r \neq \emptyset$, then $r \in \operatorname{Rng}(\bigcup F)$.

CH

There exists a surjective function $f\colon \omega_1 \to \mathbb{R}$.

$$P = \{\, f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \,\}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_1 \to \mathbb{R}$.

For $r \in \mathbb{R}$ let $D_r = \{\, f \in P : r \in \operatorname{Rng}(f) \,\}$. D_r is dense.

If $F \cap D_r \neq \emptyset$, then $r \in \operatorname{Rng}(\bigcup F)$.

If G is a V-generic filter, then $r \in \operatorname{Rng}(\bigcup G)$ for each $r \in \mathbb{R} \cap V$.

CH

There exists a surjective function $f\colon \omega_1 \to \mathbb{R}$.

$$P = \{\, f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \,\}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_1 \to \mathbb{R}$.

For $r \in \mathbb{R}$ let $D_r = \{\, f \in P : r \in \operatorname{Rng}(f) \,\}$. D_r is dense.

If $F \cap D_r \neq \emptyset$, then $r \in \operatorname{Rng}(\bigcup F)$.

If G is a V-generic filter, then $r \in \operatorname{Rng}(\bigcup G)$ for each $r \in \mathbb{R} \cap V$.

Put $\quad f = \bigcup G$.

$\neg$ CH

There exists $C \subset \mathbb{R}\ (= 2^{\omega}),\ |C| = \aleph_2$.

$\neg$ CH

There exists $C \subset \mathbb{R}\ (= 2^\omega)$, $|C| = \aleph_2$.

$$P = \{ f \colon A \to 2 : A \subset \omega_2 \times \omega, |A| < \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

$\neg$ CH

There exists $C \subset \mathbb{R}\ (= 2^\omega)$, $|C| = \aleph_2$.

$$P = \{\, f\colon A \to 2 : A \subset \omega_2 \times \omega,\ |A| < \aleph_0 \,\}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_2 \times \omega \to 2$.

$\neg$ CH

There exists $C \subset \mathbb{R}\ (= 2^\omega)$, $|C| = \aleph_2$.

$$P = \{ f \colon A \to 2 : A \subset \omega_2 \times \omega, |A| < \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_2 \times \omega \to 2$.
For $\alpha \in \omega_2, n \in \omega$ the set $D_{(\alpha,n)} = \{ f \in P : (\alpha, n) \in \mathrm{Dom}(f) \}$ is dense.

$\neg$ CH

There exists $C \subset \mathbb{R}\ (= 2^\omega)$, $|C| = \aleph_2$.

$$P = \{ f\colon A \to 2 : A \subset \omega_2 \times \omega, |A| < \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_2 \times \omega \to 2$.
For $\alpha \in \omega_2, n \in \omega$ the set $D_{(\alpha,n)} = \{ f \in P : (\alpha, n) \in \mathrm{Dom}(f) \}$ is dense.
If $F \cap D_{(\alpha,n)} \neq \emptyset$, then $(\alpha, n) \in \mathrm{Dom}(\bigcup F)$.

$\neg$ CH

There exists $C \subset \mathbb{R}\ (= 2^\omega)$, $|C| = \aleph_2$.

$$P = \{\, f \colon A \to 2 : A \subset \omega_2 \times \omega, |A| < \aleph_0 \,\}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_2 \times \omega \to 2$.
For $\alpha \in \omega_2, n \in \omega$ the set $D_{(\alpha,n)} = \{\, f \in P : (\alpha, n) \in \mathrm{Dom}(f) \,\}$ is dense.
If $F \cap D_{(\alpha,n)} \neq \emptyset$, then $(\alpha, n) \in \mathrm{Dom}(\bigcup F)$.
If G is a V-generic filter, then $\bigcup G \colon \omega_2 \times \omega \to 2$ is a total function.

