M-HARMONIC SZEGÖ KERNEL ON THE BALL

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Abstract. We give a description of the boundary singularity of the Szegö kernel of M-harmonic functions, i.e. functions annihilated by the invariant Laplacian, on the unit ball of the complex n-space, in terns of the Gauss hypergeometric functions.

1. Introduction

For a bounded domain $\Omega \subset \mathbf{C}^n$ with smooth boundary, the *Bergman space* $L^2_{\mathrm{hol}}(\Omega) \equiv L^2_{\mathrm{hol}}$ of Ω is the subspace in the standard Lebesgue space $L^2(\Omega)$ of all functions that are holomorphic on Ω. It follows from the mean value property of holomorphic functions that for each $z \in \Omega$, the evaluation functional $f \mapsto f(z)$ is bounded on L^2_{hol}, hence given by the inner product with some fixed element $K_z \in L^2_{\mathrm{hol}}$:

$$f(z) = \langle f, K_z \rangle = \int_\Omega f(w)\overline{K_z(w)}\, dw, \qquad \forall f \in L^2_{\mathrm{hol}}(\Omega).$$

The function of two variables, holomorphic in z and $\overline{w}$,

$$K(z,w) := \langle K_w, K_z \rangle = K_w(z) = \overline{K_z(w)}$$

is known as the *Bergman kernel* of Ω. Following its first appearance in the paper of Bergman one hundred years ago [Be1] (after some precursory earlier observations of the reproducing property e.g. in the work of Zaremba [Za]), and much more prominent treatment in the monograph [Be2] three decades later, the Bergman kernel has since played vital roles in complex analysis of several variables and in complex geometry. One reason for this is its transformation rule under holomorphic mappings: if $\phi : \Omega_1 \to \Omega_2$ is a biholomorphic map, then

$$K_{\Omega_1}(z,w) = J_\phi(z) K_{\Omega_2}(\phi(z), \phi(y)) \overline{J_\phi(w)},$$

where J_ϕ stands for the complex Jacobian of ϕ. It follows that the Hermitian metric

$$ds^2 = \sum_{j,k=1}^n g_{j\overline{k}}(z)\, dz_j\, d\overline{z}_k, \qquad g_{j\overline{k}}(z) := \frac{\partial^2 \log K(z,z)}{\partial z_j \partial \overline{z}_k}, \tag{1}$$

(called the *Bergman metric*) is invariant under biholomorphic maps, which makes it an extremely useful tool for studying the holomorphic equivalence problem in $\mathbf{C}^n$

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(still unresolved to this day). Of particular importance in this connection is the boundary behavior of $K(z,w)$: one has

$$K(z,z) = \frac{a(z)}{\rho(z)^{n+1}} + b(z)\log\rho(z), \qquad \forall z \in \Omega,$$

with some functions a, b smooth on the closure $\overline{\Omega}$ of Ω and $\rho \in C^\infty(\overline{\Omega})$ a *defining function* for Ω, i.e. satisfying $\rho > 0$ on Ω and $\rho = 0 < \|\nabla\rho\|$ on $\partial\Omega$. There is also an off-diagonal version of this formula,

$$K(z,w) = \frac{a(z,w)}{\rho(z,w)^{n+1}} + b(z,w)\log\rho(z,w) \qquad \forall z, w \in \Omega, \tag{2}$$

where $a \in C^\infty(\overline{\Omega\times\Omega})$, $a(z,z) = a(z)$ and $\partial_w a(z,w)$, $\overline{\partial}_z a(z,w)$ vanish to infinite order on the diagonal $z = w$, and similarly for b and ρ. This celebrated result due to Fefferman [Fe], with later different proof by Boutet de Monvel and Sjöstrand [BdS], was an impetus for an overwhelming mass of later developments in complex analysis; see Hirachi and Komatsu [HK] for a nice survey as of 1997.

Most of the above applies verbatim also to the Bergman space replaced by the *Hardy space* of holomorphic functions of Ω which are Poisson extensions of functions in L^2 on the boundary:

$$H^2(\Omega) \equiv H^2 := \{\mathbf{P}f : \ f \in L^2(\partial\Omega) \text{ and } \mathbf{P}f \text{ is holomorphic on } \Omega\},$$

and its reproducing kernel $S(z,w)$ — the *Szegö* kernel of Ω, again holomorphic in z and $\overline{w}$ — satisfying

$$\mathbf{P}f(z) = \int_{\partial\Omega} f(\zeta)S(z,\zeta)\,d\sigma(\zeta), \qquad \forall f \in H^2, \ \forall z \in \Omega;$$

here $L^2(\partial\Omega)$ is taken with respect to some surface measure $d\sigma$ on Ω with smooth density, and $\mathbf{P}$ stands for the Poisson extension operator. (Here we are abusing the notation slightly by denoting by the same letter also the radial boundary values $S(z,\zeta)$ of $S(z,w)$ on $\partial\mathbf{B}^n$.) The boundary behavior of $S(z,w)$ is again given by (2), only with the exponent $n+1$ replaced by n:

$$S(z,w) = \frac{a(z,w)}{\rho(z,w)^{n}} + b(z,w)\log\rho(z,w) \qquad \forall z, w \in \Omega, \tag{3}$$

(the functions a, b being different from the ones in (2), but again smooth on $\overline{\Omega}\times\overline{\Omega}$).

