M-HARMONIC REPRODUCING KERNELS ON THE BALL

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Abstract. Using the machinery of unitary spherical harmonics due to Koornwinder, Folland and other authors, we obtain expansions for the Szegö and the weighted Bergman kernels of M-harmonic functions, i.e. functions annihilated by the invariant Laplacian on the unit ball of the complex n-space. This yields, among others, an explicit formula for the M-harmonic Szegö kernel in terms of multivariable as well as single-variable hypergeometric functions, and also shows that most likely there is no explicit ("closed") formula for the corresponding weighted Bergman kernels.

1. Introduction

Recall that a function on the unit ball $\mathbf{B}^n$ of $\mathbf{C}^n$, $n \geq 1$, is called *Moebius-harmonic* (or *invariantly harmonic*), or *M-harmonic* for short, if it is annihilated by the invariant Laplacian

$$\widetilde{\Delta} = 4(1-|z|^2) \sum_{j,k=1}^{n} (\delta_{jk} - z_j \overline{z}_k) \frac{\partial^2}{\partial z_j \partial \overline{z}_k}. \tag{1}$$

It is well known (see e.g. Rudin [Ru], Stoll [St], or Chapter 6 in Krantz [Kr1]) that $\widetilde{\Delta}$ commutes with biholomorphic self-maps (Moebius maps) of the ball:

$$\widetilde{\Delta}(f \circ \phi) = (\widetilde{\Delta} f) \circ \phi, \qquad \forall f \in C^2(\mathbf{B}^n), \phi \in \operatorname{Aut}(\mathbf{B}^n);$$

and, accordingly, that M-harmonic functions possess the *invariant mean-value property*: namely, if $\widetilde{\Delta} f = 0$ and $z \in \mathbf{B}^n$, then $f(z)$ equals the mean value, with respect to the $\operatorname{Aut}(\mathbf{B}^n)$-invariant measure $d\tau(z) = (1-|z|^2)^{-n-1}\, dz$, over any Moebius ball in $\mathbf{B}^n$ centered at z (and similarly for spheres in the place of balls). It follows by a standard argument that the point evaluations $f \mapsto f(z)$ at any $z \in \mathbf{B}^n$ are continuous linear functionals on the subspace

$$L^2_{\mathrm{Mh}}(\mathbf{B}^n) := \{ f \in L^2(\mathbf{B}^n) : f \text{ is } M\text{-harmonic} \}$$

of all M-harmonic functions in $L^2(\mathbf{B}^n)$ (the *M-harmonic Bergman space*), and therefore there exists a reproducing kernel for $L^2_{\mathrm{Mh}}(\mathbf{B}^n)$ (the *M-harmonic Bergman kernel*), namely a function $K(x,y)$ on $\mathbf{B}^n \times \mathbf{B}^n$, M-harmonic in both variables and such that

$$f(x) = \int_{\mathbf{B}^n} f(y) K(x,y)\, dy \qquad \forall x \in \mathbf{B}^n, \forall f \in L^2_{\mathrm{Mh}}(\mathbf{B}^n).$$

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More generally, for any $s > -1$, one can consider the *weighted M-harmonic Bergman space*

$$L^2_{\mathrm{Mh}}(\mathbf{B}^n, (1-|z|^2)^s\,dz) \tag{2}$$

(where dz stands, throughout, for the Lebesgue measure of the appropriate dimension) and its reproducing kernel $K_s(x,y)$ on $\mathbf{B}^n \times \mathbf{B}^n$ (the *weighted M-harmonic Bergman kernel*).

For the analogous weighted Bergman spaces of *holomorphic*, rather than M-harmonic, functions on $\mathbf{B}^n$, the reproducing kernels have been known explicitly for a long time: one has

$$K^{\mathrm{hol}}_s(x,y) = \frac{\Gamma(n+s+1)}{\Gamma(s+1)\pi^n}(1-\langle x,y\rangle)^{-n-s-1}. \tag{3}$$

Similarly, there are formulas expressing the *harmonic* weighted Bergman kernels $K^{\mathrm{harm}}_s(x,y)$, $s > -1$, on $\mathbf{B}^n$ in terms of Appell's hypergeometric function F_1 of two variables [Bk, Section 3]:

$$\begin{aligned} K^{\mathrm{harm}}_s(x,y) &= \frac{\Gamma(n+s+1)}{\Gamma(s+1)\pi^n} F_1\left(\begin{matrix} n+s+1; n-1, n-1 \\ n-1 \end{matrix}\middle| z, \overline{z}\right), \\ & \qquad\qquad z = \langle x,y\rangle + i\sqrt{|x|^2|y|^2 - |\langle x,y\rangle|^2}. \end{aligned} \tag{4}$$

For $n=1$, the unit ball $\mathbf{B}^n$ becomes just the unit disc $\mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$, and (1) reduces to the multiple $(1-|z|^2)^2\Delta$ of the ordinary Laplacian Δ; thus M-harmonic and harmonic functions coincide for $n=1$. Also, any harmonic function on $\mathbf{D}$ can be written as $f+\overline{g}$ with f,g holomorphic and $g(0)=0$. Since holomorphic and conjugate-holomorphic functions are orthogonal in any $L^2(\mathbf{D},(1-|z|^2)^s\,dz)$ except for the constants, it follows that for $n=1$

$$K_s = K^{\mathrm{harm}}_s = 2\operatorname{Re} K^{\mathrm{hol}}_s - \frac{\Gamma(n+s+1)}{\Gamma(s+1)\pi^n}.$$

However, no explicit formula seems to be available for the M-harmonic Bergman kernels on $\mathbf{B}^n$ for $n > 1$.

If we multiply the measures in (2) by the factor $2(s+1)$ and let $s \searrow -1$, it is easily shown that $2(s+1)(1-|z|^2)^s\,dz$ converges weakly to $d\sigma$, the (unnormalized) surface measure on the topological boundary $\partial\mathbf{B}^n$ of $\mathbf{B}^n$. As a limit of the weighted Bergman spaces (2) we thus obtain the *M-harmonic Hardy space* $H^2_{\mathrm{Mh}}(\mathbf{B}^n)$ of M-harmonic functions on $\mathbf{B}^n$, whose reproducing kernel $K_{\mathrm{Sz}}(x,y)$ — the *M-harmonic Szegö kernel* — is the function on $\mathbf{B}^n \times \mathbf{B}^n$, M-harmonic in both variables, which satisfies

$$f(x) = \int_{\partial\mathbf{B}^n} f(\zeta) K_{\mathrm{Sz}}(x,\zeta)\,d\sigma(\zeta), \qquad \forall f \in H^2_{\mathrm{Mh}}(\partial\mathbf{B}^n), \forall x \in \mathbf{B}^n, \tag{5}$$

where, abusing the notation slightly, we denote by the same letter f also the radial boundary values of f on $\partial\mathbf{B}^n$, and similarly for $K_{\mathrm{Sz}}(x,\cdot)$. In the holomorphic case, one again has the explicit formula

$$K^{\mathrm{hol}}_{\mathrm{Sz}}(x,y) = \frac{\Gamma(n)}{2\pi^n}(1-\langle x,y\rangle)^{-n} \tag{6}$$

for the ordinary Szegö kernel of $\mathbf{B}^n$, and similarly from (4) we get the *harmonic Szegö kernel*

$$K^{\text{harm}}_{\text{Sz}}(x,y)=\frac{\Gamma(n)}{2\pi^n}\frac{1-|x|^2|y|^2}{(1-2\operatorname{Re}\langle x,y\rangle+|x|^2|y|^2)^n} \tag{7}$$

for the harmonic case.

The harmonic and M-harmonic Szegö kernels are intimately connected with the associated *Poisson kernels.* Namely, recall that the ordinary Poisson kernel

$$P^{\text{harm}}(x,\zeta)=\frac{\Gamma(n)}{2\pi^n}\frac{1-|x|^2}{|x-\zeta|^{2n}}$$

on $\mathbf{B}^n$ reproduces the values of a harmonic function in the interior of $\mathbf{B}^n$ from its boundary values:

$$f(x)=\int_{\partial\mathbf{B}^n} f(\zeta)P^{\text{harm}}(x,\zeta)\,d\sigma(\zeta), \qquad \forall x\in\mathbf{B}^n,$$

for any function f harmonic on $\mathbf{B}^n$ and, say, continuous on the closure $\overline{\mathbf{B}^n}$. Comparing this with the (harmonic version of) (5), we thus see that $P^{\text{harm}}(x,\cdot)$ are just the boundary values of $K^{\text{harm}}_{\text{Sz}}(x,\cdot)$; that is, $K^{\text{harm}}_{\text{Sz}}(x,\cdot)$ is just the harmonic extension of $P^{\text{harm}}(x,\cdot)$ from $\partial\mathbf{B}^n$ into $\mathbf{B}^n$, or

$$K^{\text{harm}}_{\text{Sz}}(x,y)=\int_{\partial\mathbf{B}^n} P^{\text{harm}}(x,\zeta)P^{\text{harm}}(y,\zeta)\,d\sigma(\zeta).$$

In other words, $K^{\text{harm}}_{\text{Sz}}$ is just P^{harm} extended from $\mathbf{B}^n\times\partial\mathbf{B}^n$ to a function on $\mathbf{B}^n\times\mathbf{B}^n$ harmonic in both variables.

Exactly the same argument shows that also the M-harmonic Poisson kernel (called *Poisson-Szegö kernel* in [Kr1])

$$P(x,\zeta)=\frac{\Gamma(n)}{2\pi^n}\frac{(1-|x|^2)^n}{|1-\langle x,\zeta\rangle|^{2n}}$$

(cf. [St, Chapter 5]), which reproduces any function f M-harmonic on $\mathbf{B}^n$ and continuous on $\overline{\mathbf{B}^n}$ from its boundary values:

$$f(x)=\int_{\partial\mathbf{B}^n} f(\zeta)P(x,\zeta)\,d\sigma(\zeta), \qquad \forall x\in\mathbf{B}^n, \tag{8}$$

is just the boundary value of $K_{\text{Sz}}(x,y)$ as $y\to\zeta$; that is, $K_{\text{Sz}}(x,\cdot)$ is just the M-harmonic extension of $P(x,\cdot)$ from $\partial\mathbf{B}^n$ into $\mathbf{B}^n$, or

$$K_{\text{Sz}}(x,y)=\frac{\Gamma(n)^2}{4\pi^{2n}}\int_{\partial\mathbf{B}^n}\frac{(1-|x|^2)^n(1-|y|^2)^n}{|1-\langle x,\zeta\rangle|^{2n}|1-\langle y,\zeta\rangle|^{2n}}\,d\sigma(\zeta). \tag{9}$$

For $n=1$, this is easily evaluated to

$$K_{\text{Sz}}(x,y)=\frac{1}{2\pi}\frac{1-|x|^2|y|^2}{|1-\langle x,y\rangle|^2}, \tag{10}$$

however, again, nothing seems to be known for $n\geq 2$.

The aim of this paper is, firstly, to give an explicit formula for the M-harmonic Szegö kernel $K_{\text{Sz}}(x,y)$ for any n, in terms of certain hypergeometric functions; and secondly, to give a series expansion for $K_s(x,y)$, for any n and any $s>-1$. This series expansion is sufficient to show that, on the one hand, there is probably no explicit formula for K_s when $n\geq 2$ (even for $s=0$, i.e. in the case of the unweighted M-harmonic Bergman space); and, on the other hand, to give at least a rough idea of what is the singularity of $K_s(x,y)$ when both x and y approach the boundary $\partial\mathbf{B}^n$.

To describe our results, we recall some facts about hypergeometric functions. The ordinary (Gauss) hypergeometric function ${}_2F_1$ of one variable is defined by

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\Big|z\right)=\sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\,\frac{z^j}{j!},\qquad |z|<1. $$

Here $c\notin\{0,-1,-2,\dots\}$ while a,b can be any complex numbers, and

$$ (a)_j:=a(a+1)\dots(a+j-1)=\frac{\Gamma(a+j)}{\Gamma(a)} $$

stands for the Pochhammer symbol (raising factorial). We have also already met in (4) the Appell function F_1 of two variables, defined by

$$ F_1\left(\begin{matrix}a;b_1,b_2\\c\end{matrix}\Big|x,y\right)=\sum_{j,k=0}^{\infty}\frac{(a)_{j+k}(b_1)_j(b_2)_k}{(c)_{j+k}}\,\frac{x^j}{j!}\,\frac{y^k}{k!},\qquad |x|<1,|y|<1. $$

The hypergeometric function FD_1 of four variables

$$ \begin{aligned} &FD_1\left(\begin{matrix}a,a',b_1,b_2\\c\end{matrix}\Big|x_1,x_2,y_1,y_2\right)=\\ &\sum_{i_1,i_2,j_1,j_2=0}^{\infty}\frac{(a)_{i_1+i_2}(a')_{j_1+j_2}(b_1)_{i_1+j_1}(b_2)_{i_2+j_2}}{(c)_{i_1+i_2+j_1+j_2}}\,\frac{x_1^{i_1}}{i_1!}\,\frac{x_2^{i_2}}{i_2!}\,\frac{y_1^{j_1}}{j_1!}\,\frac{y_2^{j_2}}{j_2!} \end{aligned} \tag{11} $$

has been denoted $K_{16}(a,b_1,b_2,a';c;x_1,x_2,y_1,y_2)$ in Exton [Ex1, p. 78] and $K^*_{16}(a,b_1,a',b_2;c;x_2,x_1,y_1,y_2)$ in [Ex2]; our notation follows Karlsson [Ka]. The series (11) converges for $x_1,x_2,y_1,y_2\in\mathbf{D}$.

Our main results are the following.

Theorem (Theorem 2)**.** *For any $\alpha,\beta,\gamma,\delta\in\mathbf{C}$ and $n\geq 1$,*

$$ \begin{aligned} &\int_{\partial\mathbf{B}^n}(1-\langle z,\zeta\rangle)^{-\alpha}(1-\langle\zeta,z\rangle)^{-\beta}(1-\langle w,\zeta\rangle)^{-\gamma}(1-\langle\zeta,w\rangle)^{-\delta}\,d\sigma(\zeta)\\ &\qquad=\frac{2\pi^n}{\Gamma(n)}FD_1\left(\begin{matrix}\beta,\delta,\alpha,\gamma\\n\end{matrix}\Big||z|^2,\langle z,w\rangle,\langle w,z\rangle,|w|^2\right). \end{aligned} $$

Corollary (Corollary 4)**.** *For any $n\geq 1$,*

$$ K_{\mathrm{Sz}}(z,w)=\frac{\Gamma(n)}{2\pi^n}(1-|z|^2)^n(1-|w|^2)^nFD_1\left(\begin{matrix}n,n,n,n\\n\end{matrix}\Big||z|^2,\langle z,w\rangle,\langle w,z\rangle,|w|^2\right). \tag{12} $$

The last right-hand side can be expressed in terms of ordinary ${}_2F_1$ functions.

Theorem (Theorem 6)**.** *For any $n\geq 1$, $K_{\mathrm{Sz}}(z,w)$ equals*

$$ \begin{aligned} &\frac{\Gamma(n)}{2\pi^n}\frac{(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}}\sum_{i_1=0}^{n}\sum_{i_2,j_1=0}^{n-i_1}\frac{(-n)_{i_1+i_2}(-n)_{i_1+j_1}(n)_{i_2}(n)_{j_1}}{i_1!i_2!j_1!(n)_{i_1+i_2+j_1}}\\ &\qquad\times t_1^{i_1}t_2^{i_2}t_3^{j_1}\,{}_2F_1\left(\begin{matrix}i_2+n,j_1+n\\i_1+i_2+j_1+n\end{matrix}\Big|t_4\right), \end{aligned} $$

where

$$ t_1=|z|^2,\quad t_2=\frac{|z|^2-\langle w,z\rangle}{1-\langle w,z\rangle},\quad t_3=\overline{t_2},\quad t_4=1-\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}. $$

Note that for $z, w \in \mathbf{B}^n$ we have $t_2, t_3 \in \mathbf{D}$, while $0 \leq t_1, t_4 < 1$. Note also that using standard formulas for hypergeometric functions (cf. (57) in Section 3), the last ${}_2F_1$ are actually expressible in the form $a(t_4) + b(t_4)\log(1 - t_4)$ with rational functions a, b.

Both the holomorphic Szegö kernel (6) and the harmonic Szegö kernel (7) are clearly smooth functions on the closure $\overline{\mathbf{B}^n \times \mathbf{B}^n}$ except for the boundary diagonal $\operatorname{diag}\partial\mathbf{B}^n = \{(x, y) \in \partial\mathbf{B}^n \times \partial\mathbf{B}^n : x = y\}$. This is no longer the case for the M-harmonic Szegö kernel.

Proposition (Proposition 7)**.** *For $n > 1$,*

$$K_{\mathrm{Sz}} \in C^{n-1}(\overline{\mathbf{B}^n \times \mathbf{B}^n} \setminus \operatorname{diag}\partial\mathbf{B}^n) \setminus C^n(\overline{\mathbf{B}^n \times \mathbf{B}^n} \setminus \operatorname{diag}\partial\mathbf{B}^n).$$

As for the M-harmonic kernels K_s, $s > -1$, we may actually consider more general measures $d\mu \otimes d\sigma$ on $\mathbf{B}^n$ given by

$$\int_{\mathbf{B}^n} F(z)\,(d\mu \otimes d\sigma)(z) := \int_0^1 \int_{\partial\mathbf{B}^n} F(\sqrt{t}\zeta)\, t^{n-1}\, d\mu(t)\, d\sigma(\zeta), \tag{13}$$

where $d\mu$ is any finite Borel measure on the interval $[0, 1]$ such that $1 \in \operatorname{supp} d\mu$. Denote by $L^2_{\mathrm{Mh}}(\mathbf{B}^n, d\mu \otimes d\sigma)$ the corresponding M-harmonic weighted Bergman space and let K_μ be its reproducing kernel. The spaces (2) and their kernels $K_s(x, y)$ thus correspond to the choice $d\mu(t) = \frac{1}{2}(1 - t)^s\, dt$.

Theorem (Theorem 8)**.** *For any $n \geq 1$ and μ as above, K_μ is given by*

$$K_\mu(z, w) = \frac{\Gamma(n)}{2\pi^n}(1 - |z|^2)^n(1 - |w|^2)^n \sum_{p,q,j,m=0}^{\infty} A_{pqjm}(\mu)\frac{\langle z, w\rangle^p \langle w, z\rangle^q |z|^{2j}|w|^{2m}}{p!q!j!m!}, \tag{14}$$

where

$$\begin{aligned} A_{pqjm}(\mu) := \sum_{l=0}^{\min(m,j)} &\frac{\Gamma(n+p+j)\Gamma(n+q+j)}{\Gamma(n)\Gamma(n+p+q+j+l)}\,\frac{\Gamma(n+p+m)\Gamma(n+q+m)}{\Gamma(n)\Gamma(n+p+q+m+l)} \\ &\frac{(-1)^l\Gamma(n+p+q+l-1)(n+p+q+2l-1)(-j)_l(-m)_l}{\Gamma(n)l!c_{p+l,q+l}(\mu)}, \end{aligned} \tag{15}$$

with

$$c_{pq}(\mu) := \frac{\Gamma(p+n)^2\Gamma(q+n)^2}{\Gamma(n)^2\Gamma(p+q+n)^2}\int_0^1 t^{p+q+n-1}\,{}_2F_1\biggl(\begin{matrix} p, q \\ p+q+n \end{matrix}\biggm| t\biggr)^2 d\mu(t). \tag{16}$$

We remark that for μ the unit mass at the point $t = 1$, $L^2_{\mathrm{Mh}}(\mathbf{B}^n, d\mu \otimes d\sigma)$ reduces just to $H^2_{\mathrm{Mh}}(\mathbf{B}^n)$ and K_μ to K_{Sz}, while one can then show that $c_{pq}(\mu) = 1$ for all p, q and $A_{pqjm} = (n)_{j+p}(n)_{j+q}(n)_{m+p}(n)_{m+q}/(n)_{m+j+p+q}$; thus the last theorem recovers Corollary 4 as its special case.