$\neg$ CH

There exists $C \subset \mathbb{R}\ (=2^\omega)$, $|C| = \aleph_2$.

$$P = \{\, f\colon A \to 2 : A \subset \omega_2 \times \omega,\ |A| < \aleph_0 \,\}$$

$$f \le g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_2 \times \omega \to 2$.
For $\alpha \in \omega_2$, $n \in \omega$ the set $D_{(\alpha,n)} = \{\, f \in P : (\alpha, n) \in \mathrm{Dom}(f) \,\}$
is dense.
If $F \cap D_{(\alpha,n)} \ne \emptyset$, then $(\alpha, n) \in \mathrm{Dom}(\bigcup F)$.
If G is a V-generic filter, then $\bigcup G\colon \omega_2 \times \omega \to 2$ is a total function.
For $\alpha \in \omega_2$ define $c_\alpha \in 2^\omega$ by $c_\alpha(n) = \big(\bigcup G\big)(\alpha, n)$.

$\neg$ CH

$$\text{There exists } C \subset \mathbb{R}\ (=2^\omega),\ |C| = \aleph_2.$$

$$P = \{\, f\colon A \to 2 : A \subset \omega_2 \times \omega,\ |A| < \aleph_0 \,\}$$

$$f \le g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_2 \times \omega \to 2$.
For $\alpha \in \omega_2, n \in \omega$ the set $D_{(\alpha,n)} = \{\, f \in P : (\alpha, n) \in \mathrm{Dom}(f) \,\}$ is dense.
If $F \cap D_{(\alpha,n)} \neq \emptyset$, then $(\alpha, n) \in \mathrm{Dom}(\bigcup F)$.
If G is a V-generic filter, then $\bigcup G\colon \omega_2 \times \omega \to 2$ is a total function.
For $\alpha \in \omega_2$ define $c_\alpha \in 2^\omega$ by $c_\alpha(n) = \left(\bigcup G\right)(\alpha, n)$.
For $\alpha \neq \beta \in \omega_2$ let $E_{(\alpha,\beta)} = \{\, f \in P : \exists n \in \omega\ \ f(\alpha, n) \neq f(\beta, n) \,\}$.
$E_{(\alpha,\beta)}$ is dense, i.e. $c_\alpha \neq c_\beta$.

$\neg$ CH

$$\text{There exists } C \subset \mathbb{R}\ (=2^\omega),\ |C| = \aleph_2.$$

$$P = \{\, f\colon A \to 2 : A \subset \omega_2 \times \omega,\ |A| < \aleph_0 \,\}$$

$$f \le g \quad \text{iff} \quad g \subseteq f$$

If $F \subset P$ is a filter, then $\bigcup F$ is a (partial) function $\omega_2 \times \omega \to 2$.
For $\alpha \in \omega_2, n \in \omega$ the set $D_{(\alpha,n)} = \{\, f \in P : (\alpha, n) \in \mathrm{Dom}(f) \,\}$ is dense.
If $F \cap D_{(\alpha,n)} \neq \emptyset$, then $(\alpha, n) \in \mathrm{Dom}(\bigcup F)$.
If G is a V-generic filter, then $\bigcup G\colon \omega_2 \times \omega \to 2$ is a total function.
For $\alpha \in \omega_2$ define $c_\alpha \in 2^\omega$ by $c_\alpha(n) = \left(\bigcup G\right)(\alpha, n)$.
For $\alpha \neq \beta \in \omega_2$ let $E_{(\alpha,\beta)} = \{\, f \in P : \exists n \in \omega\ \ f(\alpha, n) \neq f(\beta, n) \,\}$.
$E_{(\alpha,\beta)}$ is dense, i.e. $c_\alpha \neq c_\beta$.
Put $C = \{\, c_\alpha : \alpha \in \omega_2 \,\}$, $V[G] \models |C| = |{\omega_2}^V|$.

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Fact

For every $x \in V[G]$ *there is a name* $\dot{x} \in V$.

The name $\dot{x}$ is a 'recipe' how to build x from G.

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Fact

For every $x \in V[G]$ there is a name $\dot{x} \in V$.

