

COMPLETE NEVANLINNA-PICK PROPERTY OF \mathbb{K} -INVARIANT REPRODUCING KERNELS

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ABSTRACT. Let Ω be a Cartan domain and $K = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}$ be a \mathbb{K} -invariant kernel on Ω . In this article, we first obtain a necessary condition on K to have the complete Nevanlinna-Pick property in terms of the sequence $\{a_{\underline{s}}\}_{\underline{s}}$ with the assumption that each $a_{\underline{s}}$ is non-zero and K is non-vanishing. This generalizes the well-known Kaluza's Lemma in the context of \mathbb{K} -invariant kernels.

The notion of the characteristic function of the classical Sz.-Nagy–Foias Theory is extended to a commuting tuple of $\frac{1}{K}$ -contraction where K is an irreducible \mathbb{K} -invariant kernel. An explicit construction of the characteristic function of a $\frac{1}{K}$ -contraction is provided. A characterization of a \mathbb{K} -invariant kernel with the complete Nevanlinna-Pick property is obtained via the existence of characteristic functions associated with $\frac{1}{K}$ -contractions.

1. INTRODUCTION

Let Ω be a Cartan domain in \mathbb{C}^d of rank r . These domains are the natural generalization of open unit disc in one complex variable and open Euclidean unit ball in several complex variables. A complete classification of irreducible Cartan domains is given by É. Cartan [9]. The numerical invariants (r, a, b) determine the domain Ω uniquely up to biholomorphic equivalence. The dimension d is related to the numerical invariants (r, a, b) by the relation $\frac{d}{r} = 1 + \frac{a}{2}(r - 1) + b$. For more details on Cartan domains, we refer to [14],[1]. Let G be the connected component of identity in $\text{Aut}(\Omega)$, the group of all the biholomorphic automorphisms of Ω . Let $\mathbb{K} = \{g \in G : g(0) = 0\}$ be the maximal compact subgroup of G . Every irreducible Cartan domain Ω of rank r can be realized as an open unit ball of a Cartan factor $Z = \mathbb{C}^d$. The space of analytic polynomials $\mathcal{P}(Z)$ on Z has a natural action of the group \mathbb{K} by composition, that is, $(k \cdot p)(z) = p(k^{-1} \cdot z)$, $k \in \mathbb{K}$, $p \in \mathcal{P}(Z)$.

An r -tuple of non-negative integers $\underline{s} = (s_1, \dots, s_r)$ is called a *signature* if $s_1 \geq \dots \geq s_r \geq 0$. The set of all signatures is denoted by $\vec{\mathbb{N}}_0^r$ and $\underline{0}$ represents the signature $(0, \dots, 0)$. If $\underline{p} = (p_1, \dots, p_r)$ and $\underline{q} = (q_1, \dots, q_r)$ are two signatures, then the notation $\underline{p} \geq \underline{q}$ means that $p_i \geq q_i$ for each $1 \leq i \leq r$ and the notation $\underline{p} > \underline{q}$ means that there exists $1 \leq j \leq r$ such that $p_j > q_j$. Let $\{\varepsilon_1, \dots, \varepsilon_r\}$ denote the standard ordered basis of \mathbb{C}^r . Note that among all elements of the standard ordered basis of \mathbb{C}^r , only ε_1 is a signature. The action of \mathbb{K} on $\mathcal{P}(Z)$ is not irreducible. In fact, $\mathcal{P}(Z)$ decomposes into irreducible, mutually \mathbb{K} -inequivalent subspaces $\mathcal{P}_{\underline{s}}$, where \underline{s} is a signature. Such a decomposition of $\mathcal{P}(Z)$ is called the *Peter-Weyl decomposition* in [17, Section 3] (see also [1, page 21]).

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The space $\mathcal{P}_{\underline{s}}$ with respect to the Fischer-Fock inner product

$$\langle p, q \rangle_F := \frac{1}{\pi^d} \int_{\mathbb{C}^d} p(z) \overline{q(z)} e^{-|z|^2} dm(z)$$

is a reproducing kernel Hilbert space, where $dm(z)$ is the Lebesgue measure. The reproducing kernel of $\mathcal{P}_{\underline{s}}$ is denoted by $K_{\underline{s}}$. Let $d_{\underline{s}}$ be the dimension of $\mathcal{P}_{\underline{s}}$ for every $\underline{s} \in \vec{\mathbb{N}}_0^r$. Throughout this article, we fix an orthonormal basis $\{\psi_{\alpha}^{\underline{s}}(z)\}_{\alpha=1}^{d_{\underline{s}}}$ of $\mathcal{P}_{\underline{s}}$ with respect to the Fischer-Fock inner product. Then, the reproducing kernel $K_{\underline{s}}$ is given by

$$K_{\underline{s}}(\mathbf{z}, \mathbf{w}) = \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{z}) \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})}. \quad (1.1)$$

A non-negative definite kernel $K : \Omega \times \Omega \rightarrow \mathbb{C}$ is said to be a \mathbb{K} -invariant kernel if $K(k \cdot \mathbf{z}, k \cdot \mathbf{w}) = K(\mathbf{z}, \mathbf{w})$ holds for every $\mathbf{z}, \mathbf{w} \in \Omega$ and $k \in \mathbb{K}$. Note that, for each signature \underline{s} , the reproducing kernel $K_{\underline{s}}$ of $\mathcal{P}_{\underline{s}}$ is a \mathbb{K} -invariant kernel. If $K : \Omega \times \Omega \rightarrow \mathbb{C}$ is a \mathbb{K} -invariant kernel which is holomorphic in \mathbf{z} and $\bar{\mathbf{w}}$, then there exists a sequence of non-negative real numbers $\{a_{\underline{s}}\}_{\underline{s} \in \vec{\mathbb{N}}_0^r}$ such that

$$K(\mathbf{z}, \mathbf{w}) = \sum_{\underline{s} \in \vec{\mathbb{N}}_0^r} a_{\underline{s}} K_{\underline{s}}(\mathbf{z}, \mathbf{w}),$$

for every $\mathbf{z}, \mathbf{w} \in \Omega$ (cf. [1]). In this article, we consider non-negative definite kernels on $\Omega \times \Omega$ that are holomorphic in \mathbf{z} and $\bar{\mathbf{w}}$. In particular, if Ω is an Euclidean unit ball \mathbb{B}_d in \mathbb{C}^d , and \mathbb{K} is the group of all $d \times d$ unitary matrices $\mathcal{U}(d)$, then

$$K(\mathbf{z}, \mathbf{w}) = \sum_{n=0}^{\infty} a_n \langle \mathbf{z}, \mathbf{w} \rangle^n, \quad \mathbf{z}, \mathbf{w} \in \mathbb{B}_d,$$

where $\{a_n\}_{n \geq 0}$ is a sequence of non-negative real numbers. The $\mathcal{U}(d)$ -invariant and \mathbb{K} -invariant kernels have been studied extensively over the past few decades ([1, 7, 10, 12]).

A non-negative definite kernel $K : \Omega \times \Omega \rightarrow \mathbb{C}$ is said to be normalized at a point z_0 in Ω , if $K(z, z_0) = 1$ for every $z \in \Omega$. Note that if $K(\mathbf{z}, \mathbf{w}) = \sum_{\underline{s} \in \vec{\mathbb{N}}_0^r} a_{\underline{s}} K_{\underline{s}}(\mathbf{z}, \mathbf{w})$, $\mathbf{z}, \mathbf{w} \in \Omega$, is a \mathbb{K} -invariant kernel, then $K(\mathbf{z}, \mathbf{0}) = a_{\underline{0}}$ for every $\mathbf{z} \in \Omega$. As a consequence, $\frac{1}{a_{\underline{0}}} K$ is a \mathbb{K} -invariant kernel which is normalized at $\mathbf{0} \in \Omega$. *In this article, we always work with a normalized \mathbb{K} -invariant kernel, that is, a kernel $K = \sum_{\underline{s} \in \vec{\mathbb{N}}_0^r} a_{\underline{s}} K_{\underline{s}}$ on Ω with $a_{\underline{0}} = 1$. We also assume that $a_{\underline{s}} > 0$ for every signature \underline{s} .*

Given a non-negative definite kernel $K : \Omega \times \Omega \rightarrow \mathbb{C}$, the Moore-Aronszajn Theorem says that there exists a Hilbert space H_K such that K is the reproducing kernel of H_K . *Throughout this article, H_K represents the Hilbert space with the reproducing kernel K .* Our objective is to classify \mathbb{K} -invariant kernels that have the complete Nevanlinna-Pick property. Let us recall the definition of the complete Nevanlinna-Pick property of a reproducing kernel.

Definition 1.1. *Let X be a subset of \mathbb{C}^d . A reproducing kernel $K : X \times X \rightarrow \mathbb{C}$ is said to have $M_{p \times p}$ Nevanlinna-Pick property if, whenever $\mathbf{z}_1, \dots, \mathbf{z}_N \in X$ and W_1, \dots, W_N are $p \times p$ matrices such that*

$$\left((I - W_i W_j^*) K(\mathbf{z}_i, \mathbf{z}_j) \right)_{i,j=1}^N,$$

is non-negative definite, then there exists a multiplier ϕ in the closed unit ball of the multiplier algebra $\text{Mult}(H_K \otimes \mathbb{C}^p, H_K \otimes \mathbb{C}^p)$ such that $\phi(\mathbf{z}_i) = W_i, i = 1, \dots, N$. The kernel K is said to have the complete Nevanlinna-Pick property if it has the $M_{p \times p}$ Nevanlinna-Pick

property for all positive integers p . A reproducing kernel with the complete Nevanlinna-Pick property is called the complete Nevanlinna-Pick (CNP) kernel.

Let $K_{\mathcal{U}}(\mathbf{z}, \mathbf{w}) = \sum_{n \geq 0} a_n \langle \mathbf{z}, \mathbf{w} \rangle^n$ be a non-vanishing $\mathcal{U}(d)$ -invariant kernel on \mathbb{B}_d with each a_n being positive. Then, there exists a sequence of real numbers $\{\hat{b}_n\}_{n \geq 0}$ such that

$$\frac{1}{K_{\mathcal{U}}(\mathbf{z}, \mathbf{w})} = \sum_{n \geq 0} \hat{b}_n \langle \mathbf{z}, \mathbf{w} \rangle^n, \quad (1.2)$$

for every $\mathbf{z}, \mathbf{w} \in \mathbb{B}_d$. Equation (1.2) is one of the main ingredients of the Kaluza lemma which provides a necessary condition for $K_{\mathcal{U}}$ to be CNP (cf. [5]). The Kaluza lemma is named after Theodor Kaluza who proved it in terms of power series of one complex variable [13].