Denoting $c_{pq}(\mu)$ for $d\mu(t) = \frac{1}{2}(1 - t)^s\, dt$ by $c_{pq}(s)$, the last theorem thus gives also a series expansion for the kernels K_s, $s > -1$. We will show that even for $s = 0$ and $n = 2$ (i.e. the unweighted M-harmonic Bergman space on $\mathbf{B}^2$), $c_{11}(s)/c_{00}(s)$ and, hence, also $A_{0000}(s)/A_{1100}(s)$, is of the form $a + b\zeta(3)$, where a, b are nonzero rational numbers and ζ stands for the Riemann zeta function. That is, the Taylor coefficients of K_0 on $\mathbf{B}^2$ involve $\zeta(3)$ in a nontrivial way. This makes it pretty unlikely that K_0, and *a fortiori* K_s for general $s > -1$ and $n \geq 2$, be given by any "nice" explicit formula in terms of e.g. hypergeometric and similar functions.

The coefficients $c_{pq}(s)$ can be expressed as certain multivariable hypergeometric functions at unit argument; unfortunately, again these expressions do not seem to lend themselves to an explicit evaluation. However, we can at least describe the asymptotic behavior of $c_{pq}(s)$ for large p,q, which turns out to be sufficient for getting some idea about the boundary behavior of the kernels K_s.

Theorem (Theorem 11). *Let $p,q>0$ be fixed. Then as $\lambda\to+\infty$, we have the asymptotic expansion*

$$c_{\lambda p,\lambda q}(s)\approx\frac{\Gamma(2n+s+1)\Gamma(n+s+1)^2\Gamma(s+1)}{\Gamma(n)^2\Gamma(2n+2s+2)}\frac{\lambda^{-2s-2}}{(pq)^{s+1}}\sum_{j=0}^\infty\frac{a_j(p,q)}{\lambda^j},$$

where $a_0(p,q)=1$.

For the omitted case $pq=0$, we get directly from (16)

$$c_{0q}(s)=c_{q0}(s)=\frac{\Gamma(q+n)\Gamma(s+1)}{\Gamma(q+n+s+1)}\approx\frac{\Gamma(n)\Gamma(s+1)}{\Gamma(n+s+1)}q^{-s-1}\qquad\text{as } q\to+\infty. \tag{17}$$

Thus in all cases $c_{pq}(s)\approx\frac{\Gamma(n)\Gamma(s+1)}{\Gamma(n+s+1)}(p+1)^{-s-1}(q+1)^{-s-1}$ as $p+q\to+\infty$. Our final result, though falling short of showing the situation for K_s itself, thus at least describes the boundary behavior of a series whose "leading order term" is the same as for K_s.

Theorem (Theorem 12). *For $n>1$ and $s=0,1,2,\dots$, consider the function $F_s(z,w)$ given by* (14), (15) *but with $c_{pq}(\mu)$ replaced by $(p+\frac{n-1}{2})^{-s-1}(q+\frac{n-1}{2})^{-s-1}$. Then*

(18)

$$\begin{aligned}F_s(z,w)=\mathcal{L}^{s+1}\Big[&\frac{\Gamma(n)}{2\pi^n}\frac{(1-y_2)^n}{(1-x_2)^n(1-y_1)^n}\sum_{i_1=0}^n\sum_{i_2,j_1=0}^{n-i_1}\frac{(-n)_{i_1+i_2}(-n)_{i_1+j_1}(n)_{i_2}(n)_{j_1}}{i_1!i_2!j_1!(n)_{i_1+i_2+j_1}}\\&x_1^{i_1}\Big(\frac{x_1-y_1}{1-y_1}\Big)^{i_2}\Big(\frac{x_1-x_2}{1-x_2}\Big)^{j_1}{}_2F_1\Big(\begin{matrix}i_2+n,j_1+n\\i_1+i_2+j_1+n\end{matrix}\Big|1-\frac{(1-x_1)(1-y_2)}{(1-x_2)(1-y_1)}\Big)\Big]\\&\Big|_{x_1=|z|^2,x_2=\langle z,w\rangle,y_1=\langle w,z\rangle,y_2=|w|^2},\end{aligned}$$

where $\mathcal{L}$ is the linear differential operator

$$\mathcal{L}:=(x_2y_1-x_1y_2)\frac{\partial^2}{\partial x_2\partial y_1}+\frac{n-1}{2}\Big(x_2\frac{\partial}{\partial x_2}+y_1\frac{\partial}{\partial y_1}\Big)+\frac{(n-1)^2}{4}I.$$

Note that $\mathcal{L}$ involves differentiations only with respect to x_2 and y_1.

Recall that for each $z\in\mathbf{B}^n$, $z\neq0$, there is a (unique) biholomorphic self-map $\phi_z\in\operatorname{Aut}(\mathbf{B}^n)$ which interchanges z and the origin 0; explicitly,

$$\phi_z(w)=\frac{z-P_zw-\sqrt{1-|z|^2}(w-P_zw)}{1-\langle w,z\rangle},\qquad P_zw:=\frac{\langle w,z\rangle}{|z|^2}z.$$

For $z=0$, we set $\phi_0(w):=-w$. One has the useful formula [Ru, Theorem 2.2.2]

$$1-\langle\phi_zw_1,\phi_zw_2\rangle=\frac{(1-|z|^2)(1-\langle w_1,w_2\rangle)}{(1-\langle z,w_2\rangle)(1-\langle w_1,z\rangle)}, \tag{19}$$

from which it follows that the various quantities appearing in (18) are in fact given by

$$x_1=|z|^2,\quad\frac{x_1-x_2}{1-x_2}=\langle z,\phi_zw\rangle,\qquad\frac{x_1-y_1}{1-y_1}=\langle\phi_zw,z\rangle,$$

$$1-\frac{(1-x_1)(1-y_2)}{(1-x_2)(1-y_1)}=|\phi_z w|^2.$$

We set

$$U:=|z|^2,\quad V:=|\phi_z w|^2,\quad Z:=\langle z,\phi_z w\rangle; \tag{20}$$

it will be shown that the map $(z,w)\mapsto(U,V,Z)$ is actually a bijection of

$\mathbf{B}^n\times\mathbf{B}^n$ modulo the action of the group $U(n)$ of unitary maps of $\mathbf{C}^n$

onto the set

$$\Omega:=\{(U,V,Z):0\le U,V<1,Z\in\mathbf{C},|Z|^2\le UV\}.$$

The coordinates U,V,Z are well suited for the description of the singularity of F_s near the boundary diagonal.

Corollary (Corollaries 13 and 14)**.** *For $n>1$ and $s=0,1,2,\dots,$*

$$F_s\in C^{n-1}(\overline{\mathbf{B}^n\times\mathbf{B}^n}\setminus\operatorname{diag}\partial\mathbf{B}^n),$$

and

$$F_s(z,w)=\frac{(1-V)^n}{(1-U)^{n+s+1}}\sum_{i_1=0}^{n}\sum_{i_2,j_1=0}^{n-i_1}\sum_{k=0}^{s+1}P_{i_1i_2j_1k}(U,Z,\overline{Z},V)$$
$$\times\,{}_2F_1\left(\begin{matrix}i_2+m+k,j_1+n+k\\ i_1+i_2+j_1+n+k\end{matrix}\middle|V\right),$$

where $P_{i_1i_2j_1k}(U,Z,\overline{Z},V)$ is a polynomial of degree at most i_1+s+1, j_1+s+1, i_2+s+1 and $k+s+1$, respectively, in the indicated variables.

We expect the boundary behavior of the M-harmonic kernels K_s to be of the same nature as for F_s.

The paper is organized as follows. In Section 2, we review the necessary prerequisites on the Peter-Weyl decomposition of M-harmonic functions under the action of the unitary group $U(n)$ of $\mathbf{C}^n$. The results about the M-harmonic Szegö kernel are proved in Section 3, and those about K_μ in Section 4. The asymptotic expansion of $c_{pq}(s)$ and the assertions about F_s are derived in Section 5. Some final remarks, comments and open problems are collected in the final section, Section 6.

To make typesetting a little neater, the shorthand

$$\Gamma\binom{a_1,a_2,\dots,a_k}{b_1,b_2,\dots,b_m}:=\frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_k)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_m)}$$

is often employed throughout the paper. The inner product $\langle z,w\rangle$ of $z,w\in\mathbf{C}^n$ is sometimes also written as $z\cdot\overline{w}$; and $\phi_z(w)$ is frequently abbreviated just to $\phi_z w$ (which we actually already did a few paragraphs above).

2. Notation and preliminaries

The stabilizer of the origin $0\in\mathbf{B}^n$ in $\operatorname{Aut}(\mathbf{B}^n)$ is the group $U(n)$ of all unitary transformations of $\mathbf{C}^n$; that is, of all linear operators U that preserve inner products:

$$\langle Uz,Uw\rangle=\langle z,w\rangle\qquad\forall z,w\in\mathbf{C}^n.$$

Each $U\in U(n)$ maps the unit sphere $\partial\mathbf{B}^n$ onto itself, and the surface measure $d\sigma$ on $\partial\mathbf{B}^n$ is invariant under U. It follows that the composition with elements of $U(n)$,

$$T_U:f\mapsto f\circ U^{-1}, \tag{21}$$

is a unitary representation of $U(n)$ on $L^2(\partial\mathbf{B}^n, d\sigma)$. We will need the decomposition of this representation into irreducible subspaces. These turn out to be given by *bigraded spherical harmonics* $\mathcal{H}^{pq}$; the standard sources for this are Rudin [Ru, Chapter 12.1–12.2], or Krantz [Kr1, Chapter 6.6–6.8], with basic ingredients going back to Folland [Fo].

Namely, for integers $p, q \geq 0$, let $\mathcal{H}^{pq}$ be vector space of restrictions to $\partial\mathbf{B}^n$ of harmonic polynomials $f(z, \overline{z})$ on $\mathbf{C}^n$ which are homogeneous of degree p in z and homogeneous of degree q in $\overline{z}$. Then $\mathcal{H}^{pq}$ is invariant under the action (21) of $U(n)$, is $U(n)$-irreducible (i.e. has no proper $U(n)$-invariant subspace) and

$$L^2(\partial\mathbf{B}^n, d\sigma) = \bigoplus_{p,q=0}^{\infty} \mathcal{H}^{pq}. \tag{22}$$

Furthermore, the representations of $U(n)$ on $\mathcal{H}^{pq}$ are mutually inequivalent; that is, if $T : \mathcal{H}^{pq} \to \mathcal{H}^{kl}$ is a linear operator commuting with the action (21), then necessarily

$$\begin{cases} T = 0 & \text{if } (k,l) \neq (p,q), \\ T = cI|_{\mathcal{H}^{pq}} \text{ for some } c \in \mathbf{C} & \text{if } (k,l) = (p,q), \end{cases} \tag{23}$$

where I denotes the identity operator.

Since each space $\mathcal{H}^{pq}$ is finite-dimensional, the evaluation functional $f \mapsto f(\zeta)$ at each $\zeta \in \partial\mathbf{B}^n$ is automatically continuous on it; it follows that $\mathcal{H}^{pq}$ — with the inner product inherited from $L^2(\partial\mathbf{B}^n, d\sigma)$ — has a reproducing kernel. This reproducing kernel turns out to be given by $H^{pq}(\zeta \cdot \overline{\eta})$, where for $n \geq 2$

$$\begin{aligned} H^{pq}(re^{i\theta}) = {} & \frac{(p+q+n-1)(p+n-2)!(q+n-2)!}{p!q!(n-1)!(n-2)!} \\ & \times r^{|p-q|} e^{(p-q)i\theta} \frac{\Gamma(n)}{2\pi^n} \frac{P^{(n-2,|p-q|)}_{\min(p,q)}(2r^2-1)}{P^{(n-2,|p-q|)}_{\min(p,q)}(1)}, \end{aligned} \tag{24}$$

where

$$P^{(\alpha,\beta)}_m(x) = (-1)^n \frac{(1-x)^{-\alpha}(1+x)^{-\beta}}{m!2^m} \frac{d^m}{dx^m}[(1-x)^{\alpha+m}(1+x)^{\beta+m}]$$

are the Jacobi polynomials. Thus

$$f(\zeta) = \int_{\partial\mathbf{B}^n} f(\eta) H^{pq}(\zeta \cdot \overline{\eta})\, d\sigma(\eta), \qquad \forall \zeta \in \partial\mathbf{B}^n, \forall f \in \mathcal{H}^{pq}.$$

In particular, we have the orthogonality relations

$$\int_{\partial\mathbf{B}^n} H^{pq}(\zeta \cdot \overline{\eta}) H^{kl}(\eta \cdot \overline{\xi})\, d\sigma(\eta) = \delta_{pk}\delta_{ql} H^{pq}(\zeta \cdot \overline{\xi}). \tag{25}$$

Note that by [BE, formula 10.8(16)],

$$P^{(\alpha,\beta)}_m(x) = (-1)^m \binom{m+\beta}{m} {}_2F_1\left(\begin{matrix} -m, m+\alpha+\beta+1 \\ \beta+1 \end{matrix} \middle| \frac{1+x}{2}\right),$$

so we have

$$\begin{aligned} H^{pq}(z) = {} & \frac{\Gamma(n)}{2\pi^n} \frac{(-1)^q (n+p+q-1)(n+p-2)!}{(n-1)!q!(p-q)!} \\ & \times z^{p-q} {}_2F_1\left(\begin{matrix} -q, n+p-1 \\ p-q+1 \end{matrix} \middle| |z|^2\right) \qquad \text{for } p \geq q, \end{aligned} \tag{26}$$

while $H^{pq}(z) = H^{qp}(\overline{z})$ for $p < q$. This formula will be useful later on.

Denote

$$\begin{aligned}S^{pq}(r) &:= r^{p+q}{}_2F_1\left(\begin{matrix}p, q\\ p+q+n\end{matrix}\middle| r^2\right) \Big/ {}_2F_1\left(\begin{matrix}p, q\\ p+q+n\end{matrix}\middle| 1\right) \\ &= \Gamma\left(\begin{matrix}p+n, q+n\\ n, p+q+n\end{matrix}\right) r^{p+q}{}_2F_1\left(\begin{matrix}p, q\\ p+q+n\end{matrix}\middle| r^2\right).\end{aligned} \tag{27}$$

Then for each $f \in \mathcal{H}^{pq}$, the (unique) solution to the Dirichlet problem $\widetilde{\Delta} u = 0$ on $\mathbf{B}^n$, $u|_{\partial\mathbf{B}^n} = f$ is given by

$$u(r\zeta) = S^{pq}(r) f(\zeta), \qquad 0 \le r \le 1,\ \zeta \in \partial\mathbf{B}^n. \tag{28}$$

For $n = 1$, all the above remains in force, only the spaces $\mathcal{H}^{pq}$ reduce just to $\{0\}$ if $pq \neq 0$, while $\mathcal{H}^{p0} = \mathbf{C} z^p$, $\mathcal{H}^{0q} = \mathbf{C}\overline{z}^q$ and $H^{p0}(z) = \frac{1}{2\pi} z^p$, $H^{0q}(z) = \frac{1}{2\pi}\overline{z}^q$; note that the formula (26) still works for $n = 1$ and $pq = 0$.

For each fixed $x \in \mathbf{B}^n$, the M-harmonic Poisson kernel $P(x, \cdot)$ always belongs to $L^2(\partial\mathbf{B}^n, d\sigma)$ (it is a smooth function on the sphere), hence it can be decomposed into the $\mathcal{H}^{pq}$ components as in (22). This decomposition was obtained by Folland [Fo]:

$$P(r\zeta, \eta) = \sum_{p,q=0}^{\infty} S^{pq}(r) H^{pq}(\zeta \cdot \overline{\eta}), \qquad 0 \le r < 1,\ \zeta, \eta \in \partial\mathbf{B}^n. \tag{29}$$

Folland gave his proof for $n \ge 2$, but with the caveat from the preceding paragraph it actually holds also for $n = 1$. (We will give an alternative proof for any n in Remark 10 in Section 4). The sum converges pointwise, uniformly for $\eta \in \partial\mathbf{B}^n$ and $r\zeta$ in a compact subset of $\mathbf{B}^n$, as well as in $L^2(\partial\mathbf{B}^n, d\sigma)$ for each fixed r and ζ.

Using the orthogonality relations (25), one can get from (29) the analogous decomposition for the M-harmonic Szegö kernel. Namely, starting from (9):

$$K_{\mathrm{Sz}}(x, y) = \int_{\partial\mathbf{B}^n} P(x, \zeta)\overline{P(y, \zeta)}\, d\sigma(\zeta)$$

(note that the complex conjugation actually has no effect, since $P(y, \zeta)$ is real-valued), and substituting (29) for $P(x, \zeta)$ and $P(y, \zeta)$, we get

$$\begin{aligned}K_{\mathrm{Sz}}(r\zeta, R\xi) &= \sum_{p,q,k,l=0}^{\infty} S^{pq}(r) S^{kl}(R) \int_{\partial\mathbf{B}^n} H^{pq}(\zeta \cdot \overline{\eta}) H^{kl}(\eta \cdot \overline{\xi})\, d\sigma(\eta) \\ &= \sum_{p,q=0}^{\infty} S^{pq}(r) S^{pq}(R) H^{pq}(\zeta \cdot \overline{\xi}) \qquad \text{by (25)},\end{aligned} \tag{30}$$

the interchange of integration and summation being justified by the L^2-convergence.

We conclude this section by giving a similar formula for the reproducing kernel of any Hilbert space of M-harmonic functions on $\mathbf{B}^n$ with a $U(n)$-invariant inner product.

For each $p, q \ge 0$, let $\mathbf{H}^{pq}$ be he space of all functions on $\mathbf{B}^n$ of the form (28) with $f \in \mathcal{H}^{pq}$. In other words, while $\mathcal{H}^{pq}$ is the space of spherical harmonics on the sphere $\partial\mathbf{B}^n$, $\mathbf{H}^{pq}$ is the associated space of "solid" M-harmonic functions on $\mathbf{B}^n$. With the inner product inherited from $L^2(\partial\mathbf{B}^n, d\sigma)$, each $\mathbf{H}^{pq}$ is thus a finite-dimensional Hilbert space of M-harmonic functions on $\mathbf{B}^n$, unitarily isomorphic to the space $\mathcal{H}^{pq}$ via the isomorphism (28), and with reproducing kernel

$$K^{pq}(r\zeta, R\xi) := S^{pq}(r)S^{pq}(R)H^{pq}(\zeta \cdot \overline{\xi}). \tag{31}$$

Proposition 1. *Let $\mathcal{H}$ be any Hilbert space of M-harmonic functions on $\mathbf{B}^n$ which contains $\mathbf{H}^{pq}$ for all $p, q \geq 0$, is invariant under the action (21) (i.e. $T_U f \in \mathcal{H}$ whenever $f \in \mathcal{H}$ and $U \in U(n)$) and whose inner product is $U(n)$-invariant:*

$$\langle T_U f, T_U g\rangle_{\mathcal{H}} = \langle f, g\rangle_{\mathcal{H}} \qquad \forall f, g \in \mathcal{H}, U \in U(n). \tag{32}$$

Then

(i) *the spaces $\mathbf{H}^{pq}$ are pairwise orthogonal in $\mathcal{H}$;*
(ii) *on each $\mathbf{H}^{pq}$, the $\mathcal{H}$-inner product is a constant multiple of the $L^2(d\sigma)$-inner product: there exist finite constants $c_{pq} > 0$ such that*

$$\langle f, g\rangle_{\mathcal{H}} = c_{pq}\langle f, g\rangle_{L^2(\partial\mathbf{B}^n, d\sigma)} \qquad \forall f, g \in \mathbf{H}^{pq}. \tag{33}$$

Furthermore, if the action $U \mapsto T_U$ of $U(n)$ on $\mathcal{H}$ is strongly continuous (i.e. for each $f \in \mathcal{H}$, $U \mapsto T_U f$ is continuous from $U(n)$ into $\mathcal{H}$), then additionally

(iii) *the linear span of $\mathbf{H}^{pq}$, $p, q \geq 0$, is dense in $\mathcal{H}$;*
(iv) *if the point evaluations are bounded on $\mathcal{H}$, then the reproducing kernel $K_{\mathcal{H}}$ of $\mathcal{H}$ is given by*

$$K_{\mathcal{H}}(r\zeta, R\xi) = \sum_{p,q=0}^{\infty} \frac{S^{pq}(r)S^{pq}(R)H^{pq}(\zeta \cdot \overline{\xi})}{c_{pq}}, \tag{34}$$

with the sum converging pointwise and locally uniformly on compact subsets of $\mathbf{B}^n \times \mathbf{B}^n$, as well as in $\mathcal{H}$ as a function of $x = r\zeta$ for each fixed $y = R\xi$, or vice versa.