Despite the focus on the (difficult and beautiful) applications in complex analysis in several variables, the whole theory applies also in the context of elliptic boundary value problems on a domain $\Omega \subset \mathbf{R}^n$, and actually the excellent monograph by Bergman and Schiffer [BeS] on this topic appeared shortly after [Be2]. In particular, this applies to the setup where instead of holomorphic one considers *harmonic* functions, leading to the *harmonic Bergman space* $L^2_{\text{harm}}(\Omega) \equiv L^2_{\text{harm}}$ of all harmonic functions in $L^2(\Omega)$, and the associated *harmonic Bergman kernel* $K_{\text{harm}}(z,w)$, which is harmonic in both variables, symmetric in z, w and satisfies

$$f(z) = \int_\Omega f(w)K_{\text{harm}}(z,w)\,dw, \qquad \forall f \in L^2_{\text{harm}}, \ \forall z \in \Omega.$$

The analogue of the Hardy space in this setting is the space of Poisson extensions to Ω of all functions in $L^2(\partial\Omega)$:

$$H^2_{\text{harm}}(\Omega) := \{\mathbf{P}f : f \in L^2(\partial\Omega)\}.$$

(The domain Ω is again assumed to be bounded and with C^∞ boundary, and $L^2(\partial\Omega)$ is taken with respect to some surface measure $d\sigma$ on $\partial\Omega$ with smooth density.) This space again has a reproducing kernel, the *harmonic Szegö kernel* $S_{\rm harm}(z,w)$, which is actually related in a rather simple way to the Poisson kernel of Ω: namely, if $P(z,\zeta)$ stands for the Poisson kernel, so that the Poisson operator $\mathbf{P}$ is given just by

$$\mathbf{P}f(z)=\int_{\partial\Omega} f(\zeta)P(z,\zeta)\,d\sigma(\zeta), \qquad z\in\Omega,$$

then

$$S_{\rm harm}(z,w)=\int_{\partial\Omega} P(z,\zeta)\overline{P(w,\zeta)}\,d\sigma(\zeta). \tag{4}$$

(As the Poisson kernel is real-valued, the complex conjugation is actually superfluous.)

Here the right tool for the study of the boundary behavior of $K_{\rm harm}(z,w)$ and $S_{\rm harm}(z,w)$ turns out to be the so-called calculus of boundary pseudodifferential operators, developed again by Boutet de Monvel [BdM]; see Grubb [Gr] for a recent exposition. The analogue of (2) is

$$K_{\rm harm}(x,y)=|x-\tilde y|^{-n}a\Big(x,y,|x-\tilde y|,\frac{x-\tilde y}{|x-\tilde y|}\Big)+b(x,y)\log|x-\tilde y|$$

for x,y near the boundary, where $b\in C^\infty(\overline\Omega\times\overline\Omega)$, $a\in C^\infty(\overline\Omega\times\overline\Omega\times\overline{\mathbf{R}_+}\times\mathbf{S}^{n-1})$, and $\tilde y$ denotes the "reflection" of y with respect to $\partial\Omega$. There is also an analogous formula for $S_{\rm harm}$ (i.e. the analogue of (3)), only the exponent $-n$ gets replaced by $1-n$. We refer the reader to [Eng] for the details.

Associated to the Hermitian metric (1) on a domain $\Omega\subset\mathbf{C}^n$ is the Laplace-Beltrami operator (or *Bergman Laplacian*)

$$\widetilde\Delta := \sum_{j,k=1}^n g^{\overline{k}j}(z)\frac{\partial^2}{\partial z_j\partial\overline z_k} \tag{5}$$

where $g^{\overline{k}j}$ denotes the inverse matrix to $g_{j\overline{k}}$. The functions annihilated by $\widetilde\Delta$ are called *M-harmonic* (or *invariantly harmonic*). This class of functions lies in a way on the crossroads between the holomorphic and the harmonic case: it resembles the latter in the sense that it is preserved by complex conjugation, while resembling the former by reflecting the complex structure inherent in the definition of the Hermitian metric (1) (we will see that in more detail below).

In this paper, we will be interested in the simplest situation when Ω is the unit ball $\mathbf{B}^n$ in $\mathbf{C}^n$, so that the holomorphic Bergman kernel is just $K(z,w)=\frac{n!}{\pi^n}(1-\langle z,w\rangle)^{-n-1}$, and $\widetilde\Delta$ is given by

$$\widetilde\Delta=4(1-|z|^2)\sum_{j,k=1}^n(\delta_{jk}-z_j\overline z_k)\frac{\partial^2}{\partial z_j\partial\overline z_k}.$$

One can again consider the corresponding *M-harmonic Bergman space* $L^2_{\rm Mh}(\mathbf{B}^n)\equiv L^2_{\rm Mh}$ of all M-harmonic functions in $L^2(\mathbf{B}^n)$, and the associated *M-harmonic Bergman kernel* $K_{\rm Mh}(z,w)$; as well as the *M-harmonic Hardy space*

$$H^2_{\rm Mh}(\mathbf{B}^n):=\{\mathbf{P}_{\rm Mh}f:\ f\in L^2(\partial\mathbf{B}^n)\}$$

and its reproducing kernel, the *M-harmonic Szegö kernel* $S_{\mathrm{Mh}}(z,w)$. Here $\mathbf{P}_{\mathrm{Mh}}$ stands for the M-harmonic Poisson operator, i.e. the solution operator for the boundary value problem

$$\widetilde{\Delta}\mathbf{P}_{\mathrm{Mh}}u = 0, \qquad \mathbf{P}_{\mathrm{Mh}}u|_{\partial\mathbf{B}^n} = u.$$

Again, $\mathbf{P}_{\mathrm{Mh}}$ is actually an integral operator

$$\mathbf{P}_{\mathrm{Mh}}u(z) = \int_{\partial\Omega} u(\zeta)P_{\mathrm{Mh}}(z,\zeta)\,d\sigma(\zeta),$$

with the *M-harmonic Poisson kernel* (often called *Poisson-Szegö kernel* in the literature) given explicitly by

$$P_{\mathrm{Mh}}(z,\zeta) = \frac{\Gamma(n)}{2\pi^n}\frac{(1-|z|^2)^n}{|1-\langle z,\zeta\rangle|^{2n}},$$

and the M-harmonic Szegö kernel is related to P_{Mh} as in (4):

$$S_{\mathrm{Mh}}(z,w) = \int_{\partial\Omega} P_{\mathrm{Mh}}(z,\zeta)P_{\mathrm{Mh}}(w,\zeta)\,d\sigma(\zeta).$$

Here and throughout the rest of this paper, we take for $d\sigma$ just the (unnormalized) surface measure on $\partial\mathbf{B}^n$.