Theorem 1.2. *Let $\{c_n\}_{n \geq 0}$ be a sequence of positive real numbers with $c_0 = 1$ and M be a positive real number. If the sequence of real numbers $\left\{\frac{c_n}{c_{n-1}} : n > 0\right\}$ is non-decreasing and bounded above by M , then $f(z) = \sum_{n \geq 0} c_n z^n$ converges for all $z \in B(0, 1/M) := \{z \in \mathbb{C} : |z| < 1/M\}$ and there exists a sequence of non-negative real numbers $\{q_n\}_{n > 0}$ such that*

$$1 - \frac{1}{f(z)} = \sum_{n \geq 1} q_n z^n, \quad z \in B(0, 1/M).$$

A kernel $K : \Omega \times \Omega \rightarrow \mathbb{C}$ is said to be irreducible if $K(\cdot, \mathbf{w}_1)$ and $K(\cdot, \mathbf{w}_2)$ are linearly independent for any two distinct points $\mathbf{w}_1, \mathbf{w}_2 \in \Omega$ and $K(\mathbf{z}, \mathbf{w}) \neq 0$ for every $\mathbf{z}, \mathbf{w} \in \Omega$. A well-known characterization of an irreducible kernel $K : \Omega \times \Omega \rightarrow \mathbb{C}$ having the CNP property is that $1 - \frac{1}{K}$ must be non-negative definite (cf. [5, Theorem 7.28]). Combining this characterization with Theorem 1.2, we obtain the following necessary condition for a $\mathcal{U}(d)$ -invariant kernel to have the CNP property (cf. [5, Lemma 7.38]).

Lemma 1.3. *An irreducible $\mathcal{U}(d)$ -invariant kernel $K_{\mathcal{U}} = \sum_{n \geq 0} a_n \langle \mathbf{z}, \mathbf{w} \rangle^n$ on \mathbb{B}_d with $a_n > 0$ for each n is CNP if*

$$\frac{a_n}{a_{n-1}} \leq \frac{a_{n+1}}{a_n}$$

holds for all $n \geq 1$.

Suppose $K = \sum_{\underline{s} \in \vec{\mathbb{N}}_0^r} a_{\underline{s}} K_{\underline{s}}$ is a \mathbb{K} -invariant kernel on Ω . In Section 2, we first obtain a sequence of real numbers $\{\hat{b}_{\underline{s}}\}_{\underline{s} \in \vec{\mathbb{N}}_0^r}$ such that

$$\frac{1}{K(\mathbf{z}, \mathbf{w})} = \sum_{\underline{s} \in \vec{\mathbb{N}}_0^r} \hat{b}_{\underline{s}} K_{\underline{s}}(\mathbf{z}, \mathbf{w}) \quad (1.3)$$

holds for every $\mathbf{z}, \mathbf{w} \in \Omega$. Assuming K to be an irreducible kernel, we generalize Lemma 1.3 to find a necessary condition for K to be CNP in terms of the coefficients $a_{\underline{s}}$, $\underline{s} \in \vec{\mathbb{N}}_0^r$. We also provide examples of \mathbb{K} -invariant CNP kernels on Ω such that the multiplication operators by the coordinate functions on the corresponding Hilbert spaces are bounded.

Given a contraction T on a Hilbert space \mathcal{H} , let $D_T := (I - T^*T)^{\frac{1}{2}}$, $D_{T^*} := (I - TT^*)^{\frac{1}{2}}$ and $\mathcal{D}_T := \overline{\text{Ran } D_T}$, $\mathcal{D}_{T^*} := \overline{\text{Ran } D_{T^*}}$. The operators D_T , D_{T^*} are called the defect operators and the spaces \mathcal{D}_T , \mathcal{D}_{T^*} are called the defect spaces of T . The operator valued holomorphic function $\theta_T : \mathbb{D} \rightarrow \mathcal{B}(\mathcal{D}_T, \mathcal{D}_{T^*})$, defined by

$$\theta_T(z) = (-T + zD_{T^*}(I - zT^*)^{-1}D_T) |_{\mathcal{D}_T}, \quad z \in \mathbb{D},$$

is called the characteristic function of T . The classical Sz.-Nagy–Foias theory says that two completely nonunitary (cnu) contractions T_1 and T_2 are unitarily equivalent if and only if the characteristic functions θ_{T_1} and θ_{T_2} coincide (that is, there exist two unitaries $u : \mathcal{D}_{T_1} \rightarrow \mathcal{D}_{T_2}$ and $v : \mathcal{D}_{T_1^*} \rightarrow \mathcal{D}_{T_2^*}$ such that $v\theta_{T_1}(z) = \theta_{T_2}(z)u$, $z \in \mathbb{D}$) (cf. [15, Theorem 3.4]).

Note that T is a contraction if and only if $\frac{1}{\mathbb{S}}(T, T^*) := I - TT^*$ is a positive operator, where $\mathbb{S} : \mathbb{D} \times \mathbb{D} \rightarrow \mathbb{C}$ is the Szegő kernel $\mathbb{S}(z, w) = \frac{1}{1-z\bar{w}}$, $z, w \in \mathbb{D}$. Let $K_D : \mathbb{B}_d \times \mathbb{B}_d \rightarrow \mathbb{C}$ be defined by $K_D(\mathbf{z}, \mathbf{w}) = \frac{1}{1-\langle \mathbf{z}, \mathbf{w} \rangle}$, $\mathbf{z}, \mathbf{w} \in \mathbb{B}_d$. The kernel K_D is called the Drury–Arveson kernel. A commuting tuple of operators $\mathbf{T} = (T_1, \dots, T_d)$ is a $\frac{1}{K_D}$ -contraction if and only if $\frac{1}{K_D}(\mathbf{T}, \mathbf{T}^*) = I - \sum_{i=1}^d T_i T_i^*$ is a positive operator. Such a tuple of operators is called a row contraction. For a commuting row contraction \mathbf{T} , the characteristic function $\theta_{\mathbf{T}} : \mathbb{B}_d \rightarrow \mathcal{B}(\mathcal{D}_{\mathbf{T}}, \mathcal{D}_{\mathbf{T}^*})$ of \mathbf{T} is defined in a similar fashion to the classical characteristic function of a single contraction, where $\mathcal{D}_{\mathbf{T}}$ and $\mathcal{D}_{\mathbf{T}^*}$ are defect spaces introduced in the usual way. The characteristic function of a pure commuting row contraction \mathbf{T} determines the unitary equivalence class of \mathbf{T} (cf. [8]).

Given a non-vanishing $\mathcal{U}(d)$ -invariant kernel $K_{\mathcal{U}}(\mathbf{z}, \mathbf{w}) = \sum_{n \geq 0} a_n \langle \mathbf{z}, \mathbf{w} \rangle^n$ on \mathbb{B}_d and a commuting d -tuple of operators $\mathbf{T} = (T_1, \dots, T_d)$ on a Hilbert space \mathcal{H} , the $\frac{1}{K_{\mathcal{U}}}$ -contractivity of \mathbf{T} is defined in a similar fashion of a $\frac{1}{K_D}$ -contraction due to Equation (1.2) ([7, Definition 1.1]). A characterization of $\mathcal{U}(d)$ -invariant CNP kernels over \mathbb{B}_d is obtained in terms of the existence of the characteristic function of $\frac{1}{K_{\mathcal{U}}}$ -contractions in [7]. In particular, if $K_{\mathcal{U}}$ also has the CNP property, then an explicit construction of the characteristic function of a $\frac{1}{K_{\mathcal{U}}}$ -contraction is given in the final section of [7] and it is proved that the characteristic function of a pure $\frac{1}{K_{\mathcal{U}}}$ -contraction determines its unitary equivalence class, see [7].

Suppose $K = \sum_{\underline{s} \in \mathbb{N}_0^r} a_{\underline{s}} K_{\underline{s}}$ is a \mathbb{K} -invariant kernel on Ω . In section 3, we provide a similar definition of $\frac{1}{K}$ -contractivity of a commuting tuple of operators $\mathbf{T} = (T_1, \dots, T_d)$ on a Hilbert space \mathcal{H} . The main objective of this section is to obtain a necessary and sufficient condition of a \mathbb{K} -invariant kernel on Ω to have the CNP property in terms of the existence of the characteristic function of $\frac{1}{K}$ -contractions. In the final section, we assume that K also has the CNP property. Then, we provide an explicit construction of the characteristic function of a $\frac{1}{K}$ -contraction \mathbf{T} for certain \mathbb{K} -invariant kernels. We also prove that, for such kernels, the unitary equivalence class of a pure $\frac{1}{K}$ -contraction is determined by the characteristic function.

2. A GENERALIZATION OF KALUZA'S LEMMA FOR \mathbb{K} -INVARIANT KERNELS

In this section, we provide a criteria of a non-vanishing \mathbb{K} -invariant kernel $K = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}$ on Ω to have CNP property in terms of the sequence $\{a_{\underline{s}}\}_{\underline{s} \geq 0}$. We begin this section with a criterion of convergence of a \mathbb{K} -invariant function defined on $\Omega \times \Omega$ which is holomorphic in the first d -variables and anti-holomorphic in the last d -variables.

Proposition 2.1. *Let*

$$F(\mathbf{z}, \mathbf{w}) := \sum_{\underline{s}} c_{\underline{s}} K_{\underline{s}}(\mathbf{z}, \mathbf{w}), \quad (2.4)$$

be a \mathbb{K} -invariant function on $\Omega \times \Omega$ and e denote any maximal tripotent of Ω .

(i) If F converges absolutely on the closure of $\Omega \times \Omega$ and is bounded in modulus by 1, then

$$\sum_{\underline{s}} |c_{\underline{s}}| K_{\underline{s}}(e, e) \leq 1. \quad (2.5)$$

Conversely, if (2.5) holds, then F converges absolutely and uniformly on the closure of $\Omega \times \Omega$, and is bounded in modulus by 1 there.

(ii) If F converges absolutely on $\Omega \times \Omega$, then

$$S(t) := \sum_{\underline{s}} |c_{\underline{s}}| K_{\underline{s}}(te, te) < \infty, \quad (2.6)$$

for all $t \in (0, 1)$. Conversely, if (2.6) holds, then F converges absolutely and uniformly on compact subsets of $\Omega \times \Omega$.

Proof. (i) Taking $z = w = e$ shows that the assumption on (2.4) implies (2.5).

For the converse part, recall that a function holomorphic on Ω and continuous on its closure attains its maximum modulus on the Shilov boundary, and the Shilov boundary of Ω is precisely the orbit $\{ke : k \in \mathbb{K}\}$ of a maximal tripotent e (see [1]). Hence

$$|K_{\underline{s}}(\mathbf{z}, \mathbf{w})| \leq |K_{\underline{s}}(k_1e, k_2e)|$$

for all $\mathbf{z}, \mathbf{w} \in \bar{\Omega}$ and for some $k_1, k_2 \in \mathbb{K}$. Now by Cauchy-Schwarz

$$|K_{\underline{s}}(k_1e, k_2e)|^2 \leq K_{\underline{s}}(k_1e, k_1e) K_{\underline{s}}(k_2e, k_2e) = K_{\underline{s}}(e, e)^2$$

thanks to the \mathbb{K} -invariance of $K_{\underline{s}}$. Consequently,

$$\sum_{\underline{s}} |c_{\underline{s}} K_{\underline{s}}(\mathbf{z}, \mathbf{w})| \leq \sum_{\underline{s}} |c_{\underline{s}}| K_{\underline{s}}(e, e) \quad \forall \mathbf{z}, \mathbf{w} \in \bar{\Omega},$$

and the claim follows.