The name $\dot{x}$ is a 'recipe' how to build x from G.

Theorem

For every formula $\psi(\dot{x}, \dot{y}, \dots)$ there is a set $\mathcal{D} \subseteq P$, $\mathcal{D} \in V$ such that for every G

$$V[G] \models \psi(\dot{x}, \dot{y}, \dots) \quad \textit{iff} \quad \mathcal{D} \cap G \neq \emptyset.$$

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Fact

For every $x \in V[G]$ there is a name $\dot{x} \in V$.

The name $\dot{x}$ is a 'recipe' how to build x from G.

Theorem

For every formula $\psi(\dot{x}, \dot{y}, \dots)$ there is a set $\mathcal{D} \subseteq P$, $\mathcal{D} \in V$ such that for every G

$$V[G] \models \psi(\dot{x}, \dot{y}, \dots) \quad \textit{iff} \quad \mathcal{D} \cap G \neq \emptyset.$$

$$\mathcal{D}_\psi = \{\, p \in P : p \in G \Rightarrow V[G] \models \psi \,\} \in V$$

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Fact

For every $x \in V[G]$ there is a name $\dot{x} \in V$.

The name $\dot{x}$ is a 'recipe' how to build x from G.

Theorem

For every formula $\psi(\dot{x}, \dot{y}, \dots)$ there is a set $\mathcal{D} \subseteq P$, $\mathcal{D} \in V$ such that for every G

$$V[G] \models \psi(\dot{x}, \dot{y}, \dots) \quad \textit{iff} \quad \mathcal{D} \cap G \neq \emptyset.$$

$$\mathcal{D}_\psi = \{\, p \in P : p \in G \Rightarrow V[G] \models \psi \,\} \in V$$

$$p \in \mathcal{D}_\psi \quad \text{is denoted} \quad p \Vdash \psi$$

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Fact

For every $x \in V[G]$ there is a name $\dot{x} \in V$.

The name $\dot{x}$ is a 'recipe' how to build x from G.

Theorem

For every formula $\psi(\dot{x}, \dot{y}, \dots)$ there is a set $\mathcal{D} \subseteq P$, $\mathcal{D} \in V$ such that for every G

$$V[G] \models \psi(\dot{x}, \dot{y}, \dots) \quad \textit{iff} \quad \mathcal{D} \cap G \neq \emptyset.$$

$$\mathcal{D}_\psi = \{\, p \in P : p \in G \Rightarrow V[G] \models \psi \,\} \in V$$

$$p \in \mathcal{D}_\psi \quad \text{is denoted} \quad p \Vdash \psi$$

- $\mathcal{D}_\psi$ is downwards closed.

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Fact

For every $x \in V[G]$ there is a name $\dot{x} \in V$.

The name $\dot{x}$ is a 'recipe' how to build x from G.

Theorem

For every formula $\psi(\dot{x}, \dot{y}, \dots)$ there is a set $\mathcal{D} \subseteq P, \mathcal{D} \in V$ such that for every G

$$V[G] \models \psi(\dot{x}, \dot{y}, \dots) \quad \textit{iff} \quad \mathcal{D} \cap G \neq \emptyset.$$

$$\mathcal{D}_\psi = \{\, p \in P : p \in G \Rightarrow V[G] \models \psi \,\} \in V$$

$$p \in \mathcal{D}_\psi \quad \text{is denoted} \quad p \Vdash \psi$$

- $\mathcal{D}_\psi$ is downwards closed.
- $(p \in \mathcal{D}_\psi \ \wedge \ q \in \mathcal{D}_{\neg\psi}) \ \Rightarrow \ p \perp q$ (denote $\mathcal{D}_\psi \perp \mathcal{D}_{\neg\psi}$).

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Fact

For every $x \in V[G]$ there is a name $\dot{x} \in V$.

The name $\dot{x}$ is a 'recipe' how to build x from G.