The kernels $K_{\mathrm{Mh}}(z,w)$ and $S_{\mathrm{Mh}}(z,w)$ on $\mathbf{B}^n$ were recently studied in [EY]. It was shown there, in particular, that most likely there is no "reasonable" formula for $K_{\mathrm{Mh}}(z,w)$ when $n>1$ (even in the simplest case $n=2$, its Taylor coefficients involve the value $\zeta(3)$ of the Riemann zeta functions in a non-trivial way). For the M-harmonic Szegö kernel, the following formula was obtained,

$$S_{\mathrm{Mh}}(z,w) = \frac{\Gamma(n)}{2\pi^n}(1-|z|^2)^n(1-|w|^2)^n FD_1\left(\begin{matrix}n,n,n,n\\ n\end{matrix}\middle| |z|^2,\langle z,w\rangle,\langle w,z\rangle,|w|^2\right), \tag{6}$$

expressing it in terms of the hypergeometric function FD_1 of four variables

$$\begin{aligned} &FD_1\left(\begin{matrix}a,a',b_1,b_2\\ c\end{matrix}\middle| x_1,x_2,y_1,y_2\right) = \\ &\sum_{i_1,i_2,j_1,j_2=0}^{\infty} \frac{(a)_{i_1+i_2}(a')_{j_1+j_2}(b_1)_{i_1+j_1}(b_2)_{i_2+j_2}}{(c)_{i_1+i_2+j_1+j_2}}\frac{x_1^{i_1}}{i_1!}\frac{x_2^{i_2}}{i_2!}\frac{y_1^{j_1}}{j_1!}\frac{y_2^{j_2}}{j_2!}, \end{aligned} \tag{7}$$

which generalizes the usual Gauss hypergeometric functions

$${}_2F_1\left(\begin{matrix}a,b\\ c\end{matrix}\middle| z\right) = \sum_{j=0}^{\infty}\frac{(a)_j(b)_j}{(c)_j}\frac{z^j}{j!}, \qquad |z|<1.$$

Here $c\notin\{0,-1,-2,\dots\}$ while a,a',b,b_1,b_2 can be any complex numbers, and

$$(a)_j := a(a+1)\dots(a+j-1) = \frac{\Gamma(a+j)}{\Gamma(a)}$$

stands for the Pochhammer symbol (rising factorial); the series (7) the converges for all x_1,x_2,y_1,y_2 in the unit disc. The right-hand side of (6) can be expressed in terms of ordinary ${}_2F_1$ functions:

$$\begin{aligned} S_{\mathrm{Mh}}(z,w) = {} & \frac{\Gamma(n)}{2\pi^n}\frac{(1-|w|^2)^n}{|1-\langle z,w\rangle|^{2n}}\sum_{i_1=0}^{n}\sum_{i_2,j_1=0}^{n-i_1}\frac{(-n)_{i_1+i_2}(-n)_{i_1+j_1}(n)_{i_2}(n)_{j_1}}{i_1!i_2!j_1!(n)_{i_1+i_2+j_1}} \\ & \times t_1^{i_1}t_2^{i_2}t_3^{j_1}\,{}_2F_1\binom{i_2+n,j_1+n}{i_1+i_2+j_1+n}\Big| t_4\Big), \end{aligned} \tag{8}$$

where

$$t_1=|z|^2,\quad t_2=\frac{|z|^2-\langle w,z\rangle}{1-\langle w,z\rangle},\quad t_3=\overline{t_2},\quad t_4=1-\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}.$$

See Corollary 4 and Theorem 6 in [EY]. Note that using the formula [BaE, §2.10 (11)]

$$\begin{aligned} {}_2F_1\Big(\begin{matrix}k+1,k+m+1\\ k+m+l+2\end{matrix}\Big| z\Big) &= \frac{(k+m+l+1)!(-1)^{m+1}}{l!k!(m+k)!(m+l)!} \\ &\times\frac{d^{k+m}}{dz^{k+m}}\Big[(1-z)^{m+l}\frac{d^l}{dz^l}\frac{\log(1-z)}{z}\Big],\qquad m,k,l=0,1,2,\dots, \end{aligned}$$

and the elementary relation

$${}_2F_1\Big(\begin{matrix}a,b\\ b\end{matrix}\Big| z\Big)=(1-z)^{-a},$$

it is possible to express each ${}_2F_1$ in (8) in terms of $\log(1-t_4)$ and rational functions of t_4. The obvious drawback of (8), however, is that it completely obscures the symmetry $S_{\mathrm{Mh}}(z,w)=S_{\mathrm{Mh}}(w,z)$ of the M-harmonic Szegö kernel. This also makes it difficult to derive from it any reasonable description of the boundary singularity of $S_{\mathrm{Mh}}(z,w)$ like (2) and (3).

The aim of this paper is to obtain a simpler representation for $S_{\mathrm{Mh}}(z,w)$, and use it to get an analogue of (2) and (3) for the M-harmonic case.