Note that the last proposition applies, in particular, to $\mathcal{H} = L^2(\partial\mathbf{B}^n, d\sigma)$; in that case trivially $c_{pq} = 1$ $\forall p, q$, so we recover (30).

Proof. The restriction of the inner product in $\mathcal{H}$ to $\mathbf{H}^{pq} \times \mathbf{H}^{kl}$ is a continuous sesquilinear form on these (finite dimensional) spaces, so by the Riesz-Fischer theorem

$$\langle f, g\rangle_{\mathcal{H}} = \langle Tf, g\rangle_{L^2(d\sigma)} \qquad \forall f \in \mathbf{H}^{pq}, g \in \mathbf{H}^{kl}$$

for some linear operator $T : \mathbf{H}^{pq} \to \mathbf{H}^{kl}$. By (32), T is $U(n)$-invariant; thus by (23), $T = 0$ if $(p, q) \neq (k, l)$ while $T = c_{pq}I$ if $(p, q) = (k, l)$. This proves (i) and (ii).

To prove (iii), let $f \in \mathcal{H}$. Since f is M-harmonic, by Theorem 2.1 of [ABC], f can be expanded in the form

$$f = \sum_{p,q} f_{pq} \tag{35}$$

where $f_{pq} \in \mathbf{H}^{pq}$ for all p, q, and the series converges uniformly on compact subsets of $\mathbf{B}^n$. Furthermore, f_{pq} is actually given explicitly by

$$f_{pq}(r\zeta) = \int_{\partial\mathbf{B}^n} f(r\eta)H^{pq}(\zeta \cdot \overline{\eta})\, d\sigma(\eta)$$

(see [ABC, p. 107]). Setting $\eta = U^{-1}\zeta$, this can also be written as

$$f_{pq} = \int_{U(n)} \chi^{pq}(U)\, T_U f\, dU, \tag{36}$$

where dU is the Haar measure on the compact group $U(n)$, normalized to be of total mass $\frac{2\pi^n}{\Gamma(n)}$, and $\chi^{pq}(U) := H^{pq}(U\zeta \cdot \overline{\zeta})$ (this function — the character of the representation T_U on $\mathcal{H}^{pq}$ — does not depend on ζ). Now by the hypothesis of strong continuity of T_U, the last integral exists also as a Bochner integral, i.e. converges also in $\mathcal{H}$. Also, by an elementary estimate of the same integral, the map $P_{pq} : f \mapsto f_{pq}$ is continuous from $\mathcal{H}$ into $\mathbf{H}^{pq} \subset \mathcal{H}$. Making the change of variable $U \mapsto U^{-1}$ in (36) shows that P_{pq} is self-adjoint, and since P_{pq} clearly reduces to the identity on $\mathbf{H}^{pq}$, it follows that P_{pq} has to be the precisely the projection in $\mathcal{H}$ onto $\mathbf{H}^{pq} \subset \mathcal{H}$.

Now if $f \in \mathcal{H}$ is orthogonal to all $\mathbf{H}^{pq}$, then $f_{pq} = P_{pq} f = 0$ for all $p, q \geq 0$; thus by (35) $f = 0$. It follows that the linear span of $\mathbf{H}^{pq}$, $p, q \geq 0$, is dense in $\mathcal{H}$, proving (iii).

As for (iv), recall that for any functional Hilbert space with bounded point evaluations, the reproducing kernel is given by the formula

$$K_{\mathcal{H}}(x, y) = \sum_j f_j(x)\overline{f_j(y)} \tag{37}$$

where $\{f_j\}_j$ is any orthonormal basis of $\mathcal{H}$; see [Ar]. In our case, thanks to (i)–(ii), we can choose an orthonormal basis of the form $\{f_{pqj}/\sqrt{c_{pq}}\}_{pqj}$, where for each $p, q \geq 0$, $\{f_{pqj}\}_{j=1}^{\dim \mathbf{H}^{pq}}$ is an orthonormal basis in $\mathbf{H}^{pq}$ with respect to the $L^2(d\sigma)$ inner product. Thus

$$K_{\mathcal{H}}(x, y) = \sum_{p,q=0}^{\infty} \frac{1}{c_{pq}} \sum_{j=1}^{\dim \mathbf{H}^{pq}} f_{pqj}(x)\overline{f_{pqj}(y)}.$$

However, by (37) now applied to $\mathbf{H}^{pq}$ with the $L^2(d\sigma)$ inner product, the inner sum is precisely the reproducing kernel of $\mathbf{H}^{pq}$ with respect to the $L^2(d\sigma)$ inner product, which we know to be given by (31). This settles (34). The claim concerning convergence in $\mathcal{H}$ is immediate from the same property for (37) (cf. again [Ar]), while for the uniform convergence on compact subsets of $\mathbf{B}^n \times \mathbf{B}^n$ it is, similarly, enough to show that the norms $\|K_{\mathcal{H}}(\cdot, z)\|_{\mathcal{H}} = K_{\mathcal{H}}(z, z)^{1/2}$ stay bounded if z ranges in a compact subset of $\mathbf{B}^n$. However, this is immediate from the fact that $K_{\mathcal{H}}(x, y)$, being M-harmonic in each variable, is real-analytic in $(x, y) \in \mathbf{B}^n \times \mathbf{B}^n$ by the standard elliptic regularity theory; in particular, $K_{\mathcal{H}}(z, z)$ is a continuous function on $\mathbf{B}^n$. This completes the proof. □

We remark the Proposition 1 remains in force even when the hypothesis that $\mathbf{H}^{pq} \subset \mathcal{H}$ for all $p, q \geq 0$ is dropped. Indeed, denoting in that case

$$Y^{pq} := \mathcal{H} \cap \mathbf{H}^{pq},$$

it follows from the $U(n)$-irreducibility of $\mathcal{H}^{pq}$ (and, hence, of $\mathbf{H}^{pq}$) that if $Y^{pq} \neq \{0\}$, then already $Y^{pq} = \mathbf{H}^{pq}$ (and, so, $\mathbf{H}^{pq} \subset \mathcal{H}$). All the items (i)–(iv) then remain in force, except for the fact that in (iii) the linear span of Y^{pq} is dense in $\mathcal{H}$ and in (iv) instead of all $p, q \geq 0$ one takes only those p, q for which $Y^{pq} \neq \{0\}$. We are leaving the details to the reader.

3. The M-harmonic Szegö kernel

Theorem 2. *For any $\alpha,\beta,\gamma,\delta\in\mathbf{C}$ and $n\ge 1$,*

$$\begin{aligned}&\int_{\partial\mathbf{B}^n}(1-\langle z,\zeta\rangle)^{-\alpha}(1-\langle\zeta,z\rangle)^{-\beta}(1-\langle w,\zeta\rangle)^{-\gamma}(1-\langle\zeta,w\rangle)^{-\delta}\,d\sigma(\zeta)\\ &\qquad=\frac{2\pi^n}{\Gamma(n)}FD_1\Big(\begin{matrix}\beta,\delta,\alpha,\gamma\\ n\end{matrix}\Big||z|^2,\langle z,w\rangle,\langle w,z\rangle,|w|^2\Big).\end{aligned}\tag{38}$$

Recall that the function FD_1 has been defined in (11).

Proof. Clearly the integrand in (38) remains unchanged if z,w,ζ are replaced by $Uz,Uw,U\zeta$, respectively, with any $U\in U(n)$. Since the surface measure $d\sigma$ on $\partial\mathbf{B}^n$ is $U(n)$-invariant, it therefore follows that the integral (38) remains unchanged if z,w are replaced by Uz,Uw, for any $U\in U(n)$. Now we can pick $U\in U(n)$ which maps z into $|z|e_1$, where $e_1=(1,0,\dots,0)\in\mathbf{C}^n$. This U sends w into a point in $\mathbf{B}^n$ whose first coordinate — denote it by b — satisfies $b|z|=\langle z,w\rangle$; for $n>1$, we can then continue by choosing a suitable element of $U(n-1)$, acting on the remaining $n-1$ coordinates, so that in the end w is mapped into the point be_1+ce_2, $e_2=(0,1,0,\dots,0)$, where $c=\sqrt{|w|^2-|b|^2}$. Altogether, we thus see that for $n>1$, the integral (38) is equal to

$$\int_{\partial\mathbf{B}^n}(1-a\overline{\zeta}_1)^{-\alpha}(1-\overline{a}\zeta_1)^{-\beta}(1-b\overline{\zeta}_1-c\overline{\zeta}_2)^{-\gamma}(1-\overline{b}\zeta_1-\overline{c}\zeta_2)^{-\delta}\,d\sigma(\zeta),\tag{39}$$

where

$$\begin{cases}a=|z|,\ b=\frac{\langle z,w\rangle}{|z|},\ c=\sqrt{|w|^2-|b|^2} & \text{for } z\ne0,\\ a=b=0,\ c=\sqrt{|w|^2-|b|^2}(=|w|) & \text{for } z=0.\end{cases}\tag{40}$$

For $n=1$, this has to be replaced by

$$\begin{cases}a=|z|,\ b=\frac{\langle z,w\rangle}{|z|} & \text{for } z\ne0,\\ a=0,\ b=|w| & \text{for } z=0,\end{cases}\tag{41}$$

while $c=0$ (so that $c\overline{\zeta}_2$ and $\overline{c}\zeta_2$ in (39) both disappear).

Let us now compute the integral (39). The binomial expansion

$$(1-z)^{-\nu}=\sum_{j=0}^\infty\frac{(\nu)_j}{j!}z^j,\qquad \nu\in\mathbf{C},\tag{42}$$

converges uniformly for z in a compact subset of $\mathbf{D}$. Substituting this into (41) four times, we see that (39) equals

$$\begin{aligned}&\sum_{j,k,l,m=0}^\infty\frac{(\alpha)_j(\beta)_k(\gamma)_l(\delta)_m}{j!k!l!m!}\int_{\partial\mathbf{B}^n}(a\overline{\zeta}_1)^j(\overline{a}\zeta_1)^k(b\overline{\zeta}_1+c\overline{\zeta}_2)^l(\overline{b}\zeta_1+\overline{c}\zeta_2)^m\,d\sigma(\zeta)\\ &=\sum_{j,k,l,m=0}^\infty\frac{(\alpha)_j(\beta)_k(\gamma)_l(\delta)_m}{j!k!l!m!}\int_{\partial\mathbf{B}^n}(a\overline{\zeta}_1)^j(\overline{a}\zeta_1)^k\sum_{p=0}^l\sum_{q=0}^m\\ &\qquad\binom{l}{p}\binom{m}{q}(b\overline{\zeta}_1)^p(c\overline{\zeta}_2)^{l-p}(\overline{b}\zeta_1)^q(\overline{c}\zeta_2)^{m-q}.\end{aligned}$$

Using the familiar formula, valid for any multiindices ν,μ,

$$\int_{\partial\mathbf{B}^n}\zeta^\nu\overline{\zeta}^\mu\,d\sigma(\zeta)=\delta_{\nu\mu}\frac{\nu!}{(n)_{|\nu|}}\frac{2\pi^n}{\Gamma(n)},\tag{43}$$

it transpires that (39) equals

$$\begin{aligned}(44)\qquad \frac{2\pi^n}{\Gamma(n)} \sum_{j,k,l,m=0}^{\infty} \frac{(\alpha)_j(\beta)_k(\gamma)_l(\delta)_m}{j!k!l!m!} \int_{\partial\mathbf{B}^n} (a\overline{\zeta}_1)^j (\overline{a}\zeta_1)^k \sum_{p=0}^{l}\sum_{q=0}^{m} \binom{l}{p}\binom{m}{q} \\ \times a^j \overline{a}^k b^p \overline{b}^q c^{l-p} \overline{c}^{m-q} \delta_{p-q,l-m} \delta_{p-q,k-j} \frac{(k+q)!(m-q)!}{(n)_{k+m}}.\end{aligned}$$

We first deal with the summands for which $p \geq q$ — say, $p = q + r$ with some $r \geq 0$. The two delta functions are then nonzero only if $l = m + r$ and $k = j + r$. Thus the sum of all such summands will equal

(45)
$$\sum_{r=0}^{\infty} \overline{a}^r b^r \sum_{j,m=0}^{\infty} \frac{(\alpha)_j(\beta)_{j+r}(\gamma)_{m+r}(\delta)_m}{j!(j+r)!(m+r)!m!} \sum_{q=0}^{m} \binom{m+r}{q+r}\binom{m}{q} |a|^{2j}|b|^{2q}|c|^{2m-2q} \frac{(j+q+r)!(m-q)!}{(n)_{j+r+m}}.$$

Since $\binom{m+r}{q+r}\binom{m}{q} = \frac{(m+r)!m!}{q!(q+r)!(m-q)!^2}$, we can continue by

$$\sum_{r=0}^{\infty} \overline{a}^r b^r \sum_{j,m=0}^{\infty} \frac{(\alpha)_j(\beta)_{j+r}(\gamma)_{m+r}(\delta)_m}{j!(j+r)!} \sum_{q=0}^{m} |a|^{2j}|b|^{2q}|c|^{2m-2q} \frac{(j+q+r)!}{(n)_{j+r+m} q!(q+r)!(m-q)!}.$$

Writing $m = q + k$, this becomes

$$\sum_{r=0}^{\infty} \overline{a}^r b^r \sum_{j,q,k=0}^{\infty} \frac{(\alpha)_j(\beta)_{j+r}(\gamma)_{q+k+r}(\delta)_{q+k}}{j!(j+r)!} |a|^{2j}|b|^{2q}|c|^{2k} \frac{(j+q+r)!}{(n)_{j+r+q+k} q!(q+r)!k!}.$$

Substituting $|c|^{2k} = \sum_{l=0}^{k} \binom{k}{l}(-1)^l |b|^{2l}|w|^{2k-2l}$ from (40), this takes the form

$$\sum_{r=0}^{\infty} \overline{a}^r b^r \sum_{j,q,k=0}^{\infty} \frac{(\alpha)_j(\beta)_{j+r}(\gamma)_{q+k+r}(\delta)_{q+k}}{j!(j+r)!(q+r)!q!} \frac{(j+q+r)!}{(n)_{j+r+q+k}} \sum_{l=0}^{k} \frac{(-1)^l}{l!(k-l)!} |a|^{2j}|b|^{2q+2l}|w|^{2k-2l},$$

or, writing $k = l + m$,

$$\sum_{r=0}^{\infty} \overline{a}^r b^r \sum_{j,q,l,m=0}^{\infty} \frac{(\alpha)_j(\beta)_{j+r}(\gamma)_{q+l+m+r}(\delta)_{q+l+m}}{j!(j+r)!(q+r)!q!} \frac{(j+q+r)!}{(n)_{j+r+q+l+m}} \frac{(-1)^l}{l!m!} |a|^{2j}|b|^{2q+2l}|w|^{2m}.$$

Setting $q + l = k$, this becomes

$$\sum_{r=0}^{\infty} \overline{a}^r b^r \sum_{j,m,k=0}^{\infty} \sum_{q=0}^{k} \frac{(\alpha)_j(\beta)_{j+r}(\gamma)_{k+m+r}(\delta)_{k+m}}{j!(j+r)!(q+r)!q!} \frac{(j+q+r)!}{(n)_{j+r+k+m}} \frac{(-1)^{k-q}}{(k-q)!m!} |a|^{2j}|b|^{2k}|w|^{2m},$$

or, since $k!/(k-q)! = (-1)^q(-k)_q$,

$$\begin{aligned}&\sum_{r=0}^{\infty} \overline{a}^r b^r \sum_{j,m,k=0}^{\infty} \sum_{q=0}^{k} \frac{(\alpha)_j(\beta)_{j+r}(\gamma)_{k+m+r}(\delta)_{k+m}}{j!(j+r)!(q+r)!q!} \frac{(j+q+r)!}{(n)_{j+r+k+m}} \frac{(-1)^k(-k)_q}{k!m!} |a|^{2j}|b|^{2k}|w|^{2m} \\ &= \sum_{r=0}^{\infty} \overline{a}^r b^r \sum_{j,m,k=0}^{\infty} \frac{(\alpha)_j(\beta)_{j+r}(\gamma)_{k+m+r}(\delta)_{k+m}}{j!(j+r)!k!m!} \frac{(-1)^k}{(n)_{j+r+k+m}} |a|^{2j}|b|^{2k}|w|^{2m} \frac{(j+r)!}{r!} {}_2F_1\left(\begin{matrix}-k, j+r+1\\ r+1\end{matrix}\middle|1\right).\end{aligned}$$

By the Chu-Vandermonde formula,

$$\frac{(j+r)!}{r!} {}_2F_1\left(\begin{matrix}-k, j+r+1\\ r+1\end{matrix}\middle|1\right) = \frac{(j+r)!}{r!} \frac{(-j)_k}{(r+1)_k} = \frac{(-j)_k (j+r)!}{(r+k)!},$$

so we finally get

$$\sum_{r=0}^\infty \overline{a}^r b^r \sum_{j,m,k=0}^\infty \frac{(\alpha)_j(\beta)_{j+r}(\gamma)_{k+m+r}(\delta)_{k+m}}{j!(k+r)!k!m!} \frac{(-1)^k(-j)_k}{(n)_{j+r+k+m}} |a|^{2j}|b|^{2k}|w|^{2m}.$$

Since $(-j)_k$ vanishes for $j<k$, the sum effectively extends only over $j\ge k$, say, $j=k+l$; as $(-k-l)_k=(-1)^k(l+1)_k=(-1)^k(l+k)!/l!$, we thus obtain that (45) is equal to

$$\sum_{r=0}^\infty \overline{a}^r b^r \sum_{m,k,l=0}^\infty \frac{(\alpha)_{k+l}(\beta)_{k+l+r}(\gamma)_{k+m+r}(\delta)_{k+m}}{l!(k+r)!k!m!(n)_{l+r+2k+m}} |a|^{2k+2l}|b|^{2k}|w|^{2m}.$$

Since by (40) always $a=|z|$ and $\overline{a}b=\langle z,w\rangle$, we finally arrive at

$$\sum_{k,l,m,r=0}^\infty \frac{(\alpha)_{k+l}(\beta)_{k+l+r}(\gamma)_{k+m+r}(\delta)_{k+m}}{l!(k+r)!k!m!(n)_{l+r+2k+m}} \langle z,w\rangle^{k+r}\langle w,z\rangle^k|z|^{2l}|w|^{2m},$$

or, rechristening $k+r$, k and l to $q+r=p$, q and j, respectively,

$$\sum_{\substack{p,q,j,m=0\\ p\ge q}}^\infty \frac{(\alpha)_{q+j}(\beta)_{p+j}(\gamma)_{p+m}(\delta)_{q+m}}{p!q!j!m!(n)_{p+q+j+m}} \langle z,w\rangle^p\langle w,z\rangle^q|z|^{2j}|w|^{2m}. \tag{46}$$

This came from the summands in (44) with $p\ge q$; in the same way, the sum over $p<q$ in (44) turns out to be given again by (46), but with the summation extending over $p<q$. Putting these two pieces together, we thus see that the integral (39) is equal to

$$\begin{aligned}
&\frac{2\pi^n}{\Gamma(n)} \sum_{p,q,j,m=0}^\infty \frac{(\alpha)_{q+j}(\beta)_{p+j}(\gamma)_{p+m}(\delta)_{q+m}}{p!q!j!m!(n)_{p+q+j+m}} \langle z,w\rangle^p\langle w,z\rangle^q|z|^{2j}|w|^{2m}\\
&= \frac{2\pi^n}{\Gamma(n)} FD_1\left(\begin{matrix}\beta,\delta,\alpha,\gamma\\ n\end{matrix}\middle| |z|^2,\langle z,w\rangle,\langle w,z\rangle,|w|^2\right),
\end{aligned}$$

as claimed. This completes the proof for the case $n>1$.