(ii) Let $t \in (0, 1)$. Suppose $M_t > 0$ is such that $|F(t\mathbf{z}, t\mathbf{w})| \leq M_t$ for all $\mathbf{z}, \mathbf{w} \in \Omega$. Now, apply part (i) to the functions $F(t\mathbf{z}, t\mathbf{w})/M_t$. \square

The following proposition provides an explicit description of a \mathbb{K} -invariant function on $\Omega \times \Omega$, holomorphic in $\mathbf{z}, \bar{\mathbf{w}}$. The proof is modeled on the proof of [3, Proposition 2].

Proposition 2.2. *If $F(\mathbf{z}, \mathbf{w})$ is a \mathbb{K} -invariant function on $\Omega \times \Omega$ holomorphic in $(\mathbf{z}, \bar{\mathbf{w}})$, then it has an expansion*

$$F(\mathbf{z}, \mathbf{w}) = \sum_{\underline{s}} c_{\underline{s}} K_{\underline{s}}(\mathbf{z}, \mathbf{w}) \quad (2.7)$$

with some $c_{\underline{s}}$ satisfying (2.6).

Proof. Assume first that F extends continuously up to the boundary of $\Omega \times \Omega$, and fix a maximal tripotent e . The holomorphic function $F(\mathbf{z}, e) =: F_e(\mathbf{z})$, being bounded, then belongs to the ordinary (unweighted) Bergman space on Ω , for which $\{\psi_{\alpha}^{\underline{s}} / \sqrt{(p)_{\underline{s}}}\}_{\alpha, \underline{s}}$ (p being the genus of Ω) serves as an orthonormal basis. Thus the Fourier decomposition

$$F_e = \sum_{\underline{s}} \sum_{\alpha} \frac{\langle F_e, \psi_{\alpha}^{\underline{s}} \rangle}{(p)_{\underline{s}}} \psi_{\alpha}^{\underline{s}} =: \sum_{\underline{s}} f_{\underline{s}}$$

converges in $L^2(\Omega)$ and, by the properties of the reproducing kernels [4], absolutely and uniformly on compact subsets of Ω . If \mathbb{L} denotes the stabilizer subgroup of e in \mathbb{K} , then for any $l \in \mathbb{L}$ we have, by \mathbb{K} -invariance, $F_e(l\mathbf{z}) = F(l\mathbf{z}, e) = F(\mathbf{z}, l^{-1}e) = F(\mathbf{z}, e) = F_e(\mathbf{z})$; in other words, F_e and, hence, each $f_{\underline{s}}$ are \mathbb{L} -invariant. However, the only \mathbb{L} -invariant element in $\mathcal{P}_{\underline{s}}$, up to constant multiples, is $K_{\underline{s}}(\cdot, e)$ [11, Theorem 2.1]. Thus

$$F(\mathbf{z}, e) = \sum_{\underline{s}} c_{\underline{s}} K_{\underline{s}}(\mathbf{z}, e) \quad \forall \mathbf{z} \in \Omega,$$

with uniform convergence on compact subsets. By \mathbb{K} -invariance of both F and $K_{\underline{s}}$, this equality remains in force also for $F(\mathbf{z}, ke) = F(k^{-1}\mathbf{z}, e)$, uniformly for \mathbf{z} in compact subsets of Ω and $k \in \mathbb{K}$. Since, for each fixed \mathbf{z} , the anti-holomorphic functions $F(\mathbf{z}, \mathbf{w})$ and $K_{\underline{s}}(\mathbf{z}, \mathbf{w})$ of \mathbf{w} attain their maximum moduli on the Shilov boundary $\{ke : k \in \mathbb{K}\}$, it transpires by the same argument as in the proof of Proposition 2.1 that (2.7) holds, with convergence absolute and uniform for \mathbf{z}, \mathbf{w} in compact subsets of Ω . (In fact, \mathbf{w} could even be taken in the closure of Ω .) By part (ii) of Proposition 2.1, the claim follows.

For general F (i.e. not necessarily continuous up to the boundary), apply the argument above to $F(t\mathbf{z}, t\mathbf{w})$ for each $t \in (0, 1)$. \square

The last proposition applies, in particular, to $F = K$ with any \mathbb{K} -invariant kernel K , and to $F = 1 - 1/K$ with any such K which is in addition zero-free. If $K(\mathbf{z}, \mathbf{w}) = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}(\mathbf{z}, \mathbf{w})$ is a \mathbb{K} -invariant kernel on $\Omega \times \Omega$, then there exists a sequence of real numbers $\{b_{\underline{s}}\}$ with $b_{\mathbf{0}} = 0$ such that $1 - \frac{1}{K} = \sum_{\underline{s} > \mathbf{0}} b_{\underline{s}} K_{\underline{s}}$.

Corollary 2.3. *Any F as is the last proposition satisfies*

$$F(t\mathbf{z}, \mathbf{w}) = F(\mathbf{z}, t\mathbf{w}) \quad \forall \mathbf{z}, \mathbf{w} \in \Omega$$

for any $0 < t < 1$. In particular, for each $\mathbf{w} \in \Omega$, $F(\cdot, \mathbf{w})$ extends to a function holomorphic in a neighborhood of the closure of Ω .

Proof. For each \underline{s} and any $0 < t < 1$, $K_{\underline{s}}(t\mathbf{z}, \mathbf{w}) = t^{|\underline{s}|} K_{\underline{s}}(\mathbf{z}, \mathbf{w}) = K_{\underline{s}}(\mathbf{z}, t\mathbf{w})$. \square

Now we have all the necessary tools to formulate a criterion for a non-vanishing \mathbb{K} -invariant kernel $K = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}$ on Ω to possess the CNP property in terms of the sequence $\{a_{\underline{s}}\}_{\underline{s} > \mathbf{0}}$. We first observe that every non-vanishing \mathbb{K} -invariant kernel is irreducible.

Lemma 2.4. *Let $K = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}$ be a non-vanishing \mathbb{K} -invariant kernel on Ω . Then K is irreducible.*

Proof. Note that it is enough to prove $K(\cdot, \mathbf{w}_1)$ and $K(\cdot, \mathbf{w}_2)$ are linearly independent for any two distinct points $\mathbf{w}_1, \mathbf{w}_2 \in \Omega$. Let \mathbf{w}_1 and \mathbf{w}_2 be two distinct elements of Ω . Suppose $\mathbf{w}_j(i)$, $j = 1, 2$, $1 \leq i \leq r$, represents the i th component of the point \mathbf{w}_j . Assume that there exists a $c \in \mathbb{C}$ such that

$$K(\cdot, \mathbf{w}_1) = cK(\cdot, \mathbf{w}_2). \quad (2.8)$$

Evaluating both sides of Equation (2.8) at $\mathbf{0} \in \Omega$, we obtain $c = 1$. Substituting $c = 1$ and the expression of K in Equation (2.8), we get

$$\begin{aligned} 0 &= \sum_{\underline{s} \in \tilde{\mathbb{N}}_0^r, |\underline{s}| \geq 1} a_{\underline{s}} (K_{\underline{s}}(\mathbf{z}, \mathbf{w}_1) - K_{\underline{s}}(\mathbf{z}, \mathbf{w}_2)) \\ &= a_{\varepsilon_1} \langle \mathbf{z}, \mathbf{w}_1 - \mathbf{w}_2 \rangle + \sum_{\underline{s} \in \tilde{\mathbb{N}}_0^r, |\underline{s}| \geq 2} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{z}) (\psi_{\alpha}^{\underline{s}}(\mathbf{w}_1) - \psi_{\alpha}^{\underline{s}}(\mathbf{w}_2)), \quad \mathbf{z} \in \Omega. \end{aligned} \quad (2.9)$$

Recall that $a_{\varepsilon_1} \neq 0$. Thus, differentiating both sides of Equation (2.9) with respect to the i th variable and then evaluating at $\mathbf{z} = \mathbf{0}$, we conclude that $\mathbf{w}_1(i) = \mathbf{w}_2(i)$. This proves that $\mathbf{w}_1 = \mathbf{w}_2$. Hence, K is irreducible. \square

Henceforth, we consider only non-vanishing \mathbb{K} -invariant kernels on Ω . The following corollary is an immediate consequence of Proposition 2.2 which provides a criterion of a \mathbb{K} -invariant kernel to have CNP property.

Corollary 2.5. *A non-vanishing \mathbb{K} -invariant kernel $K : \Omega \times \Omega \rightarrow \mathbb{C}$ is CNP if and only if there exists a sequence of non-negative real numbers $\{b_{\underline{s}}\}_{\underline{s}>\underline{0}}$ such that*

$$1 - \frac{1}{K(\mathbf{z}, \mathbf{w})} = \sum_{\underline{s}>\underline{0}} b_{\underline{s}} K_{\underline{s}}(\mathbf{z}, \mathbf{w}) \quad (2.10)$$

holds for every $\mathbf{z}, \mathbf{w} \in \Omega$.

Proof. If K is a non-vanishing \mathbb{K} -invariant kernel, then the existence of a sequence of real numbers $\{b_{\underline{s}}\}_{\underline{s}>\underline{0}}$ follows from Proposition 2.2. Note that K is irreducible, thanks to Lemma 2.4. Due to [5, Theorem 7.28], it follows that K is a CNP kernel if and only if $1 - \frac{1}{K}$ is non-negative definite. Also, note that $\sum_{\underline{s}>\underline{0}} b_{\underline{s}} K_{\underline{s}}$ is a non-negative definite kernel if and only if each $b_{\underline{s}} \geq 0$ (cf. [11]). \square

Remark 2.6. *The last corollary even holds for general nonvanishing \mathbb{K} -invariant functions K , i.e. it is not necessary to assume that $K(\mathbf{z}, \mathbf{w})$ is a nonnegative-definite kernel —the nonnegative definiteness follows automatically. Indeed, if $1 - 1/K =: L$ is nonnegative-definite, then*

$$K = \frac{1}{1 - L} = 1 + L + L^2 + L^3 + \dots$$

must also be nonnegative-definite: any power L^k , $k = 0, 1, 2, \dots$, is a nonnegative definite kernel by [4, Section I.8], hence also any finite sum of these powers [4, Section I.6], and, finally, their increasing limit [4, Section I.9.B]. (For the last, observe that K being nonvanishing implies that $0 \leq L(\mathbf{z}, \mathbf{z}) < 1$ for all $\mathbf{z} \in \Omega$.)

Note that a non-vanishing unitary invariant kernel $K_{\mathbb{U}}(\mathbf{z}, \mathbf{w}) = \sum_{n \geq 0} a_n \langle \mathbf{z}, \mathbf{w} \rangle^n$ on the unit ball \mathbb{B}_d has CNP property if

$$\frac{a_{n+1}}{a_n} \geq \frac{a_n}{a_{n-1}}$$

holds for every $n \geq 1$. This is the classical Kaluza lemma [5, Lemma 7.38]. In the following theorem, we generalize Kaluza lemma providing a criteria for a \mathbb{K} -invariant kernel $K = \sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} K_{\underline{s}}$ on Ω to have the CNP property in terms of the sequence $\{a_{\underline{s}}\}_{\underline{s} \geq \underline{0}}$.