Theorem

For every formula $\psi(\dot{x}, \dot{y}, \dots)$ there is a set $\mathcal{D} \subseteq P, \mathcal{D} \in V$ such that for every G

$$V[G] \models \psi(\dot{x}, \dot{y}, \dots) \quad \textit{iff} \quad \mathcal{D} \cap G \neq \emptyset.$$

$$\mathcal{D}_\psi = \{\, p \in P : p \in G \Rightarrow V[G] \models \psi \,\} \in V$$

$$p \in \mathcal{D}_\psi \quad \text{is denoted} \quad p \Vdash \psi$$

- $\mathcal{D}_\psi$ is downwards closed.
- $(p \in \mathcal{D}_\psi \;\wedge\; q \in \mathcal{D}_{\neg\psi}) \;\Rightarrow\; p \perp q$ (denote $\mathcal{D}_\psi \perp \mathcal{D}_{\neg\psi}$).
- If $\mathcal{D}_\psi$ is dense, then $V[G] \models \psi$.

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Fact

For every $x \in V[G]$ there is a name $\dot{x} \in V$.

The name $\dot{x}$ is a 'recipe' how to build x from G.

Theorem

For every formula $\psi(\dot{x}, \dot{y}, \dots)$ there is a set $\mathcal{D} \subseteq P, \mathcal{D} \in V$ such that for every G

$$V[G] \models \psi(\dot{x}, \dot{y}, \dots) \quad \textit{iff} \quad \mathcal{D} \cap G \neq \emptyset.$$

$$\mathcal{D}_\psi = \{\, p \in P : p \in G \Rightarrow V[G] \models \psi \,\} \in V$$

$$p \in \mathcal{D}_\psi \quad \text{is denoted} \quad p \Vdash \psi$$

- $\mathcal{D}_\psi$ is downwards closed.
- $(p \in \mathcal{D}_\psi \;\wedge\; q \in \mathcal{D}_{\neg\psi}) \;\Rightarrow\; p \perp q$ (denote $\mathcal{D}_\psi \perp \mathcal{D}_{\neg\psi}$).
- If $\mathcal{D}_\psi$ is dense, then $V[G] \models \psi$.
- $(\mathcal{D}_\psi \cup \mathcal{D}_{\neg\psi})$ is dense.

Generic extensions (the forcing relation)

How to control what does hold in $V[G]$?

Fact

For every $x \in V[G]$ there is a name $\dot{x} \in V$.

The name $\dot{x}$ is a 'recipe' how to build x from G.

Theorem

For every formula $\psi(\dot{x}, \dot{y}, \dots)$ there is a set $\mathcal{D} \subseteq P$, $\mathcal{D} \in V$ such that for every G

$$V[G] \models \psi(\dot{x}, \dot{y}, \dots) \quad \textit{iff} \quad \mathcal{D} \cap G \neq \emptyset.$$

$$\mathcal{D}_\psi = \{\, p \in P : p \in G \Rightarrow V[G] \models \psi \,\} \in V$$

$$p \in \mathcal{D}_\psi \quad \text{is denoted} \quad p \Vdash \psi$$

- $\mathcal{D}_\psi$ is downwards closed.
- $(p \in \mathcal{D}_\psi \;\wedge\; q \in \mathcal{D}_{\neg\psi}) \;\Rightarrow\; p \perp q$ (denote $\mathcal{D}_\psi \perp \mathcal{D}_{\neg\psi}$).
- If $\mathcal{D}_\psi$ is dense, then $V[G] \models \psi$.
- $(\mathcal{D}_\psi \cup \mathcal{D}_{\neg\psi})$ is dense.
- If $\varphi \Rightarrow \psi$, then $\mathcal{D}_\varphi \subseteq \mathcal{D}_\psi$.

CH continued

$$P = \{ f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

CH continued

$$P = \{ f \colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

A poset is σ-closed if for each sequence $p_0 \geq p_1 \geq p_2 \geq \ldots$ exists p_ω such that $p_n \geq p_\omega$ for all $n \in \omega$.