For $n=1$ and $z\ne0$, we still have $\sqrt{|w|^2-|b|^2}=0=c$ since $|\langle z,w\rangle|/|z|=|w|$ in this case; thus the whole argument above still works without change. Finally, for $n=1$ and $z=0$, the integral (39) reduces just to

$$\int_{\partial\mathbf{B}^1} (1-|w|\zeta)^{-\gamma}(1-|w|\overline{\zeta})^{-\delta}\,d\sigma(\zeta) = 2\pi\,{}_2F_1\left(\begin{matrix}\gamma,\delta\\ 1\end{matrix}\middle| |w|^2\right)$$

by (42), while

$$FD_1\left(\begin{matrix}\beta,\delta,\alpha,\gamma\\ 1\end{matrix}\middle| 0,0,0,|w|^2\right) = {}_2F_1\left(\begin{matrix}\gamma,\delta\\ 1\end{matrix}\middle| |w|^2\right)$$

by (11). Thus the assertion holds in this case as well. □

Remark 3. If we carry out the integration over $(\zeta_3,\dots,\zeta_n)$ in (39), the integral transforms into

$$\frac{2\pi^{n-1}}{\Gamma(n-1)} \int_{\mathbf{B}^2} (1-a\overline{\zeta}_1)^{-\alpha}(1-\overline{a}\zeta_1)^{-\beta}(1-b\overline{\zeta}_1-c\overline{\zeta}_2)^{-\gamma}(1-\overline{b}\zeta_1-\overline{c}\zeta_2)^{-\delta}\,(1-|\zeta_1|^2-|\zeta_2|^2)^{n-\frac32}\,d\zeta_1\,d\zeta_2.$$

This is strangely reminiscent of the following known integral formula for FD_1, valid for $b_1, b_2 > 0$, and $c - b_1 - b_2 > 0$ [Ka, formula 4.3.(8)]:

$$\begin{aligned} FD_1\Big(\begin{matrix} a, a', b_1, b_2 \\ c \end{matrix}\Big| x_1, x_2, y_1, y_2\Big) &= \Gamma\Big(\begin{matrix} c \\ b_1, b_2, c - b_1 - b_2 \end{matrix}\Big) \\ &\times \int_{\substack{u_1, u_2 > 0 \\ u_1 + u_2 < 1}} \frac{u_1^{b_1-1} u_2^{b_2-1} (1 - u_1 - u_2)^{c-b_1-b_2-1}}{(1 - x_1 u_1 - x_2 u_2)^a (1 - y_1 u_1 - y_2 u_2)^{a'}} \, du_1 \, du_2. \end{aligned} \tag{47}$$

However, it does not seem possible to derive our Theorem 2 from (47). $\square$

The following corollary to the last theorem is immediate from (9).

Corollary 4. *For any $n \geq 1$,*

$$K_{\mathrm{Sz}}(z, w) = \frac{\Gamma(n)}{2\pi^n} (1 - |z|^2)^n (1 - |w|^2)^n FD_1\Big(\begin{matrix} n, n, n, n \\ n \end{matrix}\Big| |z|^2, \langle z, w\rangle, \langle w, z\rangle, |w|^2\Big). \tag{48}$$

Remark 5. A posteriori, it is possible to give a "direct" proof of the last corollary by checking straight away that, for each fixed $w \in \mathbf{B}^n$, the right-hand side of (48) is M-harmonic in z and its boundary value as $z \to \zeta \in \partial\mathbf{B}^n$ coincides with $P(w, \zeta)$. To see the former, denote temporarily

$$\begin{aligned} a_{pqjm} &:= \frac{(n)_{j+p}(n)_{j+q}(n)_{m+p}(n)_{m+q}}{(n)_{j+p+q+m}}, \\ I_{pqj} &:= (1 - |z|^2)^n \frac{\langle z, w\rangle^p \langle w, z\rangle^q |z|^{2j}}{p!q!j!}, \\ W_m &:= (1 - |w|^2)^n \frac{|w|^{2m}}{m!}, \end{aligned}$$

so that

$$K_{\mathrm{Sz}}(z, w) = \frac{\Gamma(n)}{2\pi^n} \sum_{p,q,j,m=0}^{\infty} a_{pqjm} I_{pqj} W_m.$$

By a routine computation (here $\widetilde{\Delta}$ applies to the z variable)

$$\frac{\widetilde{\Delta} I_{pqj}}{1 - |z|^2} = |w|^2 I_{p-1,q-1,j} - (j + n + p)(j + n + q) I_{pqj} + (p + q + n + j - 1) I_{p,q,j-1}.$$

Consequently, with the understanding that $I_{pqj} \equiv 0$ if any of the subscripts p, q, j is negative,

$$\begin{aligned} \frac{2\pi^n/\Gamma(n)}{1 - |z|^2} \widetilde{\Delta}_z K_{\mathrm{Sz}}(z, w) &= |w|^2 \sum_{pqjm} a_{pqjm} I_{p-1,q-1,j} W_m \\ &\quad - \sum_{pqjm} (j + n + p)(j + n + q) a_{pqjm} I_{pqj} W_m \\ &\quad + \sum_{pqjm} (p + q + n + j - 1) a_{pqjm} I_{p,q,j-1} W_m \\ &= |w|^2 \sum_{pqjm} a_{pqjm} I_{p-1,q-1,j} W_m \\ &\quad - \sum_{pqjm} (j + n + p)(j + n + q) a_{pqjm} I_{pqj} W_m \end{aligned}$$

$$+\sum_{pqjm}(p+q+n+j)a_{p,q,j+1,m}I_{pqj}W_m. \tag{49}$$

Since $a_{p,q,j+1.m}=\frac{(n+j+p)(n+j+q)}{n+p+q+m+j}a_{pqjm}$, the second and third sums combine into

$$\begin{aligned}
-\sum_{pqjm}&\frac{m(n+p+j)(n+q+j)}{n+p+q+j+m}a_{pqjm}I_{pqj}W_m \\
&=-|w|^2\sum_{pqjm}\frac{(n+p+j)(n+q+j)}{n+p+q+j+m}a_{pqjm}I_{pqj}W_{m-1} \\
&=-|w|^2\sum_{pqjm}\frac{(n+p+j)(n+q+j)}{n+p+q+j+m+1}a_{pqj,m+1}I_{pqj}W_m.
\end{aligned}$$

Since

$$\begin{aligned}
&\frac{(n+p+j)(n+q+j)}{n+p+q+j+m+1}a_{pqj,m+1} \\
&\qquad=\frac{(n+p+j)(n+q+j)(n+m+p)(n+m+q)}{(n+p+q+j+m)(n+p+q+j+m+1)}a_{pqjm} \\
&\qquad=a_{p+1,q+1,j,m},
\end{aligned}$$

this exactly cancels the first sum in (49). Thus, indeed, $\widetilde{\Delta}_z K_{\mathrm{Sz}}(z,w)\equiv 0$.

To verify the latter claim, let us carry out the summation over j in (55):

$$\begin{aligned}
\frac{2\pi^n}{\Gamma(n)}K_{\mathrm{Sz}}(z,w)&=(1-|z|^2)^n(1-|w|^2)^n\sum_{pqm}\frac{(n)_p(n)_q(n)_{m+p}(n)_{m+q}}{(n)_{p+q+m}} \\
&\quad\times\frac{\langle z,w\rangle^p\langle w,z\rangle^q|w|^{2m}}{p!q!m!}\,{}_2F_1\left(\begin{matrix}n+p,n+q\\ n+p+q+m\end{matrix}\middle|\,|z|^2\right).
\end{aligned} \tag{50}$$

Using the standard Euler transformation formula for ${}_2F_1$ [BE, §2.1(23)]

$${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\middle|\,t\right)=(1-t)^{c-a-b}\,{}_2F_1\left(\begin{matrix}c-a,c-b\\ c\end{matrix}\middle|\,t\right), \tag{51}$$

the right-hand side of (50) becomes

$$\begin{aligned}
(1-|w|^2)^n&\sum_{pqm}\frac{(n)_p(n)_q(n)_{m+p}(n)_{m+q}}{(n)_{p+q+m}}\frac{\langle z,w\rangle^p\langle w,z\rangle^q|w|^{2m}}{p!q!m!} \\
&\times(1-|z|^2)^m\,{}_2F_1\left(\begin{matrix}p+m,q+m\\ n+p+q+m\end{matrix}\middle|\,|z|^2\right).
\end{aligned}$$

As $z\to\zeta\in\partial\mathbf{B}^n$, only the terms with $m=0$ survive, yielding

$$\begin{aligned}
(1-|w|^2)^n&\sum_{pq}\frac{(n)_p^2(n)_q^2}{(n)_{p+q}}\frac{\langle\zeta,w\rangle^p\langle w,\zeta\rangle^q}{p!q!}\,{}_2F_1\left(\begin{matrix}p,q\\ n+p+q\end{matrix}\middle|\,1\right) \\
&=(1-|w|^2)^n\sum_{pq}\frac{(n)_p^2(n)_q^2}{(n)_{p+q}}\frac{\langle\zeta,w\rangle^p\langle w,\zeta\rangle^q}{p!q!}\Gamma\left(\begin{matrix}n+p+q,n\\ n+p,n+q\end{matrix}\right) \\
&=(1-|w|^2)^n\sum_{pq}\frac{(n)_p(n)_q}{p!q!}\langle\zeta,w\rangle^p\langle w,\zeta\rangle^q \\
&=(1-|w|^2)^n(1-\langle\zeta,w\rangle)^{-n}(1-\langle w,\zeta\rangle)^{-n}=\frac{2\pi^n}{\Gamma(n)}P(w,\zeta),
\end{aligned}$$

completing the proof. $\square$

The function FD_1 is known to satisfy an Euler-type transformation formula [Ka, formula 7.1.(4)]

$$(52)\quad \begin{aligned} &FD_1\left(\begin{matrix} a, a', b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right) = (1-x_1)^{-b_1}(1-x_2)^{-b_2} \\ &\times FD_1\left(\begin{matrix} c-a-a', a', b_1, b_2 \\ c \end{matrix}\middle| \frac{x_1}{x_1-1}, \frac{x_2}{x_2-1}, \frac{y_1-x_1}{1-x_1}, \frac{y_2-x_2}{1-x_2}\right). \end{aligned}$$

On the other hand, from its definition (47) it is apparent that FD_1 enjoys the symmetry property

$$(53)\quad FD_1\left(\begin{matrix} a, a', b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right) = FD_1\left(\begin{matrix} b_1, b_2, a, a' \\ c \end{matrix}\middle| x_1, y_1, x_2, y_2\right).$$

These formulas can be used to simplify (48).

Theorem 6. *For any $n \geq 1$, $K_{\mathrm{Sz}}(z,w)$ equals*

$$(54)\quad \begin{aligned} &\frac{\Gamma(n)}{2\pi^n}\frac{(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}} \sum_{i_1=0}^{n} \sum_{i_2,j_1=0}^{n-i_1} \frac{(-n)_{i_1+i_2}(-n)_{i_1+j_1}(n)_{i_2}(n)_{j_1}}{i_1!i_2!j_1!(n)_{i_1+i_2+j_1}} \\ &\qquad \times t_1^{i_1} t_2^{i_2} t_3^{j_1}\, {}_2F_1\left(\begin{matrix} i_2+n, j_1+n \\ i_1+i_2+j_1+n \end{matrix}\middle| t_4\right), \end{aligned}$$

where

$$(55)\quad t_1 = |z|^2, \quad t_2 = \frac{|z|^2 - \langle w,z\rangle}{1-\langle w,z\rangle}, \quad t_3 = \overline{t_2}, \quad t_4 = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}.$$

Proof. Applying (53) to the right-hand side of (52) gives

$$\begin{aligned} &FD_1\left(\begin{matrix} a, a', b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right) = (1-x_1)^{-b_1}(1-x_2)^{-b_2} \\ &\times FD_1\left(\begin{matrix} b_1, b_2, c-a-a', a' \\ c \end{matrix}\middle| \frac{x_1}{x_1-1}, \frac{y_1-x_1}{1-x_1}, \frac{x_2}{x_2-1}, \frac{y_2-x_2}{1-x_2}\right). \end{aligned}$$

Now we apply (53) one more time, to the last right-hand side; after a small computation, this yields

$$\begin{aligned} &FD_1\left(\begin{matrix} a, a', b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right) = (1-x_1)^{c-a-b}(1-x_2)^{-b_2}(1-y_1)^{-a'} \\ &\times FD_1\left(\begin{matrix} c-b_1-b_2, b_2, c-a-a', a' \\ c \end{matrix}\middle| x_1, \frac{x_1-y_1}{1-y_1}, \frac{x_1-x_2}{1-x_2}, \frac{x_1+y_2-x_2-y_1+x_2y_1-x_1y_2}{(1-x_2)(1-y_1)}\right). \end{aligned}$$

Multiplying both sides by $(1-x_1)^n(1-y_2)^n$ and setting $a = a' = b_1 = b_2 = c = n$, $x_1 = |z|^2$, $x_2 = \langle z,w\rangle$, $y_1 = \langle w,z\rangle$, $y_2 = |w|^2$, we thus get from (48)

$$(56)\quad K_{\mathrm{Sz}}(z,w) = \frac{\Gamma(n)}{2\pi^n}\frac{(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}} FD_1\left(\begin{matrix} -n, n, -n, n \\ n \end{matrix}\middle| t_1, t_2, t_3, t_4\right)$$

with t_1, t_2, t_3, t_4 as in (55). Finally, by (11)

$$FD_1\left(\begin{matrix} -n, n, -n, n \\ n \end{matrix}\middle| t_1, t_2, t_3, t_4\right) = \sum_{\substack{i_1+i_2\leq n, \\ i_1+j_1\leq n, \\ j_2\geq 0}} \frac{t_1^{i_1} t_2^{i_2} t_3^{j_1} t_4^{j_2}}{i_1!i_2!j_1!j_2!} \frac{(-n)_{i_1+i_2}(n)_{j_1+j_2}(-n)_{i_1+j_1}(n)_{i_2+j_2}}{(n)_{i_1+i_2+j_1+j_2}}$$

$$= \sum_{\substack{i_1+i_2\le n,\\ i_1+j_1\le n}} \frac{t_1^{i_1}t_2^{i_2}t_3^{j_1}}{i_1!i_2!j_1!} \frac{(-n)_{i_1+i_2}(n)_{j_1}(-n)_{i_1+j_1}(n)_{i_2}}{(n)_{i_1+i_2+j_1}} \sum_{j_2\ge 0} \frac{(n+j_1)_{j_2}(n+i_2)_{j_2}}{(n+i_1+i_2+j_1)_{j_2}} \frac{t_4^{j_2}}{j_2!}$$

$$= \sum_{\substack{i_1+i_2\le n,\\ i_1+j_1\le n}} \frac{t_1^{i_1}t_2^{i_2}t_3^{j_1}}{i_1!i_2!j_1!} \frac{(-n)_{i_1+i_2}(n)_{j_1}(-n)_{i_1+j_1}(n)_{i_2}}{(n)_{i_1+i_2+j_1}} {}_2F_1\Big(\begin{matrix} n+j_1, n+i_2 \\ n+i_1+i_2+j_1\end{matrix}\Big| t_4\Big).$$

Substituting this into (56) yields (54), completing the proof. □

Using the formula [BE, §2.10(11)]

$$\begin{aligned} {}_2F_1\Big(\begin{matrix} n+1, n+m+1 \\ n+m+l+2\end{matrix}\Big| z\Big) &= \frac{(n+m+l+1)!(-1)^{m+1}}{l!n!(m+n)!(m+l)!} \\ &\times \frac{d^{n+m}}{dz^{n+m}}\Big[(1-z)^{m+l}\frac{d^l}{dz^l}\frac{\log(1-z)}{z}\Big], \qquad m,n,l = 0,1,2,\dots, \end{aligned} \tag{57}$$

it is possible to express each ${}_2F_1$ in (54) in terms of $\log(1-t_4)$ and rational functions of t_4.

For instance, for $n=1$ we get in this way

$$FD_1\Big(\begin{matrix} 1,1,1,1 \\ 1\end{matrix}\Big| t_1,t_2,t_3,t_4\Big) = \frac{t_2t_3 + (1-t_2-t_3)t_4}{t_4(1-t_4)} + \frac{t_2t_3 - t_1t_4}{t_4^2}\log(1-t_4).$$

For $t_1 = |z|^2$, $t_2 = \overline{z}\frac{z-w}{1-\overline{z}w}$, $t_3 = \overline{t}_2$ and $t_4 = 1 - \frac{(1-|z|^2)(1-|w|^2)}{|1-w\overline{z}|^2}$, the right-hand side simplifies just to $\frac{1-|z|^2|w|^2}{1-|w|^2}$, implying that $K_{\mathrm{Sz}}(z,w) = \frac{1}{2\pi}\frac{1-|z|^2|w|^2}{|1-w\overline{z}|^2}$, in complete accordance with (10). For $n=2$, the formula for K_{Sz} already becomes quite complicated, without any apparent possibility of simplification.

Note also that the M-harmonic Szegö kernel $K_{\mathrm{Sz}}(z,w)$ is symmetric in z,w, though this is not visible at all from the formula (54).

We conclude this section by discussing the smoothness of K_{Sz} on the closure of $\mathbf{B}^n\times\mathbf{B}^n$.

Proposition 7. *For $n>1$,*

$$K_{\mathrm{Sz}} \in C^{n-1}(\overline{\mathbf{B}^n\times\mathbf{B}^n}\setminus \operatorname{diag}\partial\mathbf{B}^n) \setminus C^n(\overline{\mathbf{B}^n\times\mathbf{B}^n}\setminus \operatorname{diag}\partial\mathbf{B}^n).$$

Proof. Let U be a neighborhood of $\operatorname{diag}(\partial\mathbf{B}^n)$. Then $1-\langle z,w\rangle$ stays away from 0 on $\overline{\mathbf{B}^n\times\mathbf{B}^n}\setminus U$. Keeping our previous notation $x_1 = |z|^2$, $x_2 = \langle z,w\rangle$, $y_1 = \langle w,z\rangle$, $y_2 = |w|^2$, and denoting in addition temporarily $Q := 1/(1-\langle z,w\rangle)$, we have $t_2 = \overline{t}_3 = 1-(1-x_1)Q$, so from (54)

$$\begin{aligned} K_{\mathrm{Sz}}(z,w) &= (1-y_2)^n|Q|^{2n}\frac{\Gamma(n)}{2\pi^n}\sum_{i_1=0}^{n}\sum_{i_2,j_1=0}^{n-i_1} x_1^{i_1}(1-(1-x_1)Q)^{i_2}(1-(1-x_1)\overline{Q})^{j_1} \\ &\times \frac{(-n)_{i_1+i_2}(n)_{j_1}(-n)_{i_1+j_1}(n)_{i_2}}{(n)_{i_1+i_2+j_1}} {}_2F_1\Big(\begin{matrix} n+j_1, n+i_2 \\ n+i_1+i_2+j_1\end{matrix}\Big| 1-(1-x_1)(1-y_2)|Q|^2\Big). \end{aligned} \tag{58}$$

By the standard formulas for the analytic continuation of the hypergeometric functions ${}_2F_1$ [BE, §2.10(14)], we have for any $a,b>0$ and $m=0,1,2,\dots$,

$${}_2F_1\Big(\begin{matrix} a,b \\ a+b-m\end{matrix}\Big| z\Big) = (1-z)^{-m}A_{abm}(1-z) + B_{abm}(1-z)\log(1-z),$$

with some functions A_{abm}, B_{abm} holomorphic on $\mathbf{D}$. The right-hand side of (58) therefore equals, omitting for a moment the factor $\frac{\Gamma(n)}{2\pi^n}$,

$$\sum_{i_1=0}^{n} \sum_{i_2,j_1=0}^{n-i_1} x_1^{i_1}(1-(1-x_1)Q)^{i_2}(1-(1-x_1)\overline{Q})^{j_1} \frac{(-n)_{i_1+i_2}(n)_{j_1}(-n)_{i_1+j_1}(n)_{i_2}}{(n)_{i_1+i_2+j_1}} \\ \times \Big[\frac{(1-y_2)^{i_1}|Q|^{2i_1}}{(1-x_1)^{n-i_1}} A + (1-y_2)^n |Q|^{2n} B \log[(1-x_1)(1-y_2)|Q|^2]\Big],$$

where $A \equiv A_{n+j_1,n+i_2,n-i_1}(1-1(1-x_1)(1-y_2)|Q|^2)$ and $B \equiv B_{n+j_1,n+i_2,n-i_1}(1-1(1-x_1)(1-y_2)|Q|^2)$. In terms of Taylor series around $(x_1,y_2)=(1,1)$, the last expression has the form

$$\begin{aligned} &\sum_{j=-n}^{\infty} \sum_{k,l,m=0}^{\infty} a_{jklm}(1-x_1)^j(1-y_2)^k Q^l \overline{Q}^m \\ &+\sum_{k=n}^{\infty} \sum_{j,l,m=0}^{\infty} b_{jklm}(1-x_1)^j(1-y_2)^k Q^l \overline{Q}^m \log[(1-x_1)(1-y_2)|Q|^2], \end{aligned} \tag{59}$$

with some coefficients a_{jklm}, b_{jklm}. However, since $K_{\mathrm{Sz}}(z,w)$ is symmetric in z,w, (59) remains unchanged upon replacing x_1, y_2 and Q by y_2, x_1 and $\overline{Q}$, respectively. Consequently, a_{jkml} must vanish for $j<0$, and b_{jklm} must vanish for $j<n$. It follows that the first sum in (59) is C^∞ on $(1-x_1, 1-y_2, Q) \in \mathbf{D}\times\mathbf{D}\times(\mathbf{C}\setminus\{0\})$, while the second sum is C^{n-1} there. (Note that Q is bounded away from zero, in fact $|Q|>\frac{1}{2}$.) This in turn means that the right-hand side of (58) is C^{n-1} on $\overline{\mathbf{B}^n\times\mathbf{B}^n}\setminus U$. Thus, indeed, $K_{\mathrm{Sz}} \in C^{n-1}(\overline{\mathbf{B}^n\times\mathbf{B}^n}\setminus \operatorname{diag}\partial\mathbf{B}^n)$.