Theorem 2.7 (Generalized Kaluza lemma). *Let $K = \sum a_{\underline{s}} K_{\underline{s}}$ be a \mathbb{K} -invariant kernel. Then K is a CNP kernel if for each signature \underline{s}_0 of length $k \geq 1$, the following holds*

$$\sum_{\tilde{\underline{s}}_0 \in I(\underline{s}_0)} \sum_{|\underline{p}|=k-1-|\underline{q}|} \frac{a_{\underline{p}}}{a_{\tilde{\underline{s}}_0}} c_{\underline{p}, \underline{q}}^{\tilde{\underline{s}}_0} \geq |I(\underline{s}_0)| \sum_{|\underline{p}|=k-|\underline{q}|} \frac{a_{\underline{p}}}{a_{\underline{s}_0}} c_{\underline{p}, \underline{q}}^{\underline{s}_0}, \quad (2.11)$$

where \underline{q} is any signature with $1 \leq |\underline{q}| \leq k - 1$, $I(\underline{s}_0) = \{\underline{s}_0 - \varepsilon_i : \underline{s}_0 - \varepsilon_i \text{ is a signature}\}$ and $|I(\underline{s}_0)|$ is the cardinality of the set $I(\underline{s}_0)$.

Proof. Due to [5, Theorem 7.28], K is a CNP kernel if and only if $1 - \frac{1}{K}$ is non-negative definite. It follows from Proposition 2.2 that there exists a sequence of real numbers $\{\hat{b}_{\underline{s}}\}_{\underline{s} \geq \underline{0}}$ with $\hat{b}_{\underline{0}} = 1$ such that $\frac{1}{K} = \sum \hat{b}_{\underline{s}} K_{\underline{s}}$. This implies that

$$1 - \frac{1}{K} = - \sum_{\underline{s} > \underline{0}} \hat{b}_{\underline{s}} K_{\underline{s}}.$$

Thus, $1 - \frac{1}{K}$ is non-negative definite if each $\hat{b}_{\underline{s}}$, $\underline{s} > \underline{0}$, is a non-positive real number. Consequently, existence of a sequence of non-positive real numbers $\{\hat{b}_{\underline{s}}\}_{\underline{s} > \underline{0}}$ such that $\frac{1}{K} = 1 + \sum_{\underline{s} > \underline{0}} \hat{b}_{\underline{s}} K_{\underline{s}}$ is equivalent to the fact that K is a CNP kernel. Therefore, to prove the theorem, we have to prove that the existence of a sequence $\{\hat{b}_{\underline{s}}\}_{\underline{s} > \underline{0}}$ with $\hat{b}_{\underline{0}} = 1$ and $\hat{b}_{\underline{s}} \leq 0$, $\underline{s} > \underline{0}$ such that the equation

$$\left(\sum_{\underline{s} > \underline{0}} a_{\underline{s}} K_{\underline{s}} \right) \left(\sum_{\underline{s} > \underline{0}} \hat{b}_{\underline{s}} K_{\underline{s}} \right) = 1 \quad (2.12)$$

holds. Comparing the coefficient of $K_{\underline{\varepsilon}_1}$ from both sides of Equation (2.12), we obtain

$$\hat{b}_{\underline{\varepsilon}_1} = -a_{\underline{\varepsilon}_1}. \quad (2.13)$$

For any two signatures \underline{s} and $\tilde{\underline{s}}$, note that

$$K_{\underline{s}} K_{\tilde{\underline{s}}} = \sum_{|\underline{p}| = |\underline{s}| + |\tilde{\underline{s}}|} c_{\underline{s}, \tilde{\underline{s}}}^{\underline{p}} K_{\underline{p}}, \quad (2.14)$$

where the coefficients $c_{\underline{s}, \tilde{\underline{s}}}^{\underline{p}} \geq 0$, $\underline{p} \geq \underline{s}$ and $\underline{p} \geq \tilde{\underline{s}}$ (cf. [2, Equation 2.9]).

Let $k \geq 1$. Suppose \underline{s}_0 is a signature of length k . Comparing the coefficients of $K_{\underline{s}_0}$ from both sides of Equation (2.12), we get

$$\hat{b}_{\underline{s}_0} a_{\underline{0}} + \sum_{0 < |\underline{q}| < k} \left(\sum_{|\underline{p}| = k - |\underline{q}|} a_{\underline{p}} c_{\underline{p}, \underline{q}}^{\underline{s}_0} \right) \hat{b}_{\underline{q}} + \hat{b}_{\underline{0}} a_{\underline{s}_0} = 0. \quad (2.15)$$

Note that $|\tilde{\underline{s}}_0| = k - 1$ for every $\tilde{\underline{s}}_0 \in I(\underline{s}_0)$. Therefore, for each $\tilde{\underline{s}}_0 \in I(\underline{s}_0)$, comparing the coefficient of $K_{\tilde{\underline{s}}_0}$ from both sides of Equation (2.12), we obtain

$$\hat{b}_{\tilde{\underline{s}}_0} a_{\underline{0}} + \sum_{0 < |\underline{q}| < k-1} \left(\sum_{|\tilde{\underline{p}}| = k-1-|\underline{q}|} a_{\tilde{\underline{p}}} c_{\tilde{\underline{p}}, \underline{q}}^{\tilde{\underline{s}}_0} \right) \hat{b}_{\underline{q}} + \hat{b}_{\underline{0}} a_{\tilde{\underline{s}}_0} = 0. \quad (2.16)$$

Multiplying Equation (2.16) by $\frac{a_{\underline{s}_0}}{|I(\underline{s}_0)| a_{\tilde{\underline{s}}_0}}$, adding them up for all $\tilde{\underline{s}}_0 \in I(\underline{s}_0)$ and then subtracting it from Equation (2.15), we obtain

$$\hat{b}_{\underline{s}_0} a_{\underline{0}} = \sum_{0 < |\underline{q}| \leq k-1} \left[\sum_{\tilde{\underline{s}}_0 \in I(\underline{s}_0)} \left(\sum_{|\tilde{\underline{p}}| = k-1-|\underline{q}|} \frac{a_{\tilde{\underline{p}}} a_{\underline{s}_0}}{|I(\underline{s}_0)| a_{\tilde{\underline{s}}_0}} c_{\tilde{\underline{p}}, \underline{q}}^{\tilde{\underline{s}}_0} \right) - \sum_{|\underline{p}| = k-|\underline{q}|} a_{\underline{p}} c_{\underline{p}, \underline{q}}^{\underline{s}_0} \right] \hat{b}_{\underline{q}}. \quad (2.17)$$

Therefore, it follows from Equation (2.17) that $\hat{b}_{\underline{s}_0} \leq 0$ if Equation (2.11) holds for every signatures \underline{q} with $1 \leq |\underline{q}| \leq k-1$. \square

Remark 2.8. Denoting momentarily

$$A_{\underline{q}}^{\underline{s}} := \sum_{\underline{p}} \frac{a_{\underline{p}}}{a_{\underline{s}}} c_{\underline{p}, \underline{q}}^{\underline{s}}$$

(note that $c_{\underline{p}, \underline{q}}^{\underline{s}}$ is nonzero only for $|\underline{s}| = |\underline{p}| + |\underline{q}|$, so the last sum is automatically finite), Equation (2.11) can be rephrased as

$$\frac{1}{I(\underline{s}_0)} \sum_{\tilde{\underline{s}}_0 \in I(\underline{s}_0)} A_{\underline{q}}^{\tilde{\underline{s}}_0} \geq A_{\underline{q}}^{\underline{s}_0}, \text{ for all } |\underline{q}| \geq 1.$$

That is, $A_{\underline{q}}^{\underline{s}_0}$ has to be less than or equal to the average of the same quantities over $I(\underline{s}_0)$.

For Ω the ball, $I(\underline{s}_0)$ becomes a singleton, $A_{\underline{q}}^{\underline{s}}$ becomes just a_{s-q}/a_s , and the connection with the ordinary Kaluza Lemma becomes evident.

2.1. Examples. The space of all the square integrable holomorphic functions on Ω $\mathbb{A}^2(\Omega)$ with respect to the Lebesgue measure $dm(z)$ is known as the Bergman space over Ω . It is a reproducing kernel Hilbert space. The reproducing kernel of $\mathbb{A}^2(\Omega)$, also known as the Bergman kernel, is $B(\mathbf{z}, \mathbf{w}) = \Delta(\mathbf{z}, \mathbf{w})^{-p}$, where $p = 2 + a(r-1) + b$, is called the *genus* of the domain Ω (see [19, Theorem 2.9.8]). Here $\Delta(\mathbf{z}, \mathbf{w})$ is a \mathbb{K} -invariant sesqui-analytic polynomial on $\Omega \times \Omega$, known as the *Jordan triple determinant*.

Also, for $\nu \in \mathcal{W}_\Omega := \{0, \dots, \frac{a}{2}(r-1)\} \cup (\frac{a}{2}(r-1), \infty)$, the sesqui-analytic function $K^{(\nu)} : \Omega \times \Omega \rightarrow \mathbb{C}$ defined by

$$K^{(\nu)}(\mathbf{z}, \mathbf{w}) := \Delta(\mathbf{z}, \mathbf{w})^{-\nu} = \sum_{\underline{s}} (\nu)_{\underline{s}} K_{\underline{s}}(\mathbf{z}, \mathbf{w}), \quad \mathbf{z}, \mathbf{w} \in \Omega, \quad (2.18)$$

is non negative definite (see [11, Corollary 5.2]). Here, $(\nu)_{\underline{s}}$ denotes the *generalized Pochhammer symbol*

$$(\nu)_{\underline{s}} := \prod_{j=1}^r \left(\nu - \frac{a}{2}(j-1) \right)_{s_j} = \prod_{j=1}^r \prod_{l=1}^{s_j} \left(\nu - \frac{a}{2}(j-1) + l - 1 \right).$$

Therefore, by the Moore-Aronszajn Theorem, $K^{(\nu)}$ determines a Hilbert space $\mathbb{A}^{(\nu)}(\Omega)$, known as the weighted Bergman space. If $\nu = \frac{d}{r}$ and $\nu = \frac{a}{2}(r-1) + \frac{d}{r} + 1$, then the weighted Bergman spaces $\mathbb{A}^{(\nu)}(\Omega)$ coincide with the Hardy space over the Shilov boundary of Ω and the classical Bergman space, respectively.

Proposition 2.9. *Let Ω be a Cartan domain of rank $r > 1$. For any $\nu \in \mathcal{W}_\Omega$, the weighted Bergman kernel $K^{(\nu)}(\mathbf{z}, \mathbf{w}) := \Delta(\mathbf{z}, \mathbf{w})^{-\nu}$ is a CNP kernel if and only if $\nu = 0$.*

Proof. By the criterion recalled above, $\Delta(\mathbf{z}, \mathbf{w})^{-\nu}$ being a CNP kernel is equivalent to $1 - \Delta(\mathbf{z}, \mathbf{w})^\nu$ being a positive semi-definite kernel. By the Faraut-Koranyi binomial formula (2.18), this in turn is equivalent to

$$(-\nu)_{\underline{s}} \leq 0 \quad \forall |\underline{s}| \geq 1.$$

For $\underline{s} = (1, 0, \dots, 0)$ and $\underline{s} = (1, 1, 0, \dots, 0)$, this becomes

$$-\nu \leq 0 \quad \text{and} \quad (-\nu)(-\nu - \frac{a}{2}) \leq 0,$$

respectively. The first means that $\nu \geq 0$, while the second means that $-\frac{a}{2} \leq \nu \leq 0$ (note that the characteristic multiplicity a is always a nonnegative integer). Hence $\nu = 0$ is the only possible solution. \square

Hence, it is natural to ask whether there exists any \mathbb{K} -invariant CNP kernel on Ω . Indeed, there exist \mathbb{K} -invariant CNP kernels. In the following, we provide examples of such kernels.