In Section 2, we establish some useful facts about the FD_1 function which may be of interest in their own right, and use these to get a better formula for $S_{\mathrm{Mh}}(z,w)$ as the first main result (Theorem 5). This is then used to obtain a simple description of the singularity if $S_{\mathrm{Mh}}(z,w)$ at the boundary diagonal in Section 3 (Theorem 6). Section 4 contains some final remarks.

Throughout the rest of the paper, we abbreviate $\partial/\partial z$ etc. just to ∂_z etc., and similarly for $\overline{\partial}_z$. To make typesetting a little neater, the shorthand

$$\Gamma\binom{a_1,a_2,\dots,a_k}{b_1,b_2,\dots,b_m}:=\frac{\Gamma(a_1)\Gamma(a_2)\dots\Gamma(a_k)}{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_m)}$$

is often employed throughout the paper. Finally, to make our notation the same as in [EY], the M-harmonic Szegö kernel will be denoted K_{Sz} rather than S_{Mh} from now on.

2. Formula for the Szegö kernel

Throughout this section, unless otherwise specified, the variables x_1,x_2,y_1,y_2 range in the unit disc and $t\in[0,1]$. We will need the Appell hypergeometric function F_1 of two variables, defined by

$$F_1\Big(\begin{matrix}a;b_1,b_2\\ c\end{matrix}\Big| x,y\Big)=\sum_{j,k=0}^{\infty}\frac{(a)_{j+k}(b_1)_j(b_2)_k}{(c)_{j+k}}\frac{x^j}{j!}\frac{y^k}{k!},\qquad |x|<1,|y|<1.$$

(Our notation and conventions for hypergeometric functions follow [BaE].)

Lemma 1. *For all $n \in \mathbf{N}$,*

$$FD_1\binom{a_1, a_2, b_1, b_2}{n}\Big| tx_1, tx_2, ty_1, ty_2\Big) = \frac{(n+t\partial_t)_n}{(n)_n} FD_1\binom{a_1, a_2, b_1, b_2}{2n}\Big| tx_1, tx_2, ty_1, ty_2\Big).$$

Proof. Note that $t\partial_t t^k = kt^k$, so

$$(n+t\partial_t)_n t^k = \frac{\Gamma(2n+k)}{\Gamma(n+k)} t^k. \tag{9}$$

The claim is thus immediate from the definition (7) of the function FD_1. □

Lemma 2. *For $c = b_1 + b_2$,*

$$FD_1\binom{a_1, a_2, b_1, b_2}{c}\Big| x_1, x_2, y_1, y_2\Big) = (1-x_2)^{-a_1}(1-y_2)^{-a_2} F_1\binom{b_1; a_1, a_2}{c}\Big| \frac{x_1-x_2}{1-x_2}, \frac{y_1-y_2}{1-y_2}\Big).$$

Proof. Recall the integral representation [Ka, formula 4.3.(8)]:

$$FD_1\binom{a, a', b_1, b_2}{c}\Big| x_1, x_2, y_1, y_2\Big) = \Gamma\binom{c}{b_1, b_2, c-b_1-b_2} \times \int_{\substack{u_1,u_2>0\\ u_1+u_2<1}} \frac{u_1^{b_1-1}u_2^{b_2-1}(1-u_1-u_2)^{c-b_1-b_2-1}}{(1-x_1u_1-x_2u_2)^a(1-y_1u_1-y_2u_2)^{a'}}\, du_1\, du_2,$$

valid for $c > b_1 + b_2$ and $b_1, b_2 > 0$. Making the change of variable $u_1 = u$, $u_2 = (1-u)v$, we obtain

$$FD_1\binom{a_1, a_2, b_1, b_2}{c}\Big| x_1, x_2, y_1, y_2\Big) = \Gamma\binom{c}{b_1, b_2, c-b_1-b_2} \times \int_0^1\int_0^1 \frac{u^{b_1-1}v^{b_2-1}(1-u)^{c-b_1-1}(1-v)^{c-b_1-b_2-1}}{(1-x_1u-x_2(1-u)v)^{a_1}(1-y_1u-y_2(1-u)v)^{a_2}}\, dv\, du.$$

Integrating with respect to v and using integral representation [BaE, §5.8 (5)]

$$F_1\binom{a; b, b'}{c}\Big| x, y\Big) = \Gamma\binom{c}{a, c-a} \int_0^1 \frac{t^{a-1}(1-t)^{c-a-1}}{(1-tx)^b(1-ty)^{b'}}\, dt, \qquad c > a > 0, \tag{10}$$

for the Appell F_1 function, we obtain

$$FD_1\binom{a_1, a_2, b_1, b_2}{c}\Big| x_1, x_2, y_1, y_2\Big) = \Gamma\binom{c}{b_1, c-b_1} \times \int_0^1 \frac{u^{b_1-1}(1-u)^{c-b_1-1}}{(1-x_1u)^{a_1}(1-y_1u)^{a_2}} F_1\binom{b_2; a_1, a_2}{c-b_1}\Big| \frac{x_2(1-u)}{1-ux_1}, \frac{y_2(1-u)}{1-uy_1}\Big)\, du.$$