It remains to show that K_{Sz} does not belong to $C^n(\overline{\mathbf{B}^n\times\mathbf{B}^n}\setminus \operatorname{diag}\partial\mathbf{B}^n)$. If it did, then the function

$$\int_{\partial\mathbf{B}^n} K_{\mathrm{Sz}}(z,R\xi) H^{pq}(\xi\cdot\overline{\eta})\, d\sigma(\xi)$$

would belong to $C^n(\overline{\mathbf{B}^n})$, for any fixed $0<R<1$, $\eta\in\partial\mathbf{B}^n$ and $p,q\geq 0$. However, by (30) and the orthogonality relations (25), the last integral equals $S^{pq}(|z|)S^{pq}(R)H^{pq}(\frac{z}{|z|}\cdot\overline{\eta})$, and the hypergeometric function $S^{pq}(t)$ is well known not to be C^n at the point $t=1$ (it again contains a logarithmic singularity of the form $(1-t)^n\log(1-t)$). This completes the proof. □

4. General M-harmonic kernels

We now turn to general $U(n)$-invariant measures $d\mu\otimes d\sigma$ on $\mathbf{B}^n$ and their associated M-harmonic kernels K_μ.

Theorem 8. *For any $n\geq 1$ and μ as in* (13), *K_μ is given by*

$$K_\mu(z,w) = \frac{\Gamma(n)}{2\pi^n}(1-|z|^2)^n(1-|w|^2)^n \sum_{p,q,j,m=0}^{\infty} A_{pqjm}(\mu) \frac{\langle z,w\rangle^p \langle w,z\rangle^q |z|^{2j}|w|^{2m}}{p!q!j!m!}, \tag{60}$$

where

$$
\begin{aligned}
A_{pqjm}(\mu) :=& \sum_{l=0}^{\min(m,j)} \frac{\Gamma(n+p+j)\Gamma(n+q+j)}{\Gamma(n)\Gamma(n+p+q+j+l)} \frac{\Gamma(n+p+m)\Gamma(n+q+m)}{\Gamma(n)\Gamma(n+p+q+m+l)} \\
&\times \frac{(-1)^l \Gamma(n+p+q+l-1)(n+p+q+2l-1)(-j)_l(-m)_l}{\Gamma(n) l! c_{p+l,q+l}(\mu)},
\end{aligned}
\tag{61}
$$

with

$$
c_{pq}(\mu) := \frac{\Gamma(p+n)^2 \Gamma(q+n)^2}{\Gamma(n)^2 \Gamma(p+q+n)^2} \int_0^1 t^{p+q+n-1} {}_2F_1\left(\begin{matrix} p,q \\ p+q+n \end{matrix}\middle| t\right)^2 d\mu(t). \tag{62}
$$

Proof. Clearly, the space $\mathcal{H} = L^2_{\mathrm{Mh}}(\mathbf{B}^n, d\mu \otimes d\sigma)$ satisfies the hypotheses of Proposition 1. For any pair of functions $u(r\zeta) = S^{pq}(r) f(\zeta)$ and $v(r\zeta) = S^{pq}(r) g(\zeta)$ in $\mathbf{H}^{pq}$, we have

$$
\begin{aligned}
\langle u, v \rangle_{\mathcal{H}} &= \int_0^1 \int_{\partial \mathbf{B}^n} S^{pq}(\sqrt{t}) f(\zeta) S^{pq}(\sqrt{t}) \overline{g(\zeta)} \, (d\mu \otimes d\sigma)(t, \zeta) \\
&= \langle f, g \rangle_{L^2(\partial \mathbf{B}^n, d\sigma)} \int_0^1 S^{pq}(\sqrt{t})^2 t^{n-1} d\mu(t),
\end{aligned}
$$

so (33) holds with

$$
c_{pq} = \Gamma\binom{p+n, q+n}{n, p+q+n}^2 \int_0^1 t^{p+q+n-1} {}_2F_1\left(\begin{matrix} p,q \\ p+q+n \end{matrix}\middle| t\right)^2 d\mu(t) \equiv c_{pq}(\mu),
$$

by (62). Consequently, by (34), for $z = r\xi$ and $w = R\eta$,

$$
K_\mu(z, w) = \sum_{p,q} \frac{S^{pq}(r) S^{pq}(R) H^{pq}(\xi \cdot \overline{\eta})}{c_{pq}(\mu)}. \tag{63}
$$

To simplify the notation, let us temporarily denote $\xi \cdot \overline{\eta} =: \zeta \in \mathbf{D}$ and

$$
s^{pq}(t) = \Gamma\binom{n+p, n+q}{n, n+p+q} {}_2F_1\left(\begin{matrix} p,q \\ n+p+q \end{matrix}\middle| t\right),
$$

so that $S^{pq}(r) = r^{p+q} s^{pq}(r^2)$. Let us first consider the sum in (63) over terms with $p \geq q$ — say, $p = q + r$, $r \geq 0$. Using (26), the sum becomes (omitting momentarily the constant factor $\frac{\Gamma(n)}{2\pi^n}$)

$$
\begin{aligned}
&\sum_{q,r=0}^{\infty} \frac{|z|^{2q+r} s^{q+r,q}(|z|^2) |w|^{2q+r} s^{q+r,q}(|w|^2)}{c_{q+r,q}(\mu)} \times \\
&\qquad \frac{(-1)^q (n+2q+r-1)(n+q+r-2)!}{\Gamma(n) q! r!} \overline{\zeta}^r \sum_{j=0}^{q} \frac{(-q)_j (n+q+r-1)_j}{(r+1)_j j!} \zeta^j \overline{\zeta}^j \\
&= \sum_{q,r=0}^{\infty} \sum_{j=0}^{q} \frac{|z|^{2q+r} s^{q+r,q}(|z|^2) |w|^{2q+r} s^{q+r,q}(|w|^2)}{c_{q+r,q}(\mu)} \times \\
&\qquad \frac{(-1)^q (n+2q+r-1)(n+q+r+j-2)! (-q)_j}{\Gamma(n) q! (r+j)! j!} \zeta^j \overline{\zeta}^{j+r}.
\end{aligned}
\tag{64}
$$

Letting $q = j + l$ and noticing that $(-q)_j / q! = (-1)^j / (q-j)!$, this becomes

$$
\sum_{j,l,r \geq 0} \frac{|z|^{2j+2l+r} s^{j+l+r,j+l}(|z|^2) |w|^{2j+2l+r} s^{j+l+r,j+l}(|w|^2)}{c_{j+l+r,j+l}(\mu)}
$$

$$\times \frac{(-1)^l(n+2j+2l+r-1)(n+2j+l+r-2)!}{\Gamma(n)(r+j)!j!l!}\zeta^j\overline{\zeta}^{j+r},$$

or, writing q and $p=q+r$ instead of j and $j+r$, respectively,

$$\sum_{\substack{q,r\geq 0\\ p=q+r}}\sum_{l\geq 0}\frac{|z|^{p+q+2l}s^{p+l,q+l}(|z|^2)|w|^{p+q+2l}s^{p+l,q+l}(|w|^2)}{c_{p+l,q+l(\mu)}}$$

$$\times \frac{(-1)^l(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)p!q!l!}\zeta^q\overline{\zeta}^p.$$

The sum over $p<q$ in (63) is treated in the same way, and recalling that $\langle z,w\rangle = |z||w|\zeta$, we thus arrive at

$$K_\mu(z,w) = \frac{\Gamma(n)}{2\pi^n}\sum_{p,q,l=0}^{\infty}\frac{|z|^{2l}s^{p+l,q+l}(|z|^2)|w|^{2l}s^{p+l,q+l}(|w|^2)}{c_{p+l,q+l}(\mu)}\times$$
$$\frac{(-1)^l(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)l!}\frac{\langle z,w\rangle^q\langle w,z\rangle^p}{q!p!}.$$

On the other hand, by the Euler transformation formula (51), we have

$$\begin{aligned} s^{pq}(t) &= \Gamma\binom{p+n,q+n}{n,p+q+n}(1-t)^n{}_2F_1\left(\begin{matrix}p+n,q+n\\ p+q+n\end{matrix}\middle| t\right) \\ &= (1-t)^n\sum_{k=0}^{\infty}\Gamma\binom{p+n+k,q+n+k}{p+q+n+k,n}\frac{t^k}{k!}. \end{aligned}$$

Consequently,

$$K_\mu(z,w) = (1-|z|^2)^n(1-|w|^2)^n\frac{\Gamma(n)}{2\pi^n}\sum_{p,q,j,m=0}^{\infty}A_{pqjm}(\mu)\frac{|z|^{2j}|w|^{2k}}{j!k!}\frac{\langle z,w\rangle^q\langle w,z\rangle^p}{q!p!}$$

with

$$\begin{aligned} A_{pqjm}(\mu) &= \sum_{\substack{l,i,k\geq 0,\\ l+i=j,\\ l+k=m}}\frac{(-1)^l(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)l!c_{p+l,q+l}(\mu)}\times \\ &\qquad \Gamma\binom{p+l+n+i,q+l+n+i}{n,p+q+n+i+2l}\Gamma\binom{p+l+n+k,q+l+n+k}{n,p+q+n+k+2l}\frac{j!}{i!}\frac{m!}{k!} \\ &= \sum_{l=0}^{\min(m,j)}\frac{(-1)^l(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)l!c_{p+l,q+l}(\mu)}\times \\ &\qquad \Gamma\binom{p+n+j,q+n+j}{n,p+q+n+j+l}\Gamma\binom{p+n+m,q+n+m}{n,p+q+n+m+l}(-j)_l(-m)_l, \end{aligned}$$

since $j!/(j-l)! = (-1)^l(-j)_l$ and similarly for $m!/(m-l)!$. But this is precisely (60) and (61), completing the proof. □

Remark 9. Taking for $d\mu(t)$ the point mass at $t=1$, one can use the last theorem to give an independent proof of the formula (48) for the M-harmonic Szegö kernel (not using Theorem 1). Indeed, in that case $c_{pq}=1$ for all p,q, so, pulling out some Gamma functions,

$$A_{pqjm}(\mu)=\Gamma\binom{n+p+j,n+q+j}{n,n+p+q+j}\Gamma\binom{n+p+m,n+q+m}{n,n+p+q+m}$$
$$\sum_{l=0}^{\min(m,j)}\frac{(-1)^l(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)l!}\frac{(-j)_l(-m)_l}{(p+q+n+j)_l(p+q+n+m)_l}.$$

Denoting momentarily for brevity $n+p+q=:h$, we have

$$n+p+q+2l-1=2\Big(\frac{h-1}{2}+l\Big)=(h-1)\frac{(\frac{h+1}{2})_l}{(\frac{h-1}{2})_l},$$

so the last sum can be written as

$$\frac{h-1}{\Gamma(n)}\sum_l\frac{(-j)_l(-m)_l}{(h+j)_l(h+m)_l}\frac{(-1)^l}{l!}\frac{(\frac{h+1}{2})_l}{(\frac{h-1}{2})_l}\Gamma(h+l-1)$$
$$=\frac{\Gamma(h)}{\Gamma(n)}{}_4F_3\Big(\begin{matrix}-j,-m,\frac{h+1}{2},h-1\\ h+j,h+m,\frac{h-1}{2}\end{matrix}\Big|-1\Big). \tag{65}$$

Now by a formula of Bailey [BE, §4.5(4)]

$${}_4F_3\Big(\begin{matrix}a,1+\frac{a}{2},b,c\\ \frac{a}{2},1+a-b,1+a-c\end{matrix}\Big|-1\Big)=\Gamma\binom{1+a-b,1+a-c}{1+a,1+a-b-c},$$

so (65) is equal to

$$\frac{\Gamma(h)}{\Gamma(n)}\Gamma\binom{h+j,h+m}{h,h+j+m}=\Gamma\binom{h+j,h+m}{n,h+j+m}$$

and

$$A_{pqjm}(\mu)=\Gamma\binom{n+p+j,n+q+j}{n}\Gamma\binom{n+p+m,n+q+m}{n}\Gamma\binom{-}{n,n+p+q+j+m}$$
$$=\frac{(n)_{p+j}(n)_{q+j}(n)_{p+m}(n)_{q+m}}{(n)_{p+q+j+m}},$$

proving the claim. □

Remark 10. A similar argument as in the proof of Theorem 8 can also be used to give another proof of Folland's formula (29) for the M-harmonic Poisson kernel: for $z=r\xi$,

$$\sum_{p,q}S^{pq}(|z|)H^{pq}(\xi\cdot\overline{\eta})=\frac{\Gamma(n)}{2\pi^n}\frac{(1-|z|^2)^n}{|1-\langle z,\eta\rangle|^{2n}}. \tag{66}$$

Indeed, assuming again first that $p\geq q$ — say, $p=q+r$, $r\geq 0$ — and using (26), the sum over $p\geq q$ becomes, as in (64) (omitting yet again temporarily the constant factor $\Gamma(n)/2\pi^n$, and denoting again $\xi\cdot\overline{\eta}=:\zeta$)

$$\sum_{q,r=0}^{\infty}\sum_{j=0}^{q}|z|^{2q+r}s^{q+r,q}(|z|^2)\frac{(-1)^q(n+2q+r-1)(n+q+r+j-2)!(-q)_j}{\Gamma(n)q!(r+j)!j!}\zeta^j\overline{\zeta}^{j+r},$$

or, upon setting $q=j+l$,

$$\sum_{j,l,r\geq 0}|z|^{2j+2l+r}s^{j+l+r,j+l}(|z|^2)\frac{(-1)^l(n+2j+2l+r-1)(n+2j+l+r-2)!}{\Gamma(n)(r+j)!j!l!}\zeta^j\overline{\zeta}^{j+r},$$

that is, writing q and $q=p+r$ instead of j and $j+r$, respectively,

$$\sum_{\substack{q,r\ge 0\\ p=q+r}}\sum_{l\ge 0}|z|^{p+q+2l}s^{p+l,q+l}(|z|^2)\frac{(-1)^l(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)p!q!l!}\zeta^q\overline{\zeta}^p.$$

The sum over $p<q$ is treated in the same way, and we see that the right-hand side of (66) equals

$$\frac{\Gamma(n)}{2\pi^n}\sum_{p,q,l=0}^{\infty}|z|^{2l}s^{p+l,q+l}(|z|^2)\frac{(-1)^l(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)l!}\frac{\langle z,\eta\rangle^q\langle\eta,z\rangle^p}{q!p!}$$

$$=\frac{\Gamma(n)}{2\pi^n}(1-|z|^2)^n\sum_{p,q,m=0}^{\infty}A_{pqm}\frac{|z|^{2m}}{m!}\frac{\langle z,\eta\rangle^q\langle\eta,z\rangle^p}{q!p!},$$

where

$$\begin{aligned}A_{pqm}&=\sum_{\substack{l,k\ge 0\\ l+k=m}}\frac{(-1)^l(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)l!}\Gamma\binom{p+l+n+k,q+l+n+k}{n,p+q+n+k+2l}\frac{m!}{k!}\\&=\sum_{l=0}^{m}\frac{(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)l!}\Gamma\binom{p+n+m,q+n+m}{n,p+q+n+m+l}(-m)_l\\&=\Gamma\binom{p+n+m,q+n+m}{n,p+q+n+m}\sum_{l=0}^{m}\frac{(n+p+q+2l-1)(n+p+q+l-2)!}{\Gamma(n)l!}\frac{(-m)_l}{(p+q+n+m)_l}\\&=\Gamma\binom{p+n+m,q+n+m}{n,h+m}\Gamma\binom{h}{n}{}_3F_2\left(\begin{matrix}h-1,\frac{h+1}{2},-m\\ \frac{h-1}{2},h+m\end{matrix}\Big|1\right),\end{aligned}$$

as in (65); here we have again set $n+p+q=:h$. Now by a formula due to Dixon [BE, §4.4(5)]

$${}_3F_2\left(\begin{matrix}a,b,c\\ 1+a-b,1+a-c\end{matrix}\Big|1\right)=\Gamma\binom{1+\frac{a}{2},1+a-b,1+a-c,1+\frac{a}{2}-b-c}{1+a.1+\frac{a}{2}-b,1+\frac{a}{2}-c,1+a-b-c},$$

so the penultimate ${}_3F_2$ is equal to

$$\Gamma\binom{\frac{h+1}{2},\frac{h-1}{2},h+m,m}{h,0,\frac{h+1}{2}+m,\frac{h-1}{2}+m}.$$

This vanishes for $m\ge 1$, and reduces to 1 for $m=0$. Hence $A_{pqm}=0$ for $m\ge 1$, while

$$A_{pq0}=\Gamma\binom{p+n,q+n}{n,p+q+n}\Gamma\binom{p+q+n}{n}=(n)_p(n)_q,$$

so that

$$\sum_{p,q,m=0}^{\infty}A_{pqm}\frac{|z|^{2m}}{m!}\frac{\langle z,\eta\rangle^q\langle\eta,z\rangle^p}{q!p!}=\sum_{p,q}(n)_p(n)_q\frac{\langle z,\eta\rangle^q\langle\eta,z\rangle^p}{q!p!}=(1-\langle z,\eta\rangle)^{-n}(1-\langle\eta,z\rangle)^{-n},$$

proving (66). □

Choosing $d\mu(t)=\frac{1}{2}(1-t)^s\,dt$, the last theorem gives, in particular, a formula for the weighted M-harmonic Bergman kernels K_s, $s>-1$. The coefficients $c_{pq}(\mu)$ are then given by

$$c_{pq}(s):=\frac{1}{2}\Gamma\binom{n+p,n+q}{n,n+p+q}^2\int_0^1 t^{p+q+n-1}{}_2F_1\left(\begin{matrix}p,q\\ p+q+n\end{matrix}\Big|t\right)^2(1-t)^s\,dt. \tag{67}$$

We conclude this section by evaluating c_{00} and c_{01} in the particular case of the unweighted M-harmonic Bergman kernel $s=0$ in dimension $n=2$.