Let $\{\hat{b}_{\underline{s}}\}_{\underline{s} \geq \underline{0}}$ be a sequence of real numbers with $\hat{b}_{\underline{0}} = 1$ and $\hat{b}_{\underline{s}} \leq 0$ for all $\underline{s} > \underline{0}$. Also, assume that the series $\tilde{L} = \sum_{\underline{s} \geq \underline{0}} \hat{b}_{\underline{s}} K_{\underline{s}}$ is convergent uniformly on compact subsets of $\mathbb{B}_R \times \mathbb{B}_R$ and non-zero everywhere on $\mathbb{B}_R \times \mathbb{B}_R$ where \mathbb{B}_R is the open ball centered at 0, radius $R > 0$ in \mathbb{C}^d . For $c = \frac{r}{R}$, the function $L(\mathbf{z}, \mathbf{w}) = \tilde{L}(c\mathbf{z}, c\mathbf{w})$, $\mathbf{z}, \mathbf{w} \in \mathbb{B}_R$, defines a function which is holomorphic in \mathbf{z} and anti-holomorphic in \mathbf{w} on \mathbb{B}_r . Also, note that L

is non-zero everywhere on \mathbb{B}_r . Recall from Equation (1.1) that for each signature \underline{s} ,

$$K_{\underline{s}} = \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}} \overline{\psi_{\alpha}^{\underline{s}}},$$

where every $\psi_{\alpha}^{\underline{s}}$ is a homogeneous polynomial of degree $|\underline{s}|$. Hence, it follows that

$$L(\mathbf{z}, \mathbf{w}) = \sum_{\underline{s} \geq \underline{0}} \hat{b}_{\underline{s}} c^{2|\underline{s}|} K_{\underline{s}}(\mathbf{z}, \mathbf{w}), \quad \mathbf{z}, \mathbf{w} \in \mathbb{B}_r.$$

Using mathematical induction and Equation (2.15), we obtain a sequence of positive real numbers $\{a_{\underline{s}}\}_{\underline{s} \geq \underline{0}}$, $a_{\underline{0}} = 1$, such that Equation (2.12) holds. In fact, for every $\underline{s}_0 > \underline{0}$, Equation (2.15) provides a positive real number $a_{\underline{s}_0}$. Thus,

$$K := \frac{1}{L} = \sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} K_{\underline{s}}$$

is a non-negative definite kernel on \mathbb{B}_r , in particular on Ω . Also, it follows from Corollary 2.5 that K is CNP.

Remark 2.10. *In fact, all CNP kernels on Ω arise essentially by a construction as in the last example. Namely, let $c_{\underline{s}} \geq 0$ be such that*

$$\sum_{|\underline{s}| > 0} c_{\underline{s}} K_{\underline{s}}(\mathbf{e}, \mathbf{e}) \leq 1$$

where \mathbf{e} is any element of the Shilov boundary of Ω (a maximal tripotent, see [1]; the value of $K_{\underline{s}}(\mathbf{e}, \mathbf{e})$ is independent of the choice of \mathbf{e} and there is an explicit formula for it in terms of the signature \underline{s} and the domain invariants r , a and b , see [11, Theorem 3.4]). Setting

$$L := \sum_{|\underline{s}| > 0} c_{\underline{s}} K_{\underline{s}},$$

we then have $0 \leq L(\mathbf{z}, \mathbf{z}) < 1$ for all $\mathbf{z} \in \Omega$. The Cauchy-Schwarz inequality $|K_{\underline{s}}(\mathbf{z}, \mathbf{w})|^2 \leq K_{\underline{s}}(\mathbf{z}, \mathbf{z}) K_{\underline{s}}(\mathbf{w}, \mathbf{w})$ thus shows that the series for L converges on all of $\Omega \times \Omega$, is bounded by 1 in modulus there, defines a nonnegative definite kernel. By Remark 2.6,

$$K := \frac{1}{1 - L}$$

will therefore be a CNP kernel on Ω . Reversing this procedure also shows that all CNP kernels on Ω arise in this way.

3. CNP PROPERTY AND CHARACTERISTIC FUNCTION

A non-vanishing unitary invariant kernel K on \mathbb{B}_d is said to be an admissible kernel if the tuple $\mathbf{M} = (M_{z_1}, \dots, M_{z_d})$ of multiplication operators by the co-ordinate functions is bounded and \mathbf{M} is a $\frac{1}{K}$ -contraction. In [7], it is proved that an admissible kernel K on \mathbb{B}_d possesses the CNP property if and only if every pure $\frac{1}{K}$ -contraction admits a characteristic function. In this section, given a \mathbb{K} -invariant kernel on Ω , we provide a suitable definition of $\frac{1}{K}$ -contractivity. Our main objective of this section is to provide a necessary and sufficient condition for \mathbb{K} -invariant kernels to possess the CNP property in terms of the existence of the characteristic function of every pure $\frac{1}{K}$ -contraction.

Recall that if $K = \sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} K_{\underline{s}}$ is a \mathbb{K} -invariant kernel, then, due to Proposition 2.2, there exists a sequence of real numbers $\{b_{\underline{s}}\}_{\underline{s} > \underline{0}}$ such that $1 - \frac{1}{K} = \sum_{\underline{s} > \underline{0}} b_{\underline{s}} K_{\underline{s}}$. Unless

otherwise mentioned, henceforth, for a \mathbb{K} -invariant kernel $K = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}$, $\{b_{\underline{s}}\}_{\underline{s}>\underline{0}}$ represents a sequence of real numbers such that $1 - \frac{1}{K} = \sum_{\underline{s}>\underline{0}} b_{\underline{s}} K_{\underline{s}}$. We begin with a few necessary definitions.

Definition 3.1. (i) Let $K = \sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}$ be a non-vanishing \mathbb{K} -invariant kernel. Suppose \mathbf{T} is a commuting d -tuple of bounded operators. Assume that the series

$$\sum_{\underline{s}>\underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$$

converges strongly. If

$$I - \sum_{\underline{s}>\underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* \geq 0,$$

then \mathbf{T} is called a $\frac{1}{K}$ -contraction. Denote

$$\Delta_{\mathbf{T}} := \left(I - \sum_{\underline{s}>\underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* \right)^{\frac{1}{2}},$$

the positive square root of $I - \sum_{\underline{s}>\underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$.

(ii) A $\frac{1}{K}$ -contraction \mathbf{T} is said to be pure if $\sum_{\underline{s}>\underline{0}} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \Delta_{\mathbf{T}}^2 \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$ converges to

the identity operator I strongly.

Definition 3.2. A \mathbb{K} -invariant kernel K is called admissible if the operators of multiplication by the co-ordinate functions M_{z_i} are bounded operators on H_K for $i = 1, \dots, d$ and the d -tuple $\mathbf{M} = (M_{z_1}, \dots, M_{z_d})$ is a $\frac{1}{K}$ -contraction.

In the following lemma, we provide a sufficient condition on a \mathbb{K} -invariant CNP kernel such that the multiplication operators by the co-ordinate functions on the corresponding Hilbert space are bounded.

Lemma 3.3. Let K be a \mathbb{K} -invariant CNP kernel and H_K be the corresponding Hilbert space. Then, the multiplication operators M_{z_i} , $1 \leq i \leq d$, by the coordinate functions on H_K are bounded. Furthermore, K is an admissible kernel.

Proof. Recall that $a_{\underline{s}} > 0$ for each signature $\underline{s} \in \vec{N}_0^r$. From Equation (2.13), we have $b_{\varepsilon_1} = -\hat{b}_{\varepsilon_1} = a_{\varepsilon_1}$ and therefore, $b_{\varepsilon_1} > 0$. Due to Corollary 2.5, $b_{\underline{s}} \geq 0$ for all \underline{s} and

$$\left(1 - \sum_{\underline{s}>\underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{z}) \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \right) K(\mathbf{z}, \mathbf{w}) = 1.$$

Thus, it follows that

$$\left(1 - \sum_{\substack{|\underline{s}|=N \\ \underline{s}>\underline{0}}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{z}) \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \right) K(\mathbf{z}, \mathbf{w}) \tag{3.19}$$

is non-negative definite. Taking $N = 1$ we see that, $(\frac{1}{b_{\varepsilon_1}} - \langle \mathbf{z}, \mathbf{w} \rangle) K(\mathbf{z}, \mathbf{w})$ is non-negative definite. Hence by [6, Lemma 3.1] it follows that the multiplication operators M_{z_i} , $1 \leq i \leq d$, are bounded.

The fact that the tuple of multiplication by the coordinate functions \mathbf{M} on H_K is a $\frac{1}{K}$ -contraction follows by an argument similar to that in the proof of [10, Lemma 5.2]. \square

In the following, we provide a technical lemma.

Lemma 3.4. *Let $K = \sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} K_{\underline{s}}$ be a \mathbb{K} -invariant kernel on Ω . If $1 - \frac{1}{K} = \sum_{\underline{s} > \underline{0}} b_{\underline{s}} K_{\underline{s}}$ for a sequence of real numbers $\{b_{\underline{s}}\}_{\underline{s} > \underline{0}}$, then, for every signature $\underline{p} > \underline{0}$, we have*

$$a_{\underline{p}} = \sum_{\substack{\underline{s} > \underline{0}, \underline{\tilde{s}} > \underline{0}, \\ |\underline{s}| + |\underline{\tilde{s}}| = |\underline{p}|}} a_{\underline{s}} b_{\underline{\tilde{s}}} c_{\underline{s}, \underline{\tilde{s}}}^{\underline{p}}. \quad (3.20)$$

Proof. The existence of a sequence of real numbers $\{b_{\underline{s}}\}_{\underline{s} > \underline{0}}$ such that $1 - \frac{1}{K} = \sum_{\underline{s} > \underline{0}} b_{\underline{s}} K_{\underline{s}}$ follows from Proposition 2.2. The equation $1 - \frac{1}{K} = \sum_{\underline{s} > \underline{0}} b_{\underline{s}} K_{\underline{s}}$ is equivalent to the following equation

$$\sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} K_{\underline{s}} = 1 + \left(\sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} K_{\underline{s}} \right) \left(\sum_{\underline{\tilde{s}} > \underline{0}} b_{\underline{\tilde{s}}} K_{\underline{\tilde{s}}} \right).$$

Using Equation (2.14) and then, comparing the coefficient of $K_{\underline{p}}$ from both sides of the above equation, we obtain

$$a_{\underline{p}} = \sum_{\substack{\underline{s} > \underline{0}, \underline{\tilde{s}} > \underline{0}, \\ |\underline{s}| + |\underline{\tilde{s}}| = |\underline{p}|}} a_{\underline{s}} b_{\underline{\tilde{s}}} c_{\underline{s}, \underline{\tilde{s}}}^{\underline{p}},$$

for every signature $\underline{p} > \underline{0}$. \square

The following proposition proves that the series $\sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \Delta_{\mathbf{T}}^2 \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$ is always convergent for every $\frac{1}{K}$ -contraction \mathbf{T} .