This integral converges for $c > b_1 > 0$. We can therefore set $c := b_1 + b_2$ and using the obvious identity

$$F_1\binom{a; b_1, b_2}{a}\Big| x, y\Big) = (1-x)^{-b_1}(1-y)^{-b_2}, \tag{11}$$

we get (employing (10) one more time)

$$FD_1\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ b_1+b_2 \end{matrix}\middle| x_1, x_2, y_1, y_2\right) =$$

$$\Gamma\left(\begin{matrix} c \\ b_1, c-b_1 \end{matrix}\right) \int_0^1 \frac{u^{b_1-1}(1-u)^{c-b_1-1}}{(1-x_2-u(x_1-x_2))^{a_1}(1-y_2-u(y_1-y_2))^{a_2}}\, du$$

$$= (1-x_2)^{-a_1}(1-y_2)^{-a_2} F_1\left(\begin{matrix} b_1; a_1, a_2 \\ b_1+b_2 \end{matrix}\middle| \frac{x_1-x_2}{1-x_2}, \frac{y_1-y_2}{1-y_2}\right),$$

as claimed. □

Corollary 3. *For $c = b_1 + b_2 = a_1 + a_2$ we have*

$$FD_1\left(\begin{matrix} a_1, a_2, b_1, b_2 \\ c \end{matrix}\middle| x_1, x_2, y_1, y_2\right)$$

$$= (1-x_2)^{b_1-a_1}(1-x_1)^{-b_1}(1-y_2)^{-a_2} {}_2F_1\left(\begin{matrix} a_2, b_1 \\ c \end{matrix}\middle| 1 - \frac{(1-x_2)(1-y_1)}{(1-x_1)(1-y_2)}\right).$$

Proof. Straightforward from the well known identity

$$F_1\left(\begin{matrix} a; b_1, b_2 \\ b_1+b_2 \end{matrix}\middle| x, y\right) = (1-x)^{-a} {}_2F_1\left(\begin{matrix} a, b_2 \\ b_1+b_2 \end{matrix}\middle| \frac{x-y}{x-1}\right)$$

(see [BaE, §5.10 (1)]). □

Corollary 4.

$$FD_1\left(\begin{matrix} \alpha, \alpha, \alpha, \alpha \\ 2\alpha \end{matrix}\middle| x_1, x_2, y_1, y_2\right) = (1-x_1)^{-\alpha}(1-y_2)^{-\alpha} {}_2F_1\left(\begin{matrix} \alpha, \alpha \\ 2\alpha \end{matrix}\middle| 1 - \frac{(1-x_2)(1-y_1)}{(1-x_1)(1-y_2)}\right).$$

Proof. Take $a_1 = a_2 = b_1 = b_2 = \alpha$, $c = 2\alpha$ in Corollary 3. □

Here is our nicer formula for the M-harmonic Szegö kernel (upon evaluating at $t = 1$).

Theorem 5. *For $t \in [0,1]$,*

$$\begin{aligned} K_{Sz}(z\sqrt{t}, w\sqrt{t}) &= \frac{\Gamma(n)^2}{\Gamma(2n)2\pi^n}(1-t|z|^2)^n(1-t|w|^2)^n(n+t\partial_t)_n \\ &\quad (1-t|z|^2)^{-n}(1-t|w|^2)^{-n} {}_2F_1\left(\begin{matrix} n, n \\ 2n \end{matrix}\middle| 1 - \frac{|1-t\langle z,w\rangle|^2}{(1-t|z|^2)(1-t|w|^2)}\right). \end{aligned} \tag{12}$$

Proof. By (6), the left-hand side equals

$$\frac{\Gamma(n)}{2\pi^n}(1-t|z|^2)^n(1-t|w|^2)^n FD_1\left(\begin{matrix} n, n, n, n \\ n \end{matrix}\middle| t|z|^2, t\langle z,w\rangle, t\langle w,z\rangle, t|w|^2\right).$$

The claim thus follows by combining Lemma 1 (with $a_1 = a_2 = b_1 = b_2 = n$) and Corollary 4 (with $\alpha = n$, $x_1 = |z|^2$, $x_2 = \langle z,w\rangle$, $y_1 = \langle w,z\rangle$, $y_2 = |w|^2$). □

Since for all integers $n \geq 1$,

$${}_2F_1\left(\begin{matrix} n, n \\ 2n \end{matrix}\middle| x\right) = -\frac{(2n-1)!}{(n-1)!^4}\partial_x^{n-1}(1-x)^{n-1}\partial_x^{n-1}\frac{\ln(1-x)}{x},$$

it is immediate, in particular, that $K_{\mathrm{Sz}}(z,w)$ is an elementary function.

We also remark that, accidentally,

$$(n+t\partial_t)_n = t^{1-n}\partial_t^n t^{2n-1},$$

which however does not seem to lead to any simplifications in (12).

Likewise, one can rewrite (12) using the relation

$$(1-t|z|^2)^n(1-t|w|^2)^n(n+t\partial_t)_n(1-t|z|^2)^{-n}(1-t|w|^2)^{-n} \\ = \Big(n\frac{1-t^2|z|^2|w|^2}{(1-t|z|^2)(1-t|w|^2)}+t\partial_t\Big)_n,$$

which again seems not to yield any significant advantage.

What is nice about (12) is, of course, that unlike (8) there is now manifest symmetry in the variables z, w.

3. Boundary singularity

Let $\mathcal{F}_j$ denote the class of all functions $f(z)$ holomorphic in the cut complex plane $|\arg z|<\pi$ such that as $z\to 0$,

$$f(z)=a(z)+z^j b(z)\log z \qquad \text{with some } a,b \text{ holomorphic near } z=0.$$

Note that by the formula for the analytic continuation of the ${}_2F_1$ function [BaE, §2.10 (12)],

$$ {}_2F_1\Big(\begin{matrix}n,n\\2n\end{matrix}\Big|1-z\Big)\in\mathcal{F}_0 \tag{13}$$

for any $n\in\mathbf{N}$, $n\ge 1$.