For c_{00}, we actually have quite generally

$$c_{00}(s)=\frac{1}{2}\int_0^1 t^{n-1}(1-t)^s\,dt=\frac{\Gamma(n)\Gamma(s+1)}{2\Gamma(n+s+1)}. \tag{68}$$

For $p=q=1$ and $s=0$, (67) reads

$$c_{11}(0)=\frac{1}{2}\frac{n^2}{(n+1)^2}\int_0^1 t^{n+1}{}_2F_1\left(\begin{matrix}1,1\\ n+2\end{matrix}\Big|t\right)^2 dt.$$

Using (51) and (57),

$${}_2F_1\left(\begin{matrix}1,1\\ n+2\end{matrix}\Big|t\right)=(1-t)^n{}_2F_1\left(\begin{matrix}n+1,n+1\\ n+2\end{matrix}\Big|t\right)=(1-t)^n\frac{(n+1)!}{n!^2}\frac{d^n}{dt^n}\frac{-\log(1-t)}{t}.$$

For $n=2$, a short computation reveals that this reduces to

$${}_2F_1\left(\begin{matrix}1,1\\ 4\end{matrix}\Big|t\right)=\frac{3t(3t-2)-6(1-t)^2\log(1-t)}{2t^3}. \tag{69}$$

Differentiating the familiar Beta integral

$$\int_0^1 t^a(1-t)^b\,dt=\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)},\qquad a,b>-1,$$

with respect to b and setting $b=0$, we get

$$\int_0^1 t^a\log(1-t)\,dt=\frac{\psi(1)-\psi(a+2)}{a+1},$$
$$\int_0^1 t^a\log(1-t)^2\,dt=\frac{(\psi(1)-\psi(a+2))^2+\psi'(1)-\psi'(a+2)}{a+1},$$

where $\psi=\Gamma'/\Gamma$ is the logarithmic derivative of the Gamma function. Squaring (69), multiplying by t^{x+3}, $x>2$, and using the last two formulas yields an (unwieldy) explicit formula for $\int_0^1 t^{x+3}{}_2F_1\left(\begin{matrix}1,1\\ 4\end{matrix}\Big|t\right)^2 dt$. A tedious but utterly routine calculation reveals that the result extends analytically to $\operatorname{Re}x>-4$ (as it should!), and its value at $x=0$ is

$$\frac{117}{2}\psi(1)-\frac{171}{8}+\frac{9}{2}(\psi(3)-\psi(1))^2-36(\psi(1)-\psi(2))^2+\frac{27}{2}\psi(3)-72\psi(2)$$
$$-\frac{63}{2}\psi'(1)+36\psi'(2)-\frac{9}{2}\psi'(3)-54\psi''(1).$$

Finally, recalling that for $k,m=1,2,3,\dots$,

$$\psi(m)=-C+\sum_{j=1}^{m-1}\frac{1}{j},$$
$$\psi^{(k)}(m)=(-1)^k(k-1)!\zeta(k+1)+(-1)^k k!\sum_{j=1}^{m-1}\frac{1}{j^{k+1}},$$

where C is the Euler constant and ζ the Riemann zeta function, we finally get that for $n=2$

$$c_{11}(0)=\frac{96\zeta(3)-115}{4}$$

and

$$\frac{c_{11}(0)}{c_{00}(0)} = 96\zeta(3) - 115.$$

Since by (61)

$$A_{0000}(s) = \frac{1}{c_{00}(s)}, \qquad A_{1100}(s) = \frac{n^3}{(n+1)c_{11}(s)},$$

we also get for $n = 2$

$$A_{0000}(0) = 4, \qquad A_{1100}(0) = \frac{32}{96\zeta(3) - 115}.$$

It is somewhat unlikely for a function whose Taylor coefficient involves $96\zeta(3) - 115$ in the denominator to be given by some nice "closed" formula in terms of e.g. hypergeometric and similar functions. Thus there is probably no "explicit" expression for $K_0(z, w)$ when $n = 2$, hence *a fortiori* also for $K_s(z, w)$ for general $s > -1$ and $n \geq 2$.

5. Asymptotics of $c_{pq}(s)$

Recall that in the polar coordinates $z = r\zeta$ on $\mathbf{C}^n$ ($r > 0$, $\zeta \in \partial\mathbf{B}^n$), the Euclidean Laplacian Δ is given by

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{2n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{\mathrm{sph}},$$

where Δ_{sph} is the *spherical Laplacian*, which involves only differentiations with respect to the ζ variables. In particular, the value of $\Delta_{\mathrm{sph}}\phi$ on a sphere $|z|$ =const. depends only on the values of the function ϕ on that sphere. Another operator with this property is the *complex normal derivative* (or *Reeb vector field*)

$$\mathcal{R} := \sum_{j=1}^{n} \Big(z_j \frac{\partial}{\partial z_j} - \overline{z}_j \frac{\partial}{\partial \overline{z}_j} \Big).$$

Both Δ_{sph} and $\mathcal{R}$ commute with the action of $U(n)$, i.e. $\Delta_{\mathrm{sph}}(\phi \circ U) = (\Delta_{\mathrm{sph}}\phi) \circ U$ for any $U \in U(n)$, and similarly for $\mathcal{R}$. (In fact, the algebra of all $U(n)$-invariant linear differential operators on $\partial\mathbf{B}^n$ is generated by Δ_{sph} and $\mathcal{R}$, but we will not need this fact.) From the irreducibility of the multiplicity-free decomposition (22) it follows by abstract theory that Δ_{sph} and $\mathcal{R}$ map each $\mathcal{H}^{pq}$ (and $\mathbf{H}^{pq}$) into itself and actually reduce on it to a multiple of the identity. Evaluation on e.g. the element $\zeta_1^p \overline{\zeta}_2^q \in \mathcal{H}^{pq}$ (for $n \geq 2$) shows that, explicitly,

$$\begin{aligned} \Delta_{\mathrm{sph}}|\mathcal{H}^{pq} &= -(p+q)(p+q+2n-2)I|\mathcal{H}^{pq}, \\ \mathcal{R}|\mathcal{H}^{pq} &= (p-q)I|\mathcal{H}^{pq} \end{aligned} \tag{70}$$

(which formulas prevail also for $n = 1$; in that case $\Delta_{\mathrm{sph}} = -\mathcal{R}^2$). In view of (22), both Δ_{sph} and $\mathcal{R}$ thus give rise to self-adjoint operators on $L^2(\partial\mathbf{B}^n, d\sigma)$, and the operator

$$\mathcal{D} := [-\Delta_{\mathrm{sph}} + (n-1)^2 I]^{1/2}$$

(in the sense of functional calculus of self-adjoint operators) corresponds to multiplication by $p + q$ on $\mathcal{H}^{pq}$. Consequently, the operators $\frac{\mathcal{D}+\mathcal{R}}{2}, \frac{\mathcal{D}-\mathcal{R}}{2}$ correspond to multiplication on $\mathcal{H}^{pq}$ by p and q, respectively, and for "any" double sequence $\{f(p,q)\}_{p,q=0}^{\infty}$, the operator $f(\frac{\mathcal{D}+\mathcal{R}}{2}, \frac{\mathcal{D}-\mathcal{R}}{2})$ (again taken in the sense of functional calculus) will correspond to multiplication by $f(p,q)$ on $\mathcal{H}^{pq}$. Taking, in particular, $f(p,q)=1/c_{pq}(s)$ with the $c_{pq}(s)$ from (67), and denoting the corresponding operator $f(\frac{\mathcal{D}+\mathcal{R}}{2},\frac{\mathcal{D}-\mathcal{R}}{2})$ by M_s, we thus deduce from (34) and (30) that

$$K_s(z,w)=M_s K_{\mathrm{Sz}}(z,w), \tag{71}$$

where on the right-hand side we can choose to apply M_s to either the z or the w variable (the result will be the same). Looking at the "leading order term" of M_s, i.e. the one corresponding — loosely speaking — to highest order derivatives, we can thus get a rough idea of the behavior of K_s from that of K_{Sz} (which we are familiar with from Section 3). (This is the usual machinery of microlocal analysis.)

Since, in view of (70), the order of differentiation corresponds to the homogeneity degree of $f(p,q)$ in (p,q), we thus need to find the asymptotic behavior of $c_{pq}(s)$ as $p+q\to+\infty$.

Theorem 11. *Let $p,q>0$ be fixed. Then as $\lambda\to+\infty$, we have the asymptotic expansion*

$$c_{\lambda p,\lambda q}(s)\approx\Gamma\binom{2n+s+1,n+s+1,n+s+1,s+1}{n,n,2n+2s+2}\frac{\lambda^{-2s-2}}{(pq)^{s+1}}\sum_{j=0}^{\infty}\frac{a_j(p,q)}{\lambda^j}, \tag{72}$$

where $a_0(p,q)=1$.

Proof. Inserting the standard representation for the ${}_2F_1$ function

$${}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\Big|z\right)=\Gamma\binom{c}{b,c-b}\int_0^1\frac{t^{b-1}(1-t)^{c-b-1}}{(1-tz)^a}\,dt,\qquad c>b>0,$$

into (67) yields

$$2c_{pq}(s)=\Gamma\binom{p+n,q+n}{n,n,p,q}\int_0^1\int_0^1\int_0^1 t^{p+q+n-1}(1-t)^s\frac{x^{q-1}(1-x)^{n+p-1}y^{p-1}(1-y)^{n+q-1}}{(1-tx)^p(1-ty)^q}\,dx\,dy\,dt.$$

Using the representation [BE, §5.8 (5)]

$$F_1\left(\begin{matrix}a;b,b'\\c\end{matrix}\Big|x,y\right)=\Gamma\binom{c}{a,c-a}\int_0^1\frac{t^{a-1}(1-t)^{c-a-1}}{(1-tx)^b(1-ty)^{b'}}\,dt,\qquad c>a>0,$$

for the Appell F_1 function, this becomes

$$\begin{aligned}2c_{pq}(s)=\Gamma\binom{p+n,q+n}{n,n,p,q}\Gamma\binom{p+q+n,s+1}{p+q+n+s+1}\int_0^1\int_0^1 x^{q-1}(1-x)^{n+p-1}y^{p-1}(1-y)^{n+q-1}\\ \times F_1\left(\begin{matrix}p+q+n;p,q\\p+q+n+s+1\end{matrix}\Big|x,y\right)dx\,dy.\end{aligned}$$

By the transformation formula for F_1 [BE, §5.11(1)]

$$F_1\left(\begin{matrix}a;b,b'\\c\end{matrix}\Big|x,y\right)=(1-x)^{-b}(1-y)^{-b'}F_1\left(\begin{matrix}c-a;b,b'\\c\end{matrix}\Big|\frac{x}{x-1},\frac{y}{y-1}\right),$$

we can continue with

$$\begin{aligned}2c_{pq}(s)=\Gamma\binom{p+n,q+n}{n,n,p,q}\Gamma\binom{p+q+n,s+1}{p+q+n+s+1}\int_0^1\int_0^1 x^{q-1}(1-x)^{n-1}y^{p-1}(1-y)^{n-1}\\ \times F_1\left(\begin{matrix}s+1;p,q\\p+q+n+s+1\end{matrix}\Big|\frac{x}{x-1},\frac{y}{y-1}\right)dx\,dy.\end{aligned}$$

Applying yet another representation formula for F_1 [BE, §5.8(1)]

$$F_1\Big(\begin{matrix} a; b, b' \\ c \end{matrix}\Big| x, y\Big) = \Gamma\Big(\begin{matrix} c \\ b, b', c-b-b' \end{matrix}\Big) \iint_{\substack{u,v>0\\u+v<1}} \frac{u^{b-1}v^{b'-1}(1-u-v)^{c-b-b'-1}}{(1-ux-vy)^a}\, du\, dv,$$
$$b, b' > 0,\ c-b-b' > 0,$$

we arrive at

$$2c_{pq}(s) = \Gamma\Big(\begin{matrix} p+q+n, s+1, p+n, q+n \\ n, n, p, p, q, q, n+s+1 \end{matrix}\Big) \int_0^1 \int_0^1 \iint_{\substack{u,v>0\\u+v<1}} x^{q-1}(1-x)^{n-1}y^{p-1}(1-y)^{n-1}$$
$$\times \frac{u^{p-1}v^{q-1}(1-u-v)^{n+s}}{(1+\frac{ux}{1-x}+\frac{vy}{1-y})^{s+1}}\, du\, dv\, dx\, dy.$$

Performing the change of variable $u = \tau\sigma$, $v = (1-\tau)\sigma$, $du\, dv = \sigma\, d\sigma\, d\tau$, this transforms into

$$2c_{pq}(s) = \Gamma\Big(\begin{matrix} p+q+n, s+1, p+n, q+n \\ n, n, p, p, q, q, n+s+1 \end{matrix}\Big) \int_0^1 \int_0^1 \int_0^1 \int_0^1 x^{q-1}(1-x)^{n-1}y^{p-1}(1-y)^{n-1}$$
$$\times \frac{\sigma^{p+q-1}(1-\sigma)^{n+s}\tau^{p-1}(1-\tau)^{q-1}}{(1+\frac{\sigma\tau x}{1-x}+\frac{\sigma(1-\tau)y}{1-y})^{s+1}}\, d\sigma\, d\tau\, dx\, dy$$
$$= \Gamma\Big(\begin{matrix} p+q+n, s+1, p+n, q+n \\ n, n, p, p, q, q, n+s+1 \end{matrix}\Big) \int_0^1 \int_0^1 \int_0^1 \int_0^1 x^{q-1}(1-x)^{n+s}y^{p-1}(1-y)^{n+s}$$
$$\times \frac{\sigma^{p+q-1}(1-\sigma)^{n+s}\tau^{p-1}(1-\tau)^{q-1}}{((1-x)(1-y)+\sigma\tau x(1-y)+\sigma(1-\tau)y(1-x))^{s+1}}\, d\sigma\, d\tau\, dx\, dy.$$

To get hands on large p, q asymptotics, we make one more change of variable from x, y, σ to X, U, S via

$$x = e^{-XU/q}, \quad y = e^{-(1-X)U/p}, \quad \sigma = e^{-S/(p+q)},$$
$$dx\, dy\, d\sigma = \frac{xy\sigma U}{pq(p+q)}\, dX\, dU\, dS,$$

and also find it convenient to multiply both sides by $s+1$ and to introduce the function $G(w) := \frac{1-e^{-w}}{w}$. This leads to

$$2(s+1)c_{pq}(s) = \Gamma\Big(\begin{matrix} p+q+n, s+2, p+n, q+n \\ n, n, p, p, q, q, n+s+1 \end{matrix}\Big) \int_0^\infty \int_0^\infty \int_0^1 \int_0^1$$
$$\times e^{-U-S}\Big[\frac{XU}{q}\frac{(1-X)U}{p}\frac{S}{p+q}G\Big(\frac{XU}{q}\Big)G\Big(\frac{(1-X)U}{p}\Big)G\Big(\frac{S}{p+q}\Big)\Big]^{n+s}\tau^{p-1}(1-\tau)^{q-1}$$
$$\times \Big[\frac{XU}{q}\frac{(1-X)U}{p}G\Big(\frac{XU}{q}\Big)G\Big(\frac{(1-X)U}{p}\Big) + \tau e^{-\frac{S}{p+q}-\frac{XU}{q}}\frac{(1-X)U}{p}G\Big(\frac{(1-X)U}{p}\Big)$$
$$+ (1-\tau)e^{-\frac{S}{p+q}-\frac{(1-X)U}{p}}\frac{XU}{q}G\Big(\frac{XU}{q}\Big)\Big]^{-s-1}\frac{U\, d\tau\, dX\, dU\, dS}{pq(p+q)}.$$

Now we specialize to the situation of the theorem, i.e. take $p = P\lambda$, $q = (1-P)\lambda$, with $0 < P < 1$ fixed and $\lambda \to +\infty$. This yields the huge formula

$$
\begin{aligned}
2(s+1)c_{pq}(s) = \Gamma\binom{p+q+n, s+2, p+n, q+n}{n, n, p, p, q, q, n+s+1} \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \\
e^{-U-S} \frac{X^{n+s}(1-X)^{n+s} S^{n+s}}{[P(1-P)\lambda^3]^{n+s+1}} \Big[G\Big(\frac{U_1}{\lambda}\Big) G\Big(\frac{U_2}{\lambda}\Big) G\Big(\frac{S}{\lambda}\Big)\Big]^{n+s} \\
\frac{(\tau^P(1-\tau)^{1-P})^\lambda / (\tau(1-\tau)) U^{2n+2s+1} \lambda^{s+1} \, d\tau \, dX \, dU \, dS}{\Big[\frac{U_1 U_2}{\lambda} G\Big(\frac{U_1}{\lambda}\Big) G\Big(\frac{U_2}{\lambda}\Big) + \tau e^{-\frac{S}{\lambda} - \frac{U_1}{\lambda}} U_2 G\Big(\frac{U_2}{\lambda}\Big) + (1-\tau) e^{-\frac{S}{\lambda} - \frac{U_2}{\lambda}} U_1 G\Big(\frac{U_1}{\lambda}\Big)\Big]^{s+1}},
\end{aligned}
\tag{73}
$$

where for typographical reasons we have momentarily introduced the notations $U_1 := \frac{XU}{1-P}$, $U_2 = \frac{(1-X)U}{P}$. Let g_k momentarily stand for the Taylor coefficients of the (entire) function $G(w)^{n+s}$:

$$
G(w)^{n+s} =: \sum_{k=0}^\infty g_k w^k, \qquad g_0 = 1.
$$

Then

$$
\Big[G\Big(\frac{U_1}{\lambda}\Big) G\Big(\frac{U_2}{\lambda}\Big) G\Big(\frac{S}{\lambda}\Big)\Big]^{n+s} = \sum_{j,k,l} g_j g_k g_l \frac{U_1^j U_2^k S^l}{\lambda^{j+k+l}}. \tag{74}
$$

Also,

$$
\begin{aligned}
&\frac{U_1 U_2}{\lambda} G\Big(\frac{U_1}{\lambda}\Big) G\Big(\frac{U_2}{\lambda}\Big) + \tau e^{-\frac{S}{\lambda} - \frac{U_1}{\lambda}} U_2 G\Big(\frac{U_2}{\lambda}\Big) + (1-\tau) e^{-\frac{S}{\lambda} - \frac{U_2}{\lambda}} U_1 G\Big(\frac{U_1}{\lambda}\Big) \\
&= \Big[\tau U_2 + (1-\tau) U_1\Big] \Big[1 + \sum_{m=1}^\infty \lambda^{-m} \frac{p_{m+1}(U_1, U_2, S, \tau)}{\tau U_2 + (1-\tau) U_1}\Big],
\end{aligned}
\tag{75}
$$

where p_{m+1} is a polynomial homogeneous of degree $m+1$ in U_1, U_2, S and of degree 1 in τ. Feeding (74) and (75) into (73), we arrive at

$$
\begin{aligned}
2(s+1)c_{pq}(s) = \Gamma\binom{p+q+n, s+2, p+n, q+n}{n, n, p, p, q, q, n+s+1} \int_0^\infty \int_0^\infty \int_0^1 \int_0^1 \\
\frac{e^{-U-S} [X(1-X)S]^{n+s} U^{2n+2s+1}}{[P(1-P)]^{n+s+1} \lambda^{3n+2s+2} \tau(1-\tau) [\tau U_2 + (1-\tau) U_1]^{s+1}} \\
\sum_{j=0}^\infty \frac{a_{2j}(U_1, U_2, S, \tau)}{\lambda^j ((1-\tau) U_1 + \tau U_2)^j} (\tau^P (1-\tau)^{1-P})^\lambda \, d\tau \, dX \, dU \, dS,
\end{aligned}
\tag{76}
$$

where a_{2j} is a polynomial homogeneous of degree $2j$ in U_1, U_2, S and of degree at most j in τ, $a_0 \equiv 1$. Carrying out the X, S and U integrations in (76) already produces the desired asymptotic expansion in decreasing powers of λ; it only remains to treat the τ integral. To do that, we recall the standard asymptotics of a Laplace integral, see e.g. [Fe, §2.1, Theorem 1.3]: if $f, \mathcal{S}$ are smooth real-valued functions on a finite interval $[a, b]$, and $\mathcal{S}$ attains its maximum at a unique point $\tau_0 \in (a, b)$, with $\mathcal{S}''(\tau_0) \neq 0$, then the Laplace integral

$$
\mathcal{I}(\lambda) = \int_a^b f(\tau) e^{\lambda \mathcal{S}(\tau)} \, d\tau
$$

possesses the following asymptotics as $\lambda \to +\infty$:

$$\mathcal{I}(\lambda) \approx e^{\lambda \mathcal{S}(\tau_0)} \sum_{k=0}^{\infty} c_k \lambda^{-k-\frac{1}{2}},$$

with

$$c_k = \frac{\Gamma(k+\frac{1}{2})}{(2k)!} \frac{d^{2k}}{d\tau^{2k}} [f(\tau)\mathcal{S}(\tau,\tau_0)^{-k-\frac{1}{2}}]_{\tau=\tau_0},$$

where $\mathcal{S}(\tau,\tau_0) := \frac{\mathcal{S}(\tau_0)-\mathcal{S}(\tau)}{(\tau-\tau_0)^2/2}$; in particular,

$$c_0 = f(\tau_0)\sqrt{\frac{2\pi}{-\mathcal{S}''(\tau_0)}}.$$

Applying this to (76), taking for $\mathcal{S}$ the function $\mathcal{S}(\tau) = P\log\tau + (1-P)\log(1-\tau)$, with $\tau_0 = P$ (so, in particular, $\mathcal{S}(\tau,\tau_0) = \sum_{j=0}^{\infty} \frac{2(\tau-P)^j}{j+2}((1-P)^{-j-1} - (-P)^{-j-1})$), and for f all the remaining terms of the integrand, we get