Proposition 3.5. *For any $\frac{1}{K}$ -contraction $\mathbf{T} = (T_1, \dots, T_d)$ the series*

$$\sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \Delta_{\mathbf{T}}^2 \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$$

converges strongly to a positive contraction.

Proof. If the convergence of the series, given in the statement of the lemma, is proved, then the positivity follows immediately. For $N \geq 1$, let

$$S_N = \sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \Delta_{\mathbf{T}}^2 \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*.$$

Let $h \in \mathcal{H}$. Then,

$$\begin{aligned} & \langle S_N h, h \rangle \\ &= \sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \langle \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \Delta_{\mathbf{T}}^2 \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* h, h \rangle \\ &= \sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \|\psi_{\alpha}^{\underline{s}}(\mathbf{T})^* h\|^2 - \sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\underline{\tilde{s}} > \underline{0}} \left\langle \sum_{\alpha=1}^{d_{\underline{s}}} \sum_{\tilde{\alpha}=1}^{d_{\underline{\tilde{s}}}} b_{\underline{\tilde{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \psi_{\tilde{\alpha}}^{\underline{\tilde{s}}}(\mathbf{T}) \psi_{\tilde{\alpha}}^{\underline{\tilde{s}}}(\mathbf{T})^* \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* h, h \right\rangle. \end{aligned} \quad (3.21)$$

Note that the identity

$$\sum_{\alpha=1}^{d_s} \sum_{\bar{\alpha}=1}^{d_{\bar{s}}} \psi_{\alpha}^s(\mathbf{T}) \psi_{\bar{\alpha}}^{\bar{s}}(\mathbf{T}) \psi_{\bar{\alpha}}^{\bar{s}}(\mathbf{T})^* \psi_{\alpha}^s(\mathbf{T})^* = \sum_{|p|=|s|+|\bar{s}|} c_{s,\bar{s}}^p \sum_{\beta}^{d_p} \psi_{\beta}^p(\mathbf{T}) \psi_{\beta}^p(\mathbf{T})^*$$

follows from Equation (2.14). Using the above identity in Equation (3.21), we obtain

$$\begin{aligned} \langle S_N h, h \rangle &= \sum_{|s|=0}^N a_s \sum_{\alpha=1}^{d_s} \|\psi_{\alpha}^s(\mathbf{T})^* h\|^2 - \sum_{|s|=0}^N \sum_{\bar{s}>0} \sum_{|p|=|s|+|\bar{s}|} a_s b_{\bar{s}} c_{s,\bar{s}}^p \sum_{\beta=1}^{d_p} \|\psi_{\beta}^p(\mathbf{T})^* h\|^2 \\ &\leq \sum_{|s|=0}^N a_s \sum_{\alpha=1}^{d_s} \|\psi_{\alpha}^s(\mathbf{T})^* h\|^2 - \sum_{1 \leq |p| \leq N} \left(\sum_{\substack{0 \leq |s| \leq N \\ \bar{s} > 0 \\ |s|+|\bar{s}|=|p|}} a_s b_{\bar{s}} c_{s,\bar{s}}^p \right) \sum_{\beta=1}^{d_p} \|\psi_{\beta}^p(\mathbf{T})^* h\|^2 \\ &= \sum_{|s|=0}^N a_s \sum_{\alpha=1}^{d_s} \|\psi_{\alpha}^s(\mathbf{T})^* h\|^2 - \sum_{1 \leq |p| \leq N} a_p \sum_{\beta=1}^{d_p} \|\psi_{\beta}^p(\mathbf{T})^* h\|^2 \\ &= \|h\|^2. \end{aligned}$$

Here, the second last equality follows from Lemma 3.4. This proves that the sequence of operators $\{S_N\}$ converges strongly to a contraction. \square

Remark 3.6. For any $\frac{1}{K}$ -contraction $\mathbf{T} = (T_1, \dots, T_d)$, the proof of Proposition 3.5 implies that

$$S_N = \sum_{|s|=0}^N a_s \sum_{\alpha=1}^{d_s} \psi_{\alpha}^s(\mathbf{T}) \Delta_{\mathbf{T}}^2 \psi_{\alpha}^s(\mathbf{T})^*$$

is a contraction for every $N \geq 1$ and therefore $\{\|S_N\|\}$ is bounded above by 1.

The operator $V_{\mathbf{T}}$, defined in the following corollary, plays a significant role in the theory $\frac{1}{K}$ -calculus and also, in the theory of $\frac{1}{K}$ -contractions (cf. [2]). Following corollary is an immediate consequence of Proposition 3.5 and [2, Theorem 1.3].

Corollary 3.7. If $\mathbf{T} = (T_1, \dots, T_d)$ is a $\frac{1}{K}$ -contraction on a Hilbert space \mathcal{H} , then the operator $V_{\mathbf{T}} : \mathcal{H} \rightarrow H_K \otimes \overline{\text{Ran}} \Delta_{\mathbf{T}}$, defined by

$$V_{\mathbf{T}}(h) = \sum_{\substack{\bar{s} \geq 0 \\ |s| \leq N}} a_s \sum_{\alpha=1}^{d_s} \psi_{\alpha}^s \otimes \Delta_{\mathbf{T}} \psi_{\alpha}^s(\mathbf{T})^* h, \quad h \in \mathcal{H}, \quad (3.22)$$

is a contraction and satisfies

$$V_{\mathbf{T}}^* \left(p(\mathbf{M}_z \otimes I_{\overline{\text{Ran}} \Delta_{\mathbf{T}}}) \right) = p(\mathbf{T}) V_{\mathbf{T}}^*,$$

for every polynomials p .

Let E_0 denote the orthogonal projection of the reproducing kernel Hilbert space H_K onto the one dimensional subspace consisting of the constant functions. The following lemma asserts that the d -tuple of multiplication operators by the co-ordinate functions $\mathbf{M} = (M_{z_1}, \dots, M_{z_d})$ on H_K is pure whenever K is an admissible kernel.

Lemma 3.8. *Suppose K is an admissible kernel. Then $\Delta_{\mathbf{M}} = E_0$ and the d -tuple of multiplication operators by the co-ordinate functions $\mathbf{M} = (M_{z_1}, \dots, M_{z_d})$, acting on the Hilbert space H_K , is pure.*

Proof. Let $H_K^0 := \{\sum_{i=1}^n c_i K_{\mathbf{w}_i} : n \geq 1, c_1, \dots, c_n \in \mathbb{C}, \mathbf{w}_1, \dots, \mathbf{w}_n \in \Omega\}$. The space H_K^0 is dense in H_K . Take an arbitrary element $\sum_{i=1}^n c_i K_{\mathbf{w}_i}$ of H_K^0 . A straightforward calculation implies that

$$\Delta_{\mathbf{M}}^2 \left(\sum_{i=1}^n c_i K_{\mathbf{w}_i} \right) = \left\langle \sum_{i=1}^n c_i K_{\mathbf{w}_i}, K_0 \right\rangle K_0.$$

Since H_K^0 is a dense subset of H , it follows that $\Delta_{\mathbf{M}}$ is the projection E_0 .

For $N \geq 1$, let S_N denote $\sum_{|\underline{s}| \leq N} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{M}) \Delta_{\mathbf{M}}^2 \psi_{\alpha}^{\underline{s}}(\mathbf{M})^*$. Suppose $\mathbf{w} \in \Omega$ be an arbitrary element. Since the sequence of partial sums $\{\sum_{|\underline{s}| \leq N} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\cdot)\}_{N \geq 1}$ converges to $K_{\mathbf{w}}$ in the norm topology, given an $\varepsilon > 0$, there exists a natural number m such that

$$\|K_{\mathbf{w}} - \sum_{|\underline{s}| \leq N} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\cdot)\| < \varepsilon \quad (3.23)$$

holds for every $N \geq m$. A straightforward computation implies that

$$S_N K_{\mathbf{w}} = \sum_{|\underline{s}| \leq N} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\cdot). \quad (3.24)$$

Combining Equations (3.23) and (3.24), we obtain

$$\|K_{\mathbf{w}} - S_N K_{\mathbf{w}}\| < \varepsilon$$

holds for every $N \geq m$. This implies that the sequence $\{S_N K_{\mathbf{w}}\}$ converges to $K_{\mathbf{w}}$. Consequently, for each h in H_K^0 , the sequence $\{S_N h\}$ is convergent. Combining the facts that H_K^0 is a dense subset of H_K and that the sequence $\{\|S_N\|\}$ is bounded - a consequence of Proposition 3.5 - we obtain that $\{S_N\}$ converges to the identity operator in the strong operator topology of H_K . \square

The following theorem is due to Arazy and Engliš. For the proof of the following theorem, see [2, Theorem 1.3].

Theorem 3.9. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a commuting tuple of operators on a Hilbert space \mathcal{H} . If \mathbf{T} is a pure $\frac{1}{K}$ -contraction, then the map $V_{\mathbf{T}} : \mathcal{H} \rightarrow H_K \otimes \overline{\text{Ran } \Delta_{\mathbf{T}}}$, given by*

$$h \rightarrow \sum_{\underline{s} > 0} \sum_{\alpha=1}^{d_{\underline{s}}} a_{\underline{s}} \psi_{\alpha}^{\underline{s}} \otimes \Delta_{\mathbf{T}} (\psi_{\alpha}^{\underline{s}}(\mathbf{T}))^* h,$$

is an isometry satisfying

$$V_{\mathbf{T}}^* \left(p(\mathbf{M}) \otimes I_{\overline{\text{Ran } \Delta_{\mathbf{T}}}} \right) = p(\mathbf{T}) V_{\mathbf{T}}^*$$

for all $p \in \mathbb{C}[z_1, \dots, z_d]$.

We recall the definition of a (K, \mathbf{T}) -factorable positive operator from [7, Definition 2.3]. Suppose K is an admissible kernel and $\mathbf{T} = (T_1, \dots, T_d)$ is a commuting d -tuple of operators on \mathcal{H} . A positive operator $X \in \mathcal{B}(\mathcal{H})$ is said to be (K, \mathbf{T}) -factorable if X has closed range and there exists a Hilbert space \mathcal{L} with a bounded linear transformation $\Theta : H_K \otimes \mathcal{L} \rightarrow \mathcal{H}$ such that $X = \Theta \Theta^*$ and $\Theta(M_{z_i} \otimes I_{\mathcal{L}}) = T_i \Theta$ holds for each i . The

following proposition is the key ingredient of the main theorem of this section. Both the proposition and proof are motivated from [7, Proposition 2.4].