Theorem 6. *There exist $G_j\in\mathcal{F}_j$ and polynomials a_j on $\mathbf{C}^4$ of degree at most $4n$ such that*

$$FD_1\Big(\begin{matrix}n,n,n,n\\n\end{matrix}\Big|x_1,x_2,y_1,y_2\Big)= \\ (1-x_2)^{-2n}(1-y_1)^{-2n}\sum_{j=0}^n a_j(x_1,x_2,y_1,y_2)G_j\Big(\frac{(1-x_1)(1-y_2)}{(1-x_2)(1-y_1)}\Big)$$

for all x_1,x_2,y_1,y_2 in the unit disc.

Feeding the last expression into (6), the theorem gives a fairly simple and explicit description of the boundary singularity of K_{Sz}:

$$K_{\mathrm{Sz}}(z,w)=\frac{\Gamma(n)}{2\pi^n}\frac{(1-x_1)^n(1-y_2)^n}{(1-x_2)^{2n}(1-y_1)^{2n}}\times \\ \sum_{j=0}^n a_j(x_1,x_2,y_1,y_2)G_j\Big(\frac{(1-x_1)(1-y_2)}{(1-x_2)(1-y_1)}\Big),$$

with $x_1=|z|^2$, $x_2=\langle z,w\rangle$, $y_1=\langle w,z\rangle$, and $y_2=|w|^2$, and a_j and G_j as in the theorem.

Proof. Let again x_1,x_2,y_1,y_2 lie in the unit disc and t lie in the interval $[0,1]$. As we saw in the last proof, by Lemma 1 and Corollary 4,

$$(n)_nFD_1\Big(\begin{matrix}n,n,n,n\\n\end{matrix}\Big|tx_1,tx_2,ty_1,ty_2\Big)= \\ (n+t\partial_t)_n(1-tx_1)^{-n}(1-ty_2)^{-n}{}_2F_1\Big(\begin{matrix}n,n\\2n\end{matrix}\Big|1-\frac{(1-tx_2)(1-ty_1)}{(1-tx_1)(1-ty_2)}\Big).$$

Applying the Pfaff transform

$$ {}_2F_1\left(\begin{matrix}a,b\\c\end{matrix}\Big|y\right) = (1-y)^{-a}{}_2F_1\left(\begin{matrix}a,c-b\\c\end{matrix}\Big|\frac{y}{y-1}\right), $$

the right-hand side becomes

$$ (n+t\partial_t)_n(1-tx_2)^{-n}(1-ty_1)^{-n}{}_2F_1\left(\begin{matrix}n,n\\2n\end{matrix}\Big|1-\frac{(1-tx_1)(1-ty_2)}{(1-tx_2)(1-ty_1)}\right). \tag{14} $$

Let us temporarily denote

$$ Z(t) := \frac{(1-tx_1)(1-ty_2)}{(1-tx_2)(1-ty_1)}, \qquad F(z) := {}_2F_1\left(\begin{matrix}n,n\\2n\end{matrix}\Big|1-z\right). $$

Let c denote a real constant, not necessarily the same one on each occurrence. Consider the expression

$$ V_{pqrs,F} := (1-tx_1)^{-p}(1-tx_2)^{-q}(1-ty_1)^{-r}(1-ty_2)^{-s}F(Z(t)). $$

Note that

$$ t\partial_t(1-tz)^m = -\frac{mtz}{1-tz}(1-tz)^m = m\Big(1-\frac{1}{1-tz}\Big)(1-tz)^m. $$

Similarly,

$$ t\partial_t F(Z(t)) = t\frac{Z'(t)}{Z(t)}Z(t)F'(Z(t)) = V_{0100,TF} + V_{0010,TF} - V_{1000,TF} - V_{0001,TF}, $$

where we have introduced the notation $TF(z) := zF'(z)$. Thus by the Leibniz rule

$$ \begin{aligned} &(c+t\partial_t)V_{pqrs,F} = \\ &\quad cV_{pqrs,F} + cV_{p+1,q,r,s,F} + cV_{p,q+1,r,s,F} + cV_{p,q,r+1,s,F} + cV_{p,q,r,s+1,F} \\ &\qquad + cV_{p+1,q,r,s,TF} + cV_{p,q+1,r,s,TF} + cV_{p,q,r+1,s,TF} + cV_{p,q,r,s+1,TF}. \end{aligned} $$

Iterating the last formula n times, we obtain

$$ (n+t\partial_t)_n V_{0,n,n,0,F} = \sum_{\substack{m_1+m_2+m_3+m_4=m,\\ p+q+r+s+m\le n}} c\, V_{p+m_1,n+q+m_2,n+r+m_3,s+m_4,T^mF}. $$

Hence finally

$$ \begin{aligned} &(1-tx_1)^n(1-ty_2)^n FD_1\left(\begin{matrix}n,n,n,n\\n\end{matrix}\Big|tx_1,tx_2,ty_1,ty_2\right) \\ &\qquad = \sum_{\substack{m_1+m_2+m_3+m_4=m,\\ p+q+r+s+m\le n}} c\, V_{p+m_1-n,q+m_2+n,r+m_3+n,s+m_4-n,T^mF}, \end{aligned} $$

i.e.

$$ \begin{aligned} &(1-tx_1)^n(1-ty_2)^n FD_1\left(\begin{matrix}n,n,n,n\\n\end{matrix}\Big|tx_1,tx_2,ty_1,ty_2\right) \\ &\quad = (1-tx_2)^{-2n}(1-ty_1)^{-2n}\sum_{m=0}^{n}\sum_{\substack{m_1+m_2+m_3+m_4=m,\\ p+q+r+s\le n-m}} c(1-tx_1)^{n-p-m_1} \\ &\qquad \times(1-tx_2)^{n-q-m_2}(1-ty_1)^{n-r-m_3}(1-ty_2)^{n-s-m_4}\,T^mF(Z(t)) \end{aligned} $$

$$=: (1-tx_2)^{-2n}(1-ty_1)^{-2n}\sum_{m=0}^{n}\tilde a_m(tx_1,tx_2,ty_1,ty_2)\,T^mF(Z), \tag{15}$$

for some polynomials $\tilde a_m$ of degree $\le 4n$ in the indicated variables. Observe now that $T^mF(z)$ comes as a sum of $cz^jF^{(j)}(z)$, $j\le m$, and a simple check also shows that $z^jF^{(j)}(z)\in\mathcal F_j$. Setting $t=1$ in (15), the theorem thus follows. □