$$\begin{aligned}
2(s+1)c_{pq}(s) &= \Gamma\binom{p+q+n, s+2, p+n, q+n}{n,n,p,p,q,q,n+s+1} (P^P(1-P)^{(1-P)})^\lambda \\
&\quad \int_0^\infty\int_0^\infty\int_0^1 \frac{e^{-U-S}[X(1-X)S]^{n+s}U^{2n+2s+1}}{[P(1-P)]^{n+s+1}\lambda^{3n+2s+2}} \\
&\quad \sum_{j,k=0}^{\infty} \Gamma\binom{k+\frac{1}{2}}{2k+1} \frac{d^{2k}}{d\tau^{2k}} \Big[\frac{a_{2j}(U_1,U_2,S,\tau)}{\tau(1-\tau)[(1-\tau)U_1+\tau U_2]^{j+s+1}} \mathcal{S}(\tau,P)^{-k-\frac{1}{2}}\Big]_{\tau=P} \lambda^{-k-\frac{1}{2}-j}\,dX\,dU\,dS \\
&= \Gamma\binom{p+q+n, s+2, p+n, q+n}{n,n,p,p,q,q,n+s+1} \sqrt{\pi}(P^P(1-P)^{(1-P)})^\lambda \\
&\quad \sum_{j,k=0}^{\infty} \lambda^{-3n-2s-\frac{5}{2}-j-k} \int_0^\infty\int_0^\infty\int_0^1 \frac{e^{-U-S}[X(1-X)S]^{n+s}U^{2n+s-j}}{[P(1-P)]^{n+s+\frac{3}{2}}} \Phi_{jk}(X,\tfrac{1}{P},\tfrac{1}{1-P},U,S)\,dX\,dU\,dS
\end{aligned}$$

with some polynomials Φ_{jk} in the indicated variables, $\Phi_{00} \equiv 1$. Carrying out the S, U and X integrations, we obtain

$$\begin{aligned}
2(s+1)c_{pq}(s) &= \Gamma\binom{p+q+n, s+2, p+n, q+n}{n,n,p,p,q,q,n+s+1} \frac{\sqrt{\pi}(P^P(1-P)^{1-P})^\lambda}{[P(1-P)]^{n+s+\frac{3}{2}}} \\
&\qquad \lambda^{-3n-2s-\frac{5}{2}} \sum_{m=0}^{\infty} b_m(\tfrac{1}{P},\tfrac{1}{1-P})\lambda^{-m},
\end{aligned} \tag{77}$$

with some polynomials b_m, $b_0 \equiv \frac{\Gamma(n+s+1)^3\Gamma(2n+s+1)}{\Gamma(2n+2s+2)}$. (All of p_{m+1}, a_{2j}, Φ_{jk} and b_m depends also on s, although this is not explicitly reflected by the notation.) Now we employ the fact that $2(s+1)c_{pq}(s) \to 1$ as $s \searrow 1$ (since the measures $(s+1)(1-t)^s\,dt$ converge weakly to the Dirac mass at $t=1$); dividing both sides by (77) by the same expressions with $s=-1$, and restoring the variables $p = P\lambda$, $q = (1-P)\lambda$, we finally arrive at

$$(s+1)c_{pq}(s) \approx \Gamma\binom{2n+s+1, n+s+1, n+s+1, s+2}{n,n,2n+2s+2} (pq)^{-s-1}\Big[1+\sum_{j=1}^{\infty} A_j(p,q)\Big],$$

with some functions A_j homogeneous of degree $-j$. But this is precisely (72), completing the proof. □

Combining the last theorem with (17) from the Introduction, we see that $c_{pq}(s) \approx (p+1)^{-s-1}(q+1)^{-s-1}$ as $p+q \to +\infty$ (for fixed s). As argued in the beginning of this section, the "leading order" term of $K_s(z,w)$ can thus be expected to be the same as for the function

$$F_s(r\zeta, R\eta) := \sum_{p,q} S^{pq}(r) S^{pq}(R) H^{pq}(\zeta\cdot\overline{\eta})(p+\tfrac{n-1}{2})^{s+1}(q+\tfrac{n-1}{2})^{s+1}. \tag{78}$$

We conclude this section by discussing the boundary singularity of F_s.

Theorem 12. *For $n>1$ and $s=0,1,2,\dots$*
(79)

$$F_s(z,w) = \mathcal{L}^{s+1}\Big[\frac{\Gamma(n)}{2\pi^n}\frac{(1-y_2)^n}{(1-x_2)^n(1-y_1)^n}\sum_{i_1=0}^{n}\sum_{i_2,j_1=0}^{n-i_1}\frac{(-n)_{i_1+i_2}(-n)_{i_1+j_1}(n)_{i_2}(n)_{j_1}}{i_1!i_2!j_1!(n)_{i_1+i_2+j_1}}$$

$$x_1^{i_1}\Big(\frac{x_1-y_1}{1-y_1}\Big)^{i_2}\Big(\frac{x_1-x_2}{1-x_2}\Big)^{j_1}{}_2F_1\Big(\begin{matrix}i_2+n, j_1+n\\ i_1+i_2+j_1+n\end{matrix}\Big|1-\frac{(1-x_1)(1-y_2)}{(1-x_2)(1-y_1)}\Big)\Big]\Big|_{\substack{x_1=|z|^2, x_2=\langle z,w\rangle,\\ y_1=\langle w,z\rangle, y_2=|w|^2}},$$

where $\mathcal{L}$ is the linear differential operator

$$\mathcal{L} := (x_2y_1 - x_1y_2)\frac{\partial^2}{\partial x_2\partial y_1} + \frac{n-1}{2}\Big(x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1}\Big) + \frac{(n-1)^2}{4}I. \tag{80}$$

Proof. Consider the differential operator

$$D := -\frac{\Delta_{\mathrm{sph}}}{4} - \frac{\mathcal{R}^2}{4} + \frac{(n-1)^2}{4}I.$$

By (70),

$$D|\mathbf{H}^{pq} = (p+\tfrac{n-1}{2})(q+\tfrac{n-1}{2})I|\mathbf{H}^{pq}.$$

Consequently, using (30),

$$F_s(z,w) = D^{s+1}K_{\mathrm{Sz}}(z,w), \tag{81}$$

with the understanding that, to fix ideas, D is always applied to the z variable. Note that since Δ_{sph} and $\mathcal{R}$ are "tangential" operators — that is, they act only on the ζ variable in the polar coordinates $z=r\zeta$ — we have $D(fg)=fDg$ for any function f depending on $|z|$ only. Substituting (12) for the K_{Sz} in (81), we thus obtain

$$F_s(z,w) = \frac{\Gamma(n)}{2\pi^n}(1-|z|^2)^n(1-|w|^2)^n\sum_{p,q,j,m=0}^{\infty}\frac{(n)_{p+j}(n)_{q+j}(n)_{p+m}(n)_{q+m}}{(n)_{p+q+j+m}}$$
$$\times D^{s+1}\frac{\langle z,w\rangle^p\langle w,z\rangle^q|z|^{2j}|w|^{2m}}{p!q!j!m!}.$$

Now by direct computation

$$D[\langle z,w\rangle^p\langle w,z\rangle^q|z|^{2j}|w|^{2m}]$$
$$= \Big[\Big(1-\frac{|z|^2|w|^2}{|\langle z,w\rangle|^2}\Big)pq + \frac{n-1}{2}(p+q) + \Big(\frac{n-1}{2}\Big)^2\Big]\langle z,w\rangle^p\langle w,z\rangle^q|z|^{2j}|w|^{2m}$$
$$= \mathcal{L}[x_1^j x_2^p y_1^q y_2^m]_{x_1=|z|^2, x_2=\langle z,w\rangle, y_1=\langle w,z\rangle, y_2=|w|^2}.$$

Hence

$$F_s(z,w) = \mathcal{L}^{s+1}\Big[\frac{\Gamma(n)}{2\pi^n}(1-x_1)^n(1-y_2)^n FD_1\Big(\begin{matrix}n,n,n,n\\ n\end{matrix}\Big|x_1,x_2,y_1,y_2\Big)\Big]_{\substack{x_1=|z|^2, x_2=\langle z,w\rangle,\\ y_1=\langle w,z\rangle, y_2=|w|^2}}.$$

Replacing the expression in the square brackets by the one from Corollary 4 yields the desired claim. $\square$

Corollary 13. *For $n > 1$ and $s = 0, 1, 2, \ldots,$ $F_s \in C^{n-1}(\overline{\mathbf{B}^n \times \mathbf{B}^n} \setminus \operatorname{diag} \partial \mathbf{B}^n)$.*

Proof. We have seen in course of the proof of Proposition 7, cf. (59), that

$$K_{\mathrm{Sz}}(z,w) = \sum_{j,k=0}^{\infty} a_{jk}(Q,\overline{Q})(1-x_1)^j(1-y_2)^k + \sum_{j,k=0}^{\infty} b_{jk}(Q,\overline{Q})(1-x_1)^j(1-y_2)^k \log[(1-x_1)(1-y_2)], \tag{82}$$

where we have again set $x_1 = |z|^2$, $x_2 = \langle z, w\rangle$, $y_1 = \langle w, z\rangle$, $y_2 = |w|^2$ and $Q := 1/(1-x_2)$, $\overline{Q} = 1/(1-y_1)$, and a_{jk}, b_{jk} are holomorphic functions of $Q, \overline{Q}$ in the right half-plane $\operatorname{Re} Q > 0$. Now since $\mathcal{L}$ involves only differentiations with respect to the x_2 and y_1 variables, the application of $\mathcal{L}^{s+1}$ to (82) preserves this form of the right-hand side (only the functions a_{jk}, b_{jk} will get changed). Since, again as in the proof of Proposition 7, the first summand is C^∞ on $|1-x_1| < 1$, $|1-y_2| < 1$ and $\operatorname{Re} Q > 0$, while the second summand is C^{n-1} there, the claim follows. $\square$

For $z, w \in \mathbf{B}^n$, the three quantities

$$x_1 = |z|^2, \quad x_2 = \overline{y}_1 = \langle z, w\rangle, \quad y_2 = |w|^2,$$

are preserved upon replacing z, w by Uz, Uw with any $U \in U(n)$, and satisfy

$$0 \le x_1, y_2 < 1, \qquad x_2 \in \mathbf{C},\ |x_2|^2 \le x_1 y_2. \tag{83}$$

Conversely, all triples x_1, x_2, y_2 satisfying (83) arise in this way, and if $z', w' \in \mathbf{B}^n$ give rise to the same triple x_1, x_2, y_2 as z, w, then there is some $U \in U(n)$ for which $z' = Uz$ and $w' = Uw$. In other words, the map $(z, w) \mapsto (|z|^2, \langle z, w\rangle, |w|^2)$ is a bijection of the equivalence classes of $\mathbf{B}^n \times \mathbf{B}^n$ modulo the diagonal action of $U(n)$ onto the set of all triples x_1, x_2, y_2 satisfying (83).

Now instead of z, w, we can apply the observation in the preceding paragraph to the pair $z, \phi_z w$. Since

$$\phi_{Uz}(Uw) = U(\phi_z w), \qquad \forall z, w \in \mathbf{B}^n,\ U \in U(n),$$

it again transpires that the map

$$(z, w) \mapsto (U, V, Z), \qquad U := |z|^2,\ V := |\phi_z w|^2,\ Z := \langle z, \phi_z w\rangle, \tag{84}$$

is a bijection of the equivalence classes of $\mathbf{B}^n \times \mathbf{B}^n$ modulo the diagonal action of $U(n)$ onto the set

$$\Omega := \{(U, V, Z) : 0 \le U, V < 1,\ Z \in \mathbf{C},\ |Z|^2 \le UV\}. \tag{85}$$

We conclude by expressing the singularity of $F_s(z, w)$ at the boundary diagonal in terms of the "coordinates" (U, V, Z).

Corollary 14. *For $n > 1$ and $s = 0, 1, 2, \ldots,$*

$$F_s(z,w) = \frac{(1-V)^n}{(1-U)^{n+s+1}} \sum_{i_1=0}^{n} \sum_{i_2, j_1=0}^{n-i_1} \sum_{k=0}^{s+1} P_{i_1 i_2 j_1 k}(U, Z, \overline{Z}, V)\, {}_2F_1\left(\begin{matrix} i_2+m+k, j_1+n+k \\ i_1+i_2+j_1+n+k \end{matrix} \middle| V \right),$$

where $P_{i_1 i_2 j_1 k}(U, Z, \overline{Z}, V)$ is a polynomial of degree at most $i_1 + s + 1$, $j_1 + s + 1$, $i_2 + s + 1$ and $k + s + 1$, respectively, in the indicated variables.

Proof. We still keep the previous notation $x_1 = |z|^2$, $x_2 = \langle z, w\rangle$, $y_1 = \langle w, z\rangle$, $y_2 = |w|^2$. Taking 0 and w for w_1 and w_2, respectively, in (19), we get

$$Z = \langle \phi_z 0, \phi_z w\rangle = 1 - \frac{1-|z|^2}{1-\langle z,w\rangle} = \frac{x_1 - x_2}{1-x_2},$$

and similarly

$$\overline{Z} = \frac{x_1 - y_1}{1-y_1}, \qquad V = 1 - \frac{(1-x_1)(1-y_2)}{(1-x_2)(1-y_1)}$$

(which establishes (20)). In terms of U, V, Z, (54) therefore becomes simply

$$K_{\mathrm{Sz}}(z,w) = \frac{(1-V)^n}{(1-U)^n} \sum_{\substack{i_1+i_2\le n,\\ i_1+j_1\le n}} a_{i_1 i_2 j_1} U^{i_1} \overline{Z}^{i_2} Z^{j_1} {}_2F_1\Big(\begin{matrix} i_2+n, j_1+n \\ i_1+i_2+j_1+n \end{matrix}\Big| V\Big),$$

with $a_{i_1 i_2 j_1} := \frac{\Gamma(n)}{2\pi^n} \frac{(-n)_{i_1+i_2}(-n)_{i_1+j_1}(n)_{i_2}(n)_{j_1}}{i_1! i_2! j_1! (n)_{i_1+i_2+j_1}}$. On the other hand, by a tedious but routine computation, the operator $\mathcal{L}$ expressed in the coordinates U, V, Z takes the form

$$\mathcal{L} = \frac{|Z|^2 - UV}{1-U}[|1-Z|^2 \partial_Z \partial_{\overline{Z}} + (1-\overline{Z})(1-V)\partial_{\overline{Z}}\partial_V + (1-Z)(1-V)\partial_Z\partial_V + (1-V)^2\partial_V^2 - (1-V)\partial_V]$$
$$- \frac{n-1}{2(1-U)}[(1-Z)(U-Z)\partial_Z + (1-\overline{Z})(U-\overline{Z})\partial_{\overline{Z}} + (1-V)(2U-Z-\overline{Z})\partial_V] + \Big(\frac{n-1}{2}\Big)^2 I.$$

Combining these two facts with the formula

$$\partial_V\, {}_2F_1\Big(\begin{matrix} a,b \\ c \end{matrix}\Big| V\Big) = \frac{ab}{c}\, {}_2F_1\Big(\begin{matrix} a+1, b+1 \\ c+1 \end{matrix}\Big| V\Big) \tag{86}$$

for the derivative of a hypergeometric function, the claim follows by a simple induction argument. □

The advantage of the coordinates U, V, Z is that when (z,w) approaches the boundary diagonal — that is, essentially, when both z and w approach the same point $\zeta \in \partial\mathbf{B}^n$ — then of course all of $x_1 = |z|^2$, $x_2 = \langle z,w\rangle$, $y_1 = \langle w,z\rangle$ and $y_2 = |w|^2$ tend to 1, but $|\phi_z w|^2 = V$ can behave in many ways: or instance, for $z = w$ one has $V = 0$, while e.g. for $z = (1-h)e_1$, $w = (1-h^2)e_1$ one has $V \to 1$ as $h \searrow 0$. Thus the coordinates U, V, Z capture how z approaches ζ "relative" to w. Of course, the downside of the formula in the last corollary is that it completely hides the symmetry $z \leftrightarrow w$. At the moment, we do not know how to express the boundary singularity of F_s in a manner that would make this symmetry evident.

6. Concluding remarks

6.1. **Formulas for c_{pq}.** The coefficients $c_{pq}(s)$ can be expressed in terms of various multivariable hypergeometric functions. One such expression comes from using the formula for the Taylor coefficients of the square of a ${}_2F_1$ function [BE, §4.3(14)]

$${}_2F_1\Big(\begin{matrix} a,b \\ c \end{matrix}\Big| z\Big)^2 = \sum_{m=0}^{\infty} {}_4F_3\Big(\begin{matrix} a,b,1-c-m,-m \\ c,1-a-m,1-b-m \end{matrix}\Big| 1\Big) \frac{(a)_m (b)_m}{(c)_m m!} z^m,$$

yielding

$$\begin{aligned}2c_{pq}(s) = \Gamma\binom{n+p, n+q}{n, n+p+q}^2 \sum_{m=0}^{\infty} {}_4F_3\left(\left.\begin{matrix} p, q, 1-n-p-q-m, -m \\ n+p+q, 1-p-m, 1-q-m \end{matrix}\right| 1\right) \\ \times \frac{(p)_m (q)_m}{(n+p+q)_m m!} \Gamma\binom{n+p+q+m, s+1}{n+p+q+m+s+1},\end{aligned} \tag{87}$$

which however is not very useful. A somewhat nicer expression is obtained upon expanding both ${}_2F_1$ factors in (67) into Taylor series and integrating term by term; this gives

(88)

$$\begin{aligned}2c_{pq}(s) &= \Gamma\binom{n+p, n+q}{n, n+p+q}^2 \sum_{j,k=0}^{\infty} \frac{(p)_j (q)_j}{(n+p+q)_j j!} \frac{(p)_k (q)_k}{(n+p+q)_k k!} \Gamma\binom{n+p+q+j+k, s+1}{n+p+q+j+k+s+1} \\ &= \Gamma\binom{n+p, n+q}{n, n+p+q}^2 \Gamma\binom{n+p+q, s+1}{n+p+q+s+1} \\ &\qquad \times \sum_{j,k=0}^{\infty} \frac{(p)_j (q)_j}{(n+p+q)_j j!} \frac{(p)_k (q)_k}{(n+p+q)_k k!} \frac{(n+p+q)_{j+k}}{(n+p+q+s+1)_{j+k}}.\end{aligned}$$

In terms of the higher order hypergeometric function of two variables of Appell and Kampé de Fériet [AK, p. 150]

$$F\left(\begin{matrix}\mu \\ \nu \\ \rho \\ \sigma\end{matrix}\left|\begin{matrix} a_1, \dots, a_\mu \\ b_1, b_1', \dots, b_\nu, b_\nu' \\ c_1, \dots, c_\rho \\ d_1, d_1', \dots, d_\sigma, d_\sigma' \end{matrix}\right| x, y\right) := \sum_{j,k=0}^{\infty} \frac{\prod_{i=1}^{\mu} (a_i)_{j+k}}{\prod_{i=1}^{\rho} (c_i)_{j+k}} \frac{\prod_{i=1}^{\nu} (b_i)_j (b_i')_k}{\prod_{i=1}^{\sigma} (d_i)_j (d_i')_k} \frac{x^j y^k}{j! k!}$$

this becomes

(89)

$$2c_{pq}(s) = \Gamma\binom{n+p, n+p, n+q, n+q, s+1}{n, n, n+p+q, n+p+q+s+1} F\left(\begin{matrix}1 \\ 2 \\ 1 \\ 1\end{matrix}\left|\begin{matrix} n+p+q \\ p, p, q, q \\ n+p+q+s+1 \\ n+p+q, n+p+q \end{matrix}\right| 1, 1\right).$$

In the notation of Karlsson and Srivastava [KS, p. 27], the last function is denoted $F^{1:2,2}_{1:1,1}\left(\begin{matrix} p+q+n & : & p, q; p, q; \\ p+q+n+s+1 & : & p+q+n; p+q+n; \end{matrix} 1, 1\right)$. However, an explicit expression for the value of these seems again not to be available. In fact, our computations at the end of Section 4 amount to an evaluation of (89) for the special case of $n = 2$, $s = 0$ and $p = q = 1$.

6.2. **Uniformity of asymptotic expansions.** In Theorem 11, we have found the asymptotics of $c_{\lambda P, \lambda(1-P)}(s)$ as $\lambda \to +\infty$, for fixed s and fixed $0 < P < 1$; also, the case of $P \in \{0, 1\}$ has been handled separately. We expect that the asymptotic expansion obtained is actually uniform in $P \in [0, 1]$, and hence in fact yields the unrestricted asymptotics of $c_{pq}(s)$ as $p + q \to +\infty$ (with s fixed); however, at the moment we can offer no proof that this is indeed the case. This kind of difficulty arises also in other situations where two-parameter asymptotics are involved, cf. e.g. §II.7 in [Fe].