CH continued

$$P = \{ f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

A poset is σ-closed if for each sequence $p_0 \geq p_1 \geq p_2 \geq \ldots$ exists p_ω such that $p_n \geq p_\omega$ for all $n \in \omega$.

Claim

P is σ-closed.

CH continued

$$P = \{ f : A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

A poset is σ-closed if for each sequence $p_0 \geq p_1 \geq p_2 \geq \ldots$ exists p_ω such that $p_n \geq p_\omega$ for all $n \in \omega$.

Claim

P is σ-closed.

Theorem

If P is σ-closed, then $V[G] \models \mathbb{R} = (\mathbb{R} \cap V)$.

CH continued

$$P = \{ f \colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

A poset is σ-closed if for each sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ exists p_ω such that $p_n \geq p_\omega$ for all $n \in \omega$.

Claim

P is σ-closed.

Theorem

If P is σ-closed, then $V[G] \models \mathbb{R} = (\mathbb{R} \cap V)$.

Proof.

Take $c \in V[G] \cap \mathbb{R}$, investigate $\dot{c} \dots$

CH continued

$$P = \{\, f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \,\}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

A poset is σ-closed if for each sequence $p_0 \geq p_1 \geq p_2 \geq \dots$ exists p_ω such that $p_n \geq p_\omega$ for all $n \in \omega$.

Claim

P is σ-closed.

Theorem

If P is σ-closed, then $V[G] \models \;\; \mathbb{R} = (\mathbb{R} \cap V)$.

Proof.

Take $c \in V[G] \cap \mathbb{R}$, investigate $\dot{c} \dots$

Show that $D_c = \bigcup \{\, \mathcal{D}_{\dot{c}=x} : x \in V \cap \mathbb{R} \,\}$ is dense.

CH continued

$$P = \{ f\colon A \to \mathbb{R} : A \subset \omega_1, |A| \leq \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

A poset is σ-closed if for each sequence $p_0 \geq p_1 \geq p_2 \geq \ldots$ exists p_ω such that $p_n \geq p_\omega$ for all $n \in \omega$.

Claim

P is σ-closed.

Theorem

If P is σ-closed, then $V[G] \models \mathbb{R} = (\mathbb{R} \cap V)$.

Proof.

Take $c \in V[G] \cap \mathbb{R}$, investigate $\dot{c} \ldots$
Show that $D_c = \bigcup \{ \mathcal{D}_{\dot{c}=x} : x \in V \cap \mathbb{R} \}$ is dense.
$G \cap D_c \neq \emptyset$ implies $G \cap \mathcal{D}_{\dot{c}=x} \neq \emptyset$ for some $x \in V$,
and $V[G] \models c = x$.

¬ CH continued

$$P = \{ f : A \to 2 : A \subset \omega_2 \times \omega, |A| < \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

$\neg$ CH continued

$$P = \{ f\colon A \to 2 : A \subset \omega_2 \times \omega, |A| < \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

A poset P is c.c.c. if $A \subset P$, A antichain implies $|A| < \aleph_1$.

$\neg$ CH continued

$$P = \{ f : A \to 2 : A \subset \omega_2 \times \omega, |A| < \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

A poset P is c.c.c. if $A \subset P$, A antichain implies $|A| < \aleph_1$.

Fact

P is c.c.c.

$\neg$ CH continued

$$P = \{ f : A \to 2 : A \subset \omega_2 \times \omega, |A| < \aleph_0 \}$$

$$f \leq g \quad \text{iff} \quad g \subseteq f$$

A poset P is c.c.c. if $A \subset P$, A antichain implies $|A| < \aleph_1$.

Fact

P is c.c.c.

Theorem

If P is c.c.c. and $V \models |\kappa| < |\lambda|$, then $V[G] \models |\kappa| < |\lambda|$.

Corollary

If P is c.c.c., then $V[G] \models |{\omega_2}^V| = \aleph_2$.

c.c.c. posets preserve cardinals

Theorem

If P is c.c.c. and $V \models |\kappa| < |\lambda|$, then $V[G] \models |\kappa| < |\lambda|$.