Note that if z tends to a point on the boundary while w stays inside or tends to a different boundary point, then $Z(1)\equiv Z\to 0$ while $1/(1-x_2)$, $1/(1-y_1)$ stay bounded and smooth. Thus $K_{\mathrm{Sz}}=a+bZ^n\log Z$, with a,b smooth up to the boundary. Consequently, K_{Sz} is C^{n-1} up to the boundary away from the boundary diagonal, in full agreement with Proposition 7 in [EY].

If, on the other hand, $z=w$, then $Z=1$ so K_{Sz} is a polynomial in $1/(1-|z|^2)$ of degree n. This is in full agreement with formula (99) in [EY].

For z,w both approaching the same point on the boundary, Z can range over the entire interval $0<Z\le 1$, and it seems unclear whether (15) can be simplified or brought to a more tangible form.

4. Concluding remarks

Using the standard formulas

$$\int_1^\infty e^{-tp}t^s\,dt=\begin{cases}\dfrac{\Gamma(s+1)}{p^{s+1}}+\mathcal O(p), & s\in\mathbf C\setminus\{-1,-2,\dots\},\\[2mm] \dfrac{(-1)^{k+1}}{k!}p^k(\log p+\mathcal O(p)), & s=-1-k,\ k\in\mathbf N,\end{cases} \tag{16}$$

valid for $\operatorname{Re}p>0$, where $\mathcal O(p)$ denotes a function of p which is smooth (in fact — holomorphic) in a neighborhood of the origin, the boundary singularity (3) of the holomorphic Szegö kernel S can also be rewritten as

$$S(x,y)\sim\int_0^\infty e^{-t\rho(x,y)}\,b(x,y,t)\,dt,\qquad x,y\in\Omega,$$

where b is a classical symbol in the Hörmander class $S^{n-1}(\overline\Omega\times\overline\Omega\times\mathbf R_+)$ with asymptotic expansion

$$b(x,y,t)\sim\sum_{j=0}^\infty t^{n-1-j}\,b_j(x,y)\qquad\text{for } t>1,$$

with some functions $b_j\in C^\infty(\overline\Omega\times\overline\Omega)$. In other words,

$$S(x,y)\approx\int_0^\infty\sum_{j=0}^\infty t^{n-1-j}\,e^{-t\rho(x,y)}\,b_j(x,y)\,dt \tag{17}$$

on $\overline\Omega\times\overline\Omega$. (Here the integrals need to be understood as "finite parts"; see [BdS] for the details.) For the harmonic Szegö kernel, the analogue of (3) — as mentioned in the Introduction — becomes

$$S_{\mathrm{harm}}(x,y)\approx\int_0^\infty\sum_{j=0}^\infty t^{n-1-j}\,e^{-t|x-\tilde y|}\,b_j(x,|x-\tilde y|,\tfrac{x-\tilde y}{|x-\tilde y|})\,dt \tag{18}$$

again on $\overline\Omega\times\overline\Omega$ (this time with Ω a bounded domain with smooth boundary in $\mathbf R^n$ rather than $\mathbf C^n$), now with $b_j\in C^\infty(\overline\Omega\times\overline{\mathbf R_+}\times\mathbf S^{n-1})$. In other words, for each fixed x, $\tilde{b}_j(x, x+w) := b_j(x, |w|, \frac{w}{|w|})$ comes as an asymptotic expansion of homogeneous distributions in w of higher and higher degree. The role of the "sesquiholomorphic extension" $\rho(x, y)$ of the defining function is thus played simply by the Euclidean distance function $|x - \tilde{y}|$.

From Theorem 6, using again (16), one can get an expansion akin to (17) and (18) also for our M-harmonic Szegö kernel K_{Sz} on $\mathbf{B}^n$:

$$K_{\mathrm{Sz}}(z,w) \approx |1-\langle z,w\rangle|^{-2n} \int_0^\infty \sum_{j=0}^\infty t^{-1-j}\, e^{-tZ(z,w)}\, a_j(z,w)\, dt, \tag{19}$$

with

$$Z(z,w) := \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \tag{20}$$

and $a_j(z,w)$ a polynomial in $z, \overline{z}, w, \overline{w}$ on $\mathbf{C}^n$ of degree at most $4n$. The marked difference is that now the coefficients of the asymptotic expansion are not smooth up to the closure $\overline{\mathbf{B}^n \times \mathbf{B}^n}$, due to the extra factor $|1-\langle z,w\rangle|^{-2n}$. Another difference is that the highest power of t in the integrand is independent of n. A drawback of (19), however, is that while the expansions (17) and (18) are "sharp" in the sense that the top order term b_0 is positive everywhere on $\partial\Omega \times \partial\Omega$ (in the holomorphic case, it is given by the Monge-Ampére determinant of $\rho(x)$, and a similar formula is available also in the harmonic case, see [Eng]), this is no longer clear for (19): the top order term there corresponds to $m = 0$ and $Z = 0$ in (15), hence is given by (after setting $t = 1$)

$$(1-x_2)^{-2n}(1-y_1)^{-2n} \sum_{p+q+r+s\le n} c\,(1-x_1)^{n-p}(1-x_2)^{n-q}(1-y_1)^{n-r}(1-y_2)^{n-s}.$$

Now at least one of the exponents $n-q$ or $n-r$ is always positive, hence lowering the degree of singularity at $(x_1, x_2, y_1, y_2) = (1,1,1,1)$ by partially canceling the term in front of the sum. Or, put differently, the last sum is a polynomial of degree at least $3n$ in the variables $1-x_1, 1-x_2, 1-y_1, 1-y_2$, hence always vanishes at $(x_1, x_2, y_1, y_2) = (1,1,1,1)$. It is as yet unclear to the current authors how to remove this deficiency.