Proposition 3.10. *$X \in \mathcal{B}(\mathcal{H})$ is (K, \mathbf{T}) -factorable if and only if*

- (1) *for all i , $\|M_{z_i}\|^2 X - T_i X T_i^* \geq 0$.*
- (2) *$P_{\mathbf{T}}(X) = \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) X \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$ converges strongly such that $X - P_{\mathbf{T}}(X) \geq 0$.*
- (3) *$\sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) (X - P_{\mathbf{T}}(X)) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$ converges to X .*

Proof. Let X be (K, \mathbf{T}) -factorable, that is, there exists a Hilbert space \mathcal{L} with a bounded linear transformation $\Theta : H_K \otimes \mathcal{L} \rightarrow \mathcal{H}$ such that $X = \Theta \Theta^*$ and $\Theta(M_{z_i} \otimes I_{\mathcal{L}}) = T_i \Theta$ holds for each i . Then

$$\begin{aligned} \|M_{z_i}\|^2 X - T_i X T_i^* &= \|M_{z_i}\|^2 \Theta \Theta^* - T_i \Theta \Theta^* T_i^* \\ &= \|M_{z_i}\|^2 \Theta \Theta^* - \Theta(M_{z_i} \otimes I_{\mathcal{L}})(M_{z_i}^* \otimes I_{\mathcal{L}})\Theta^* \\ &= \Theta \left(\|M_{z_i}\|^2 I - (M_{z_i} \otimes I_{\mathcal{L}})(M_{z_i}^* \otimes I_{\mathcal{L}}) \right) \Theta^* \geq 0. \end{aligned}$$

This proves (1). The following computation implies (2):

$$\begin{aligned} X - P_{\mathbf{T}}(X) &= X - \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) X \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* \\ &= \Theta \left(I - \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} (\psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I_{\mathcal{L}}) (\psi_{\alpha}^{\underline{s}}(\mathbf{M})^* \otimes I_{\mathcal{L}}) \right) \Theta^* \geq 0. \end{aligned}$$

Also, observe that

$$\begin{aligned} &\sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) (X - P_{\mathbf{T}}(X)) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* \\ &= \sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) (\Theta \Theta^* - P_{\mathbf{T}}(\Theta \Theta^*)) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* \\ &= \Theta \left[\sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} (\psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I_{\mathcal{L}}) \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I_{\mathcal{L}})^* - \sum_{|\underline{s}|=0}^N \sum_{\alpha=1}^{d_{\underline{s}}} a_{\underline{s}} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I_{\mathcal{L}}) \right. \\ &\quad \left. \left(\sum_{\tilde{\underline{s}} > \underline{0}} b_{\tilde{\underline{s}}} \sum_{\alpha=1}^{d_{\tilde{\underline{s}}}} (\psi_{\alpha}^{\tilde{\underline{s}}}(\mathbf{M} \otimes I_{\mathcal{L}}) \psi_{\alpha}^{\tilde{\underline{s}}}(\mathbf{M} \otimes I_{\mathcal{L}})^* \right) \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I)^* \right] \Theta^*. \end{aligned}$$

Since \mathbf{M} is a pure $\frac{1}{K}$ -contraction, it follows that

$$\left[\sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I_{\mathcal{L}}) \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I_{\mathcal{L}})^* - \sum_{|\underline{s}|=0}^N \sum_{\alpha=1}^{d_{\underline{s}}} a_{\underline{s}} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I_{\mathcal{L}}) \right. \\ \left. \left(\sum_{\underline{\tilde{s}} > \underline{0}} b_{\underline{\tilde{s}}} \sum_{\alpha=1}^{d_{\underline{\tilde{s}}}} \psi_{\alpha}^{\underline{\tilde{s}}}(\mathbf{M} \otimes I_{\mathcal{L}}) \psi_{\alpha}^{\underline{\tilde{s}}}(\mathbf{M} \otimes I_{\mathcal{L}})^* \right) \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I)^* \right]$$

converges to $I_{H_K \otimes \mathcal{L}}$ in the strong operator topology and therefore, (3) holds.

Conversely, let us assume (1), (2), and (3). For each $i = 1, \dots, d$, define the operator A_i on the range of X , $\text{Ran } X$, by $A_i X^{1/2} h = X^{1/2} T_i^* h$, $h \in \mathcal{H}$. It is easy to see that $\mathbf{A} = (A_1, \dots, A_d)$ is a commuting tuple of bounded linear operators. Now, using the condition (2), we get

$$\left\langle \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{A}^*) \psi_{\alpha}^{\underline{s}}(\mathbf{A}) X^{1/2} h, X^{1/2} h \right\rangle \\ = \left\langle \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}^*) X \psi_{\alpha}^{\underline{s}}(\mathbf{T}) h, h \right\rangle \leq \langle Xh, h \rangle = \langle X^{1/2} h, X^{1/2} h \rangle.$$

Thus \mathbf{A}^* is a $\frac{1}{K}$ -contraction on $\text{Ran } X$. Also, a similar argument with condition (3) shows that \mathbf{A}^* is pure:

$$\left\langle \sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{A}^*) \Delta_{\mathbf{A}^*}^2 \psi_{\alpha}^{\underline{s}}(\mathbf{A}) X^{1/2} h, X^{1/2} h \right\rangle \\ = \left\langle \sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}^*) (X - P_{\text{Ran } X}) \psi_{\alpha}^{\underline{s}}(\mathbf{T}) h, h \right\rangle = \langle Xh, h \rangle.$$

Now Theorem 3.9 yields there exists an isometry $V : \text{Ran } X \rightarrow H_K \otimes \overline{\text{Ran } \Delta_{\mathbf{A}}}$ such that

$$V^* \left(M_i \otimes I_{\overline{\text{Ran } \Delta_{\mathbf{A}}}} \right) = A_i V^*.$$

Finally, if we define $\Theta = X^{1/2} V^*$ it is easy to check $X = \Theta \Theta^*$ and for each i

$$\Theta \left(M_i \otimes I_{\overline{\text{Ran } \Delta_{\mathbf{A}}}} \right) = A_i \Theta.$$

□

Below, we provide the definition of the characteristic function of a $\frac{1}{K}$ -contraction, following [7, Definition 2.5]. The definition of the characteristic function of a $\frac{1}{K}$ -contraction \mathbf{T} in [7, Definition 2.5] is given by assuming that \mathbf{T} is pure. In our context, we give the definition of the characteristic function for any $\frac{1}{K}$ -contraction.

Definition 3.11. *Let $\mathbf{T} = (T_1, \dots, T_d)$ be a $\frac{1}{K}$ -contraction. The tuple \mathbf{T} is said to admit a characteristic function if there exists a Hilbert space \mathcal{E} and an analytic function $\Theta_{\mathbf{T}} : \Omega \rightarrow \mathcal{B}(\mathcal{E}, \overline{\text{Ran } \Delta_{\mathbf{T}}})$ such that the corresponding multiplication operator $M_{\Theta_{\mathbf{T}}} : H_K \otimes \mathcal{E} \rightarrow H_K \otimes \overline{\text{Ran } \Delta_{\mathbf{T}}}$ fulfills the following identity*

$$I - V_{\mathbf{T}} V_{\mathbf{T}}^* = M_{\Theta_{\mathbf{T}}} M_{\Theta_{\mathbf{T}}}^*.$$

If K is an admissible kernel, it follows from Theorem 3.9 that $\ker V_{\mathbf{T}}^*$ is an invariant subspace for each $M_{z_i} \otimes I_{\overline{\text{Ran } \Delta_{\mathbf{T}}}}$, $1 \leq i \leq d$.

Definition 3.12. For a pure $\frac{1}{K}$ -contraction $\mathbf{T} = (T_1, \dots, T_d)$, associated d -tuple of commuting operators $B_{\mathbf{T}}$ is defined as

$$B_{\mathbf{T}} := \left((M_{z_1} \otimes I_{\overline{\text{Ran } \Delta_{\mathbf{T}}}})|_{\ker V_{\mathbf{T}}^*}, \dots, (M_{z_d} \otimes I_{\overline{\text{Ran } \Delta_{\mathbf{T}}}})|_{\ker V_{\mathbf{T}}^*} \right). \quad (3.25)$$

The theorem below is one of the important theorems of this section. It gives a necessary and sufficient criterion for the existence of the characteristic function of a pure $\frac{1}{K}$ -contraction.

Theorem 3.13. Let K be an admissible \mathbb{K} -invariant kernel over the Cartan domain Ω . A pure $\frac{1}{K}$ -contraction $\mathbf{T} = (T_1, \dots, T_d)$ on a Hilbert Space \mathcal{H} admits a characteristic function if and only if $B_{\mathbf{T}}$ is a $\frac{1}{K}$ -contraction.

Proof. Let \mathbf{T} admit a characteristic function. Then there exists a $B(\mathcal{L}, \overline{\text{Ran } \Delta_{\mathbf{T}}})$ -valued bounded analytic function $\theta_{\mathbf{T}}$ on Ω such that

$$I - V_{\mathbf{T}}V_{\mathbf{T}}^* = M_{\theta_{\mathbf{T}}}M_{\theta_{\mathbf{T}}}^*.$$

Thus for any $N \geq 1$,

$$\begin{aligned} (I - V_{\mathbf{T}}V_{\mathbf{T}}^*) - \left(\sum_{|s|=1}^N b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} (\psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I)(I - V_{\mathbf{T}}V_{\mathbf{T}}^*)(\psi_{\alpha}^{\underline{s}}(\mathbf{M})^* \otimes I) \right) \\ = M_{\theta_{\mathbf{T}}} \left(I - \sum_{|s|=1}^N b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} (\psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I)(\psi_{\alpha}^{\underline{s}}(\mathbf{M})^* \otimes I) \right) M_{\theta_{\mathbf{T}}}^*. \end{aligned} \quad (3.26)$$

Since K is an admissible kernel, it follows that the tuple of multiplication operators \mathbf{M} is $\frac{1}{K}$ -contraction and therefore, the series $\sum_{\underline{s} > \mathbf{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} (\psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I)(\psi_{\alpha}^{\underline{s}}(\mathbf{M})^* \otimes I)$ converges

strongly and $\left(I - \sum_{\underline{s} > \mathbf{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} (\psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I)(\psi_{\alpha}^{\underline{s}}(\mathbf{M})^* \otimes I) \right) \geq 0$. This with (3.26) implies that $B_{\mathbf{T}}$ is a $\frac{1}{K}$ -contraction.