Proof.

WLOG show that there is no surjection $b \colon \omega \to {\omega_1}^V$ in $V[G]$, other cases are analogous.

c.c.c. posets preserve cardinals

Theorem

If P is c.c.c. and $V \models |\kappa| < |\lambda|$, then $V[G] \models |\kappa| < |\lambda|$.

Proof.

WLOG show that there is no surjection $b\colon \omega \to {\omega_1}^V$ in $V[G]$, other cases are analogous.

Assume $b\colon \omega \to {\omega_1}^V$ in $V[G]$, investigate $\dot{b}$.

$\mathcal{D}_{\dot{b}(n)=\alpha} \perp \mathcal{D}_{\dot{b}(n)=\beta}$ for each $n \in \omega$ and $\alpha \neq \beta \in \omega_1$.

c.c.c. posets preserve cardinals

Theorem

If P is c.c.c. and $V \models |\kappa| < |\lambda|$, then $V[G] \models |\kappa| < |\lambda|$.

Proof.

WLOG show that there is no surjection $b\colon \omega \to {\omega_1}^V$ in $V[G]$, other cases are analogous.

Assume $b\colon \omega \to {\omega_1}^V$ in $V[G]$, investigate $\dot{b}$.

$\mathcal{D}_{\dot{b}(n)=\alpha} \perp \mathcal{D}_{\dot{b}(n)=\beta}$ for each $n \in \omega$ and $\alpha \neq \beta \in \omega_1$.

Thus $R_n = \{\, \alpha : \mathcal{D}_{\dot{b}(n)=\alpha} \neq \emptyset \,\}$ is countable.

c.c.c. posets preserve cardinals

Theorem

If P is c.c.c. and $V \models |\kappa| < |\lambda|$, then $V[G] \models |\kappa| < |\lambda|$.

Proof.

WLOG show that there is no surjection $b\colon \omega \to {\omega_1}^V$ in $V[G]$, other cases are analogous.

Assume $b\colon \omega \to {\omega_1}^V$ in $V[G]$, investigate $\dot{b}$.

$\mathcal{D}_{\dot{b}(n)=\alpha} \perp \mathcal{D}_{\dot{b}(n)=\beta}$ for each $n \in \omega$ and $\alpha \neq \beta \in \omega_1$.

Thus $R_n = \{\, \alpha : \mathcal{D}_{\dot{b}(n)=\alpha} \neq \emptyset \,\}$ is countable.

If $\mathcal{D}_{\dot{b}(n)=\alpha} = \emptyset$, then $V[G] \models b(n) \neq \alpha$.
I.e. $b(n) \in R_n$, and $\operatorname{Rng}(b) \subset \bigcup\{\, R_n : n \in \omega \,\}$.
$\bigcup\{\, R_n : n \in \omega \,\}$ is in V and countable, and b is not a surjection.

Δ-system lemma

Lemma

Suppose $\gamma \in \mathrm{On}$, $\{ a_\alpha : \alpha \in \omega_1 \} \subset [\gamma]^{<\omega}$*. There exists* $I \in [\omega_1]^{\omega_1}$ *and* $\Delta \in [\gamma]^{<\omega}$ *such that* $a_\alpha \cap a_\beta = \Delta$ *for each* $\alpha \neq \beta \in I$*.*

Δ-system lemma

Lemma

Suppose $\gamma \in \mathrm{On}$, $\{ a_\alpha : \alpha \in \omega_1 \} \subset [\gamma]^{<\omega}$. *There exists* $I \in [\omega_1]^{\omega_1}$ *and* $\Delta \in [\gamma]^{<\omega}$ *such that* $a_\alpha \cap a_\beta = \Delta$ *for each* $\alpha \neq \beta \in I$.
Moreover if $\alpha < \beta$, $\chi \in a_\alpha \setminus \Delta$, $\xi \in a_\beta \setminus \Delta$, *then* $\chi < \xi$.