6.3. **The Wallach set.** In the holomorphic case, the weighted Bergman kernels $K_s^{\rm hol}(z,w)$, $s>-1$, on $\mathbf{B}^n$ actually extend by analytic continuation to a meromorphic function of $s\in\mathbf{C}$, and continue to be positive defnite functions on $\mathbf{B}^n\times\mathbf{B}^n$ for all $s\in(-n-1,+\infty)$; the last interval is called the *Wallach set* of $\mathbf{B}^n$, and there is a similar story for $\mathbf{B}^n$ replaced by any irreducible bounded symmetric domain in $\mathbf{C}^n$ [RV]. More precisely, if one first normalizes the measures $(1-|z|^2)^s\,dz$ on $\mathbf{B}^n$ to be of total mass one — that is, in other words, multiplies them by $1/c_{00}(s)$ — then the normalized reproducing kernels

$$c_{00}(s)K_s^{\rm hol}(z,w)=(1-\langle z,w\rangle)^{-n-1-s} \tag{90}$$

extend to a holomorphic function of $z,\overline{w}\in\mathbf{B}^n$ and $s\in\mathbf{C}$, and this analytic continuation is still a positive definite kernel in (z,w) on $\mathbf{B}^n\times\mathbf{B}^n$ as long as $s>-n-1$ (and only for these s).

Similar phenomenon prevails for the harmonic case [E2]. We show that the M-harmonic case is, likewise, no exception.

Proposition 15. *The normalized M-harmonic weighted Bergman kernels*

$$c_{00}(s)K_s(z,w)$$

extend by analytic continuation in s to a meromorphic function on $\mathbf{B}^n\times\mathbf{B}^n\times\mathbf{C}$, and remain positive definite in (z,w) as long as $s>-n-1$ (and only for such s).

Thus, the "M-harmonic Wallach set" for $\mathbf{B}^n$ is the interval $(-n-1,+\infty)$.

Proof. By (63), we have

$$c_{00}(s)K_s(r\zeta,R\eta)=\sum_{p,q}\frac{c_{00}(s)}{c_{pq}(s)}S^{pq}(r)S^{pq}(R)H^{pq}(\zeta\cdot\overline{\eta}). \tag{91}$$

It is therefore enough to exhibit an analytic continuation of the functions $f_{pq}(s):=\frac{c_{00}(s)}{c_{pq}(s)}$ in s for all p,q, and show that the "Wallach set"

$$\mathcal{W}:=\{s\in\mathbf{C}:0<f_{pq}(s)<+\infty\ \forall p,q\geq0\}$$

coincides with the interval $(-n-1,+\infty)$. A routine convergence check then yields the desired analytic continuation of (91), and we are done.

First of all, from either (87) or (88) it is immediate that $c_{pq}(s)$ and, hence, $f_{pq}(s)$ extend meromorphically to all $s\in\mathbf{C}$ (because the Gamma function does).

Next, we have already seen in (68) that

$$2c_{00}(s)=\Gamma\binom{n,s+1}{n+s+1}=\frac{\Gamma(n)}{(s+1)_n}.$$

Similarly

$$2c_{10}(s)=\Gamma\binom{n+1,s+1}{n+s+2}.$$

Hence

$$f_{10}(s)=\frac{c_{00}(s)}{c_{10}(s)}=\Gamma\binom{n,s+1,n+s+2}{n+s+1,n+1,s+1}=\frac{n+s+1}{n}.$$

This is positive only for $n+s+1>0$, so $\mathcal{W}\subset(-n-1,+\infty)$.

Let us momentarily denote

$$G_{pq}(t):=\Gamma\binom{n+p,n+q}{n,n+p+q}^2 t^{p+q+n-1}{}_2F_1\left(\begin{matrix}p,q\\n+p+q\end{matrix}\middle|t\right)^2, \tag{92}$$

so that

$$2c_{pq}(s) = \int_0^1 G_{pq}(t)(1-t)^s\,dt, \qquad \operatorname{Re} s > -1. \tag{93}$$

Integrating by parts n times, we obtain

$$2c_{pq}(s) = \sum_{j=0}^{n-1} \frac{[-G_{pq}^{(j)}(t)(1-t)^{s+j+1}]_{t=0}^{t=1}}{(s+1)_{j+1}} + \frac{1}{(s+1)_n}\int_0^1 G_{pq}^{(n)}(t)(1-t)^{s+n}\,dt, \tag{94}$$

provided all the terms are finite. Now by (92), the derivative $G_{pq}^{(k)}(t)$ is continuous up to $t = 1$ for $k < n$, while $G_{pq}^{(n)}(t) \approx \log\frac{1}{1-t}$. Also, $G_{pq}^{(j)}(0) = 0$ whenever $j < p+q+n-1$. Thus all the terms in the sum in (94) actually vanish for $s > -n-1$, and we thus get for any $(p,q) \neq (0,0)$ and $s > -n-1$ the representation

$$2c_{pq}(s) = \frac{1}{(s+1)_n}\int_0^1 G_{pq}^{(n)}(t)(1-t)^{s+n}\,dt$$

and

$$\frac{c_{pq}(s)}{c_{00}(s)} = \frac{1}{\Gamma(n)}\int_0^1 G_{pq}^{(n)}(t)(1-t)^{s+n}\,dt. \tag{95}$$

Furthermore, in view of (86), all derivatives of G_{pq} are positive on $(0,1)$, hence the integrand in (95) is positive, and $f_{pq}(s)$ is positive and finite, for all $s > -n-1$. Consequently, $(-n-1,+\infty) \subset \mathcal{W}$, completing the proof. $\square$

6.4. **Semiclassical asymptotics.** In the holomorphic case, the behavior of $K_s(z,w)$ as $s \to +\infty$ is of importance in certain quantization procedures [E1]; the weight parameter s plays the role of the reciprocal of the Planck constant, and one therefore speaks of "semiclassical" limits. The following result is again the same as for the holomorphic, as well as for some harmonic [E3], situations.

Proposition 16. *For all $z \in \mathbf{B}^n$,*

$$\lim_{s\to+\infty} K_s(z,z)^{1/s} = (1-|z|^2)^{-1}.$$

Proof. From the definition of the reproducing kernel $K_{\mathcal{H}}(z,z)$ as the norm square of the evaluation functional,

$$K_{\mathcal{H}}(z,z) = \sup\{|f(z)|^2 : f \in \mathcal{H},\ \|f\|_{\mathcal{H}} \leq 1\}, \tag{96}$$

and the fact that holomorphic functions are M-harmonic it follows that

$$K_s(z,z) \geq K_s^{\mathrm{hol}}(z,z).$$

Hence

$$\liminf_{s\to+\infty} K_s(z,z)^{1/s} \geq \liminf_{s\to+\infty} K_s^{\mathrm{hol}}(z,z)^{1/s} = (1-|z|^2)^{-1}, \tag{97}$$

by the known result for the holomorphic case (cf. (90)).

On the other hand, by the invariant mean value property of M-harmonic functions, we have for any M-harmonic f and $0 < r < 1$

$$\begin{aligned} f(z) = (f\circ\phi_z)(0) &= \frac{n!}{\pi^n r^{2n}}\int_{|x|<r}(f\circ\phi_z)(x)\,dx \\ &= \frac{n!}{\pi^n r^{2n}}\int_{|\phi_z y|<r} f(y)\,J_z(y)\,dy, \end{aligned}$$

where J_z is the Jacobian of ϕ_z. By Cauchy-Schwarz,

$$|f(z)| \le \frac{n!}{\pi^n r^{2n}} \Big(\int_{|\phi_z y|<r} |f(y)|^2 (1-|y|^2)^s \, dy \Big)^{1/2} \Big(\int_{|\phi_z y|<r} \frac{J_z(y)^2}{(1-|y|^2)^s} \, dy \Big)^{1/2}$$
$$\le \frac{n!}{\pi^n r^{2n}} \|f\|_s \Big(\int_{|\phi_z y|<r} \frac{J_z(y)^2}{(1-|y|^2)^s} \, dy \Big)^{1/2}.$$

Taking supremum over all f with $\|f\|_s \le 1$, we get by (96)

$$K_s(z,z)^{1/2} \le \frac{n!}{\pi^n r^{2n}} \Big(\int_{|\phi_z y|<r} \frac{J_z(y)^2}{(1-|y|^2)^s} \, dy \Big)^{1/2},$$

whence

$$K_s(z,z)^{1/s} \le \Big(\frac{n!}{\pi^n r^{2n}} \Big)^{2/s} \Big(\int_{|\phi_z y|<r} \frac{J_z(y)^2}{(1-|y|^2)^s} \, dy \Big)^{1/s}.$$

Letting $s \to +\infty$ and using the fact that

$$\|F\|_{L^s(d\mu} \to \|F\|_{L^\infty(d\mu)} \qquad \text{as } s \to +\infty$$

for any finite measure μ and bounded function F, we obtain

$$\limsup_{s\to+\infty} K_s(z,z)^{1/s} \le \sup_{|\phi_z(y)|<r} (1-|y|^2)^{-1}.$$

Finally, since $r \in (0,1)$ was arbitrary, letting $r \searrow 0$ yields

$$\limsup_{s\to+\infty} K_s(z,z)^{1/s} \le (1-|z|^2)^{-1}.$$

Combining this with (97) completes the proof. □

It would be interesting to know what is the limit of $|K_s(z,w)|^{1/s}$ as $s \to +\infty$ for $z \neq w$, or whether there is any asymptotic expansion of $K_s(z,z)(1-|z|^2)^s$ in decreasing powers of s, as is the case for some harmonic Bergman kernels [E3].

6.5. **Particular cases.** For $z \perp w$, the formula (9) for the Szegö kernel $K_{\mathrm{Sz}}(z,w)$ can also be evaluated in a different way: namely, using $U(n)$-invariance, we can assume without loss of generality that $z = |z|e_1$ and $w = |w|e_2$. The integral in (9) then takes the form

$$(1-|z|^2)^n (1-|w|^2)^n \int_{\partial\mathbf{B}^n} |1-|z|\zeta_1|^{-2n} |1-|w|\zeta_2|^{-2n} \, d\sigma(\zeta).$$

Using the binomial expansion for $(1-t)^{-n}$ and integrating term by term shows that the last integral equals

$$\frac{2\pi^n}{\Gamma(n)} \sum_{k,l=0}^{\infty} \frac{(n)_k^2 (n)_l^2}{(n)_{k+l}} \frac{|z|^{2k}|w|^{2l}}{k!l!} = \frac{2\pi^n}{\Gamma(n)} F_3 \Big(\begin{matrix} n,n,n,n \\ n \end{matrix} \Big| |z|^2, |w|^2 \Big), \tag{98}$$

where F_3 is the third Appell hypergeometric function

$$F_3 \Big(\begin{matrix} a,a',b,b' \\ c \end{matrix} \Big| x,y \Big) = \sum_{j,k} \frac{(a)_j (a')_k (b)_j (b')_k}{(c)_{j+k} j!k!} x^j y^k.$$

Naturally, (98) agrees with the formula (12) for $\langle z,w\rangle = \langle w,z\rangle = 0$, as it should.

Another case when $K_{\mathrm{Sz}}(z,w)$ can be evaluated independently is when $z = w$. The integral in (9) then becomes

$$(1-|z|^2)^{2n} \int_{\partial\mathbf{B}^n} |1-\langle z,\zeta\rangle|^{-4n} \, d\sigma(\zeta).$$

Upon substituting again the binomial expansion and integrating term by term, this produces

$$\frac{2\pi^n}{\Gamma(n)}(1-|z|^2)^{2n}\sum_k \frac{(2n)_k^2}{(n)_k}\frac{|z|^{2k}}{k!} = \frac{2\pi^n}{\Gamma(n)}(1-|z|^2)^{2n}{}_2F_1\left(\begin{matrix}2n, 2n\\ n\end{matrix}\middle| |z|^2\right)$$

$$= \frac{2\pi^n}{\Gamma(n)}(1-|z|^2)^{-n}{}_2F_1\left(\begin{matrix}-n, -n\\ n\end{matrix}\middle| |z|^2\right) \tag{99}$$

(note that the last ${}_2F_1$ is actually a polynomial; we have used the Euler transformation formula (51)).

It is entertaining to compare (99) with direct application of the formula (63), where in the latter the Szegö case corresponds, as we have already noted repeatedly, to $c_{pq} \equiv 1\ \forall p, q$. Namely, (63) gives

$$K_{\mathrm{Sz}}(z, z) = \sum_{p,q} S^{pq}(|z|)^2 H^{pq}(1).$$

Now by (24)

$$H^{pq}(1) = \frac{(n+p+q-1)(n+p-2)!(n+q-2)!}{p!q!(n-1)!(n-2)!}.$$

Substituting (27) for S^{pq}, we thus get from (99)

$$\sum_{p,q} t^{p+q}\Gamma\binom{n+p, n+q}{n, n+p+q}^2 {}_2F_1\left(\begin{matrix}p, q\\ n+p+q\end{matrix}\middle| t\right)^2 H^{pq}(1) = (1-t)^{-n}{}_2F_1\left(\begin{matrix}-n, -n\\ n\end{matrix}\middle| t\right),$$

or, using again (51),

$$\sum_{p,q} t^{p+q}\Gamma\binom{n+p, n+q}{n, n+p+q}^2 {}_2F_1\left(\begin{matrix}n+p, n+q\\ n+p+q\end{matrix}\middle| t\right)^2 H^{pq}(1) = {}_2F_1\left(\begin{matrix}2n, 2n\\ n\end{matrix}\middle| t\right).$$

Looking at the coefficients at like powers of t, this is equivalent to

$$\sum_{\substack{p,q,j,k\geq 0\\ p+q+j+k=m}} \Gamma\binom{p+n+j, q+n+j}{n, p+q+n+j}\Gamma\binom{p+n+k, q+n+k}{n, p+q+n+k} H^{pq}(1) = \frac{(2n)_m^2}{m!(n)_m},$$

for all $m = 0, 1, 2, \dots$. We do not know a direct proof of this (valid) formula.

6.6. **Weighted M-harmonic Green functions.** It has been known since the monograph of Bergman and Schiffer [BS] that the harmonic Bergman kernel $H(z, w)$ on a domain is intimately connected with the Green function $G(z, w)$ for the biharmonic operator Δ^2 with Dirichlet boundary conditions: namely,

$$H(z, w) = -\Delta_z\Delta_w G(z, w),$$

where the subscript at Δ indicates the variable that the operator is applied to. An analogous formula holds also for the weighted case. We conclude this paper by pointing out a similar connection in the M-harmonic case.

Let

$$d\tau(z) := (1-|z|^2)^{-n-1}\,dz$$

be the invariant measure on $\mathbf{B}^n$. Given a positive C^∞ weight functions ρ on $\mathbf{B}^n$, consider the "invariant weighted biharmonic" operator

$$\widetilde{\Delta}\rho\widetilde{\Delta}$$

with our $\widetilde{\Delta}$ from (1). By its Green function at a point $w \in \mathbf{B}^n$ we mean, by definition, a function $G(z,w) \equiv G_w(z)$ such that, firstly,

$$u(w) = \int_{\mathbf{B}^n} G_w \widetilde{\Delta}\rho\widetilde{\Delta}u \, d\tau$$

for all smooth functions u whose support is a compact subset of $\mathbf{B}^n$ (that is, $\widetilde{\Delta}\rho\widetilde{\Delta}G_w = \delta_w$, the Dirac point mass at w with respect to $d\tau$, in the sense of distributions); and secondly, both G_w and its normal derivative $\partial G_w/\partial n$ vanish at $\partial\mathbf{B}^n$. (More precisely — they grow sufficiently slowly as $|z| \nearrow 1$; we are skipping some technical details here.) It is then a fact that such G_w exists, is uniquely determined, $G(z,w) = G(w,z)$, and the following proposition holds.

Proposition 17. *The weighted M-harmonic Bergman kernel on $\mathbf{B}^n$ with respect to the measure $\rho(z)^{-1}\,d\tau(z)$ satisfies*

$$K_{d\tau/\rho}(z,w) = -\rho(z)\rho(w)\widetilde{\Delta}_z\widetilde{\Delta}_w G(z,w). \tag{100}$$

We omit the proof, which is the same as for the original harmonic case in [BS], only using the Green formula for $d\tau$ from [Zh, Section 1.6] in the place of the ordinary Green formula.

Choosing, in particular, $\rho(z) = (1-|z|^2)^{-n-s-1}$, the M-harmonic kernels $K_{d\tau/\rho}$ in (100) will be precisely our K_s. One of our earlier ideas how to compute $K_s(z,w)$ was to find $G(z,w)$ first and then apply (100); a possible approach to finding $G(z,w)$ being along the lines of the one used in [EP1] for the ordinary biharmonic Green function on the annulus, and in [EP2] for the unweighted invariant biharmonic Green function on the ball. In both cases, the ordinary decomposition of a function into its Fourier components would need to be replaced by the decomposition (22) into the (p,q) components with respect to the action of $U(n)$. Unfortunately, so far we have not been able to carry out this program.

References

[ABC] P. Ahern, J. Bruna and C. Cascante, *H^p-theory for generalized M-harmonic functions in the unit ball*, Indiana Univ. Math. J. **45** (1996), 103–135.

[AK] P. Appell, J. Kampé de Fériet, *Fonctions hypergéométriques et hypersphériques,* Gauthier-Villars, Paris 1926.

[Ar] N. Aronszajn: *Theory of reproducing kernels,* Trans. Amer. Math. Soc. **68** (1950), 337–404.

[BE] H. Bateman, A. Erdélyi, *Higher transcendental functions, vol. I,* McGraw-Hill, New York - Toronto - London 1953.

[BS] S. Bergman, M. Schiffer, *Kernel functions and elliptic differential equations in mathematica physics,* Academic Press 1953.

[Bk] P. Blaschke: *Berezin transform on harmonic Bergman spaces on the real ball,* J. Math. Anal. Appl. **411** (2014), 607–630.

[E1] M. Engliš: *A Forelli-Rudin construction and asymptotics of weighted Bergman kernels,* J. Funct. Anal. **177** (2000), 257–281.

[E2] M. Engliš: *Analytic continuation of weighted Bergman kernels,* J. Math. Pures Appl. **94** (2010), 622–650.

[E3] M. Engliš: *High-power asymptotics of some weighted harmonic Bergman kernels,* J. Funct. Anal. **271** (2016), 1243–1261.

[EP1] M. Engliš, J. Peetre: *A Green's function for the annulus,* Ann. Mat. Pura Appl. **171** (1996), 313–377.

[EP2] M. Engliš, J. Peetre: *Green's functions for powers of the invariant Laplacian,* Canadian J. Math. **50** (1998), 40–73.

[Ex1] H. Exton, *Multiple hypergeometric functions and applications,* Wiley, 1976.

[Ex2] H. Exton: *On a hypergeometric function of four variables with a new aspect of SL-symmetry,* Ann. Mat. Pura Appl. **161** (1992), 315–343.

[Fe] M.V. Fedoryuk, *Asymptotics, integrals, series* (in Russian), Nauka, Moscow 1987.

[Fo] G.B. Folland: *Spherical harmonic expansion of the Poisson-Szegö kernel for the ball,* Proc. Amer. Math. Soc. **47** (1975), 401–408.

[Ka] P.W. Karlsson, *On certain generalizations of hypergeometric functions of two variables,* Lyngby 1976.

[KS] P.W. Karlsson, H.M. Srivastava, *Multiple Gaussian hypergeometric series,* Wiley 1985.

[Kr1] S.G. Krantz, *Partial differential equations and complex analysis,* CRC Press, 1992.

[RV] H. Rossi, M. Vergne: *Analytic continuation of the holomorphic discrete series of a semi-simple Lie group,* Acta Math. **136** (1976), 1–59.

[Ru] W. Rudin, *Function theory in the unit ball of* $\mathbf{C}^n$*,* Springer, Berlin - Heidelberg 2008.

[St] M. Stoll, *Invariant potential theory in the unit ball of* $\mathbf{C}^n$*,* Cambridge University Press, 1994.

[Zh] K. Zhu, *Spaces of holomorphic functions on the unit ball,* Springer 2005.

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