Here is an explicit formula for the "leading order" coefficient $a_0(z,w)$ in (19).

Proposition 7. *We have*

$$a_0(z,w) = \frac{1}{\Gamma(2n)2\pi^n}(1-|z|^2)^n(1-|w|^2)^n Q(\langle z,w\rangle, \langle w,z\rangle),$$

where Q is the polynomial

$$Q(x_2,y_1) := (1-x_2)^n(1-y_1)^n F_1\Big(\begin{matrix}-n;n,n\\ n\end{matrix}\Big|\frac{x_2}{x_2-1}, \frac{y_1}{y_1-1}\Big).$$

Proof. By (16) for $s=-1$ and (12), a_0 equals $-\frac{\Gamma(n)^2}{\Gamma(2n)2\pi^n}$ times the coefficient at $Z^0 \log Z$ in the sum in (15). Since $T^m(z^k) = kz^k$ and $T^m(z^k \log z) = kz^k \log z + z^k$, this coefficient is nonzero only in the term $m = 0$; hence, looking back at (14), it equals

$$(1-tx_1)^n(1-ty_2)^n(1-tx_2)^{2n}(1-ty_1)^{2n}(n+t\partial_t)_n(1-tx_2)^{-n}(1-ty_1)^{-n}\Big|_{t=1} \tag{21}$$

times the coefficient at $Z^0 \log Z$ in $F(Z)$, which according to [BaE, §2.10 (12)] is equal to $-1/\Gamma(n)^2$. To compute (21), revoke again (11) (taking $a = n$ there) and (9) to conclude that

$$\begin{aligned}(n + t\partial_t)_n (1 - tx_2)^{-n}(1 - ty_1)^{-n} &= (n + t\partial_t)_n F_1\Big(\begin{matrix} n; n, n \\ n \end{matrix}\Big| tx_2, ty_1\Big) \\ &= F_1\Big(\begin{matrix} 2n; n, n \\ n \end{matrix}\Big| tx_2, ty_1\Big) \\ &= (1 - tx_2)^{-n}(1 - ty_1)^{-n} F_1\Big(\begin{matrix} -n; n, n \\ n \end{matrix}\Big| \frac{tx_2}{tx_2 - 1}, \frac{ty_1}{ty_1 - 1}\Big),\end{aligned}$$

where on the last line we have used [BaE, §5.11 (1)]. Putting everything together and setting $t = 1$, we thus get

$$a_0(z, w) = \frac{1}{\Gamma(2n)2\pi^n}(1-x_1)^n(1-y_2)^n(1-x_2)^n(1-y_1)^n F_1\Big(\begin{matrix} -n; n, n \\ n \end{matrix}\Big| \frac{x_2}{x_2 - 1}, \frac{y_1}{y_1 - 1}\Big)$$

with $x_1 = |z|^2$, $x_2 = \langle z, w\rangle$, $y_1 = \langle w, z\rangle$ and $y_2 = |w|^2$, and the assertion follows. □

Example 8. For $n = 2$, the polynomial Q in the last proposition is given by

$$\begin{aligned}Q(x_2, y_1) = \frac{1}{3}\Big[&3(1 - x_2)^2 + 4(1 - x_2)(1 - y_1) - 4(1 - x_2)^2(1 - y_1) \\ &+ 3(1 - y_1)^2 - 4(1 - x_2)(1 - y_1)^2 + (1 - x_2)^2(1 - y_1)^2\Big].\end{aligned}$$

It is not clear how to express this in a simpler manner as $(x_2, y_1) \to (1, 1)$. □

Probably all one can say in general is that $\frac{a_j(z,w)}{(1-|z|^2)^n|1-\langle z,w\rangle|^{2n}(1-|w|^2)^n}$ is a polynomial of total degree $\le n$ in the variables $\frac{1}{1-|z|^2}, \frac{1}{1-\langle z,w\rangle}, \frac{1}{1-\langle w,z\rangle}$ and $\frac{1}{1-|w|^2}$.

We remark that, returning from the ball to a general domain $\Omega \subset \mathbf{C}^n$, one is tempted to speculate that the general analogue of (19) may be

$$K_{\mathrm{Sz}}(z, w) \approx |\rho(z, w)|^{-2n} \int_0^\infty \sum_{j=0}^\infty t^{-1-j}\, e^{-tZ(z,w)}\, a_j(z, w)\, dt,$$

with

$$Z(z, w) := \frac{\rho(z, z)\rho(w, w)}{|\rho(z, w)|^2}$$

and $a_j \in C^\infty(\overline{\Omega} \times \overline{\Omega})$. Note that $Z(z, w) = e^{D(z,w)}$, where

$$D(z, w) := \log \rho(z, z) + \log \rho(w, w) - \log \rho(z, w) - \log \rho(w, z)$$

resembles the famous *Calabi diastasis function*, which was introduced in [Ca] in connection with isometric imbeddings of complex manifolds and which also plays a prominent role e.g. in some quantization procedures on Kähler manifolds; however, these applications do not involve its behavior near the boundary diagonal.

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