Conversely, assume that $B_{\mathbf{T}}$ is a $\frac{1}{K}$ -contraction. Let $c_i = \|M_{z_i}\|^2$. Suppose $P_{\ker V_{\mathbf{T}}^*} := I - V_{\mathbf{T}}V_{\mathbf{T}}^*$ is the projection of $H_K \otimes \overline{\text{Ran } \Delta_{\mathbf{T}}}$ onto $\ker V_{\mathbf{T}}^*$. For each i , define a linear operator $R_i := M_{z_i} \otimes I|_{\ker V_{\mathbf{T}}^*}$. Then $\|R_i\|^2 \leq c_i$. In other words, $c_i I \geq R_i R_i^*$, i.e.,

$$c_i(I - V_{\mathbf{T}}V_{\mathbf{T}}^*) - (M_{z_i} \otimes I)(I - V_{\mathbf{T}}V_{\mathbf{T}}^*)(M_{z_i} \otimes I)^* \geq 0.$$

This proves condition (1) of Proposition 3.10. Since \mathbf{M} is a $\frac{1}{K}$ -contraction, it follows that the series

$$\begin{aligned} \sum_{\underline{s} > \mathbf{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I|_{\ker V_{\mathbf{T}}^*}) \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I|_{\ker V_{\mathbf{T}}^*})^* \\ = \sum_{\underline{s} > \mathbf{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I)(I - V_{\mathbf{T}}V_{\mathbf{T}}^*) \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I)^* \end{aligned}$$

converges strongly and

$$\begin{aligned} (I - V_{\mathbf{T}}V_{\mathbf{T}}^*) - \sum_{\underline{s} > 0} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I)(I - V_{\mathbf{T}}V_{\mathbf{T}}^*)\psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I)^* \\ = (I - V_{\mathbf{T}}V_{\mathbf{T}}^*) - P_{\mathbf{M} \otimes I}(I - V_{\mathbf{T}}V_{\mathbf{T}}^*) \geq 0, \end{aligned}$$

where the map $P_{\mathbf{M} \otimes I}$ is defined in Proposition 3.10 (2). This proves condition (2) of Proposition 3.10.

Let $N \geq 1$ be a natural number. Suppose

$$S_N = \sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I) \left((I - V_{\mathbf{T}}V_{\mathbf{T}}^*) - P_{\mathbf{M} \otimes I}(I - V_{\mathbf{T}}V_{\mathbf{T}}^*) \right) \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I)^*.$$

We prove that S_N converges strongly to $(I - V_{\mathbf{T}}V_{\mathbf{T}}^*)$. Consider an element $\sum_{i=1}^n (K_{\mathbf{w}_i} \otimes x_i) \in H_K \otimes \overline{\text{Ran}} \Delta_{\mathbf{T}}$. Then

$$\begin{aligned} & \left\langle S_N \sum_{i=1}^n (K_{\mathbf{w}_i} \otimes x_i), \sum_{i=1}^n (K_{\mathbf{w}_i} \otimes x_i) \right\rangle \\ &= \sum_{i,j=1}^n \left[\left\langle \sum_{|\underline{s}| \leq N} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I) \left((I - V_{\mathbf{T}}V_{\mathbf{T}}^*) \right) \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I)^* K_{\mathbf{w}_i} \otimes x_i, K_{\mathbf{w}_j} \otimes x_j \right\rangle \right. \\ & \quad \left. - \left\langle \sum_{|\underline{s}| \leq N} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I) \left(P_{\mathbf{M} \otimes I}(I - V_{\mathbf{T}}V_{\mathbf{T}}^*) \right) \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I)^* K_{\mathbf{w}_i} \otimes x_i, K_{\mathbf{w}_j} \otimes x_j \right\rangle \right] \\ &= \sum_{i,j=1}^n \left[\left\langle \sum_{|\underline{s}| \leq N} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w}_i)} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I) \left((I - V_{\mathbf{T}}V_{\mathbf{T}}^*) \right) K_{\mathbf{w}_i} \otimes x_i, K_{\mathbf{w}_j} \otimes x_j \right\rangle \right. \\ & \quad \left. - \left\langle \sum_{|\underline{s}| \leq N} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w}_i)} \psi_{\alpha}^{\underline{s}}(\mathbf{M} \otimes I) \sum_{\tilde{\underline{s}} \geq 0} b_{\tilde{\underline{s}}} \sum_{\tilde{\alpha}=1}^{d_{\tilde{\underline{s}}}} \overline{\psi_{\tilde{\alpha}}^{\tilde{\underline{s}}}(\mathbf{w}_i)} \psi_{\tilde{\alpha}}^{\tilde{\underline{s}}}(\mathbf{M} \otimes I) (I - V_{\mathbf{T}}V_{\mathbf{T}}^*) K_{\mathbf{w}_j} \otimes x_j, \right. \right. \\ & \quad \left. \left. K_{\mathbf{w}_j} \otimes x_j \right\rangle \right] \\ &= \sum_{i,j=1}^n \left[\sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w}_i)} \psi_{\alpha}^{\underline{s}}(\mathbf{w}_j) \left\langle \left((I - V_{\mathbf{T}}V_{\mathbf{T}}^*) \right) K_{\mathbf{w}_i} \otimes x_i, K_{\mathbf{w}_j} \otimes x_j \right\rangle \right. \\ & \quad \left. - \sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w}_i)} \psi_{\alpha}^{\underline{s}}(\mathbf{w}_j) \sum_{\tilde{\underline{s}} \geq 0} b_{\tilde{\underline{s}}} \sum_{\tilde{\alpha}=1}^{d_{\tilde{\underline{s}}}} \overline{\psi_{\tilde{\alpha}}^{\tilde{\underline{s}}}(\mathbf{w}_i)} \psi_{\tilde{\alpha}}^{\tilde{\underline{s}}}(\mathbf{w}_j) \left\langle \left((I - V_{\mathbf{T}}V_{\mathbf{T}}^*) \right) K_{\mathbf{w}_i} \otimes x_i, K_{\mathbf{w}_j} \otimes x_j \right\rangle \right] \\ &= \sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w}_i)} \psi_{\alpha}^{\underline{s}}(\mathbf{w}_j) \left[1 - \sum_{\tilde{\underline{s}} \geq 0} b_{\tilde{\underline{s}}} \sum_{\tilde{\alpha}=1}^{d_{\tilde{\underline{s}}}} \overline{\psi_{\tilde{\alpha}}^{\tilde{\underline{s}}}(\mathbf{w}_i)} \psi_{\tilde{\alpha}}^{\tilde{\underline{s}}}(\mathbf{w}_j) \right] \left\langle \left((I - V_{\mathbf{T}}V_{\mathbf{T}}^*) \right) \sum_{i=1}^n K_{\mathbf{w}_i} \otimes x_i, \right. \\ & \quad \left. \sum_{i=1}^n K_{\mathbf{w}_i} \otimes x_i \right\rangle. \end{aligned}$$

The equation above implies that $S_N \leq (I - V_{\mathbf{T}}V_{\mathbf{T}}^*)$. Also, note that the series

$$\sum_{|\underline{s}|=0}^N a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w}_i)} \psi_{\alpha}^{\underline{s}}(\mathbf{w}_j) \left[1 - \sum_{\underline{\tilde{s}} \geq 0} b_{\underline{\tilde{s}}} \sum_{\tilde{\alpha}=1}^{d_{\underline{\tilde{s}}}} \overline{\psi_{\tilde{\alpha}}^{\underline{\tilde{s}}}(\mathbf{w}_i)} \psi_{\tilde{\alpha}}^{\underline{\tilde{s}}}(\mathbf{w}_j) \right]$$

converges to 1 as $N \rightarrow \infty$, which implies that $\{S_N\}$ converges to $(I - V_{\mathbf{T}}V_{\mathbf{T}}^*)$ for every element of the set $(H_K \otimes \overline{\text{Ran } \Delta_{\mathbf{T}}})_0 := \{\sum_{i=1}^n K_{\mathbf{w}_i} \otimes x_i : n \geq 1, \mathbf{w}_1, \dots, \mathbf{w}_n \in \Omega, x_1, \dots, x_n \in \overline{\text{Ran } \Delta_{\mathbf{T}}}\}$. Since $(H_K \otimes \overline{\text{Ran } \Delta_{\mathbf{T}}})_0$ is a dense subset of $H_K \otimes \overline{\text{Ran } \Delta_{\mathbf{T}}}$ and $\{S_N\}$ is bounded above by $(I - V_{\mathbf{T}}V_{\mathbf{T}}^*)$, it follows that S_N converges to $(I - V_{\mathbf{T}}V_{\mathbf{T}}^*)$ in strong operator topology. This proves condition (3) of the Proposition 3.10. Therefore, $(I - V_{\mathbf{T}}V_{\mathbf{T}}^*)$ is $(K, \mathbf{M} \otimes I)$ -factorable. In other words, \mathbf{T} admits a characteristic function. \square

Recall from [7] that an admissible kernel K is said to admit a characteristic function if every pure $\frac{1}{K}$ -contraction admits a characteristic function. A sufficient condition for a \mathbb{K} -invariant kernel to have CNP property is obtained in the theorem below.

Theorem 3.14. *If K is a \mathbb{K} -invariant CNP kernel, then K admits a characteristic function.*

Proof. Let \mathbf{T} be a pure $\frac{1}{K}$ -contraction. To show that \mathbf{T} admits a characteristic function, it is enough to show $B_{\mathbf{T}}$ is a $\frac{1}{K}$ -contraction (see Theorem 3.13). Since K is an admissible kernel, \mathbf{M} on H_K is a $\frac{1}{K}$ -contraction. Thus $\mathbf{M} \otimes I$ on $H_K \otimes \overline{\text{Ran } \Delta_{\mathbf{T}}}$ is also $\frac{1}{K}$ -contraction. For any $N \geq 1$,

$$\begin{aligned} \sum_{|\underline{s}|=1}^N b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(B_{\mathbf{T}}) \psi_{\alpha}^{\underline{s}}(B_{\mathbf{T}})^* &= \sum_{|\underline{s}|=1}^N b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} P_{\ker V_{\mathbf{T}}^*}(\psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I) P_{\ker V_{\mathbf{T}}^*}(\psi_{\alpha}^{\underline{s}}(\mathbf{M})^* \otimes I)|_{\ker V_{\mathbf{T}}^*} \\ &\leq \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} P_{\ker V_{\mathbf{T}}^*}(\psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I) (\psi_{\alpha}^{\underline{s}}(\mathbf{M})^* \otimes I)|_{\ker V_{\mathbf{T}}^*} \\ &\leq I_{\ker V_{\mathbf{T}}^*}. \end{aligned}$$

Thus $B_{\mathbf{T}}$ is a $\frac{1}{K}$ -contraction. \square

Now, we have all the necessary ingredients to prove the main theorem of this section. It professes that the CNP property for an admissible kernel on a Cartan domain is characterized by the requirement of the existence of the characteristic function of every pure $\frac{1}{K}$ -contraction. A similar characterization of a unitary invariant kernel on a unit ball is proved in [7, Theorem 3.4].

Theorem 3.15. *Suppose K is an admissible kernel on a Cartan domain Ω . Then, the kernel K is CNP if and only if any pure $\frac{1}{K}$ -contraction admits a characteristic function.*

Proof. The proof of the forward direction follows from Theorem 3.14. Therefore, we prove the converse direction. Assume that any pure $\frac{1}{K}$ -contraction admits a characteristic function. Let \mathcal{H} be Hilbert space. It is trivial to verify that the d -tuple $\mathbf{T} = (0, \dots, 0)$ of zero operators on \mathcal{H} is a pure $\frac{1}{K}$ -contraction and $\Delta_{\mathbf{T}} = I_{\mathcal{H}}$. Consequently, we have $\text{Ran } \Delta_{\mathbf{T}} = \mathcal{H}$. From Theorem 3.9, it follows that the operator $V_{\mathbf{T}} : \mathcal{H} \rightarrow H_K \otimes \overline{\text{Ran } \Delta_{\mathbf{T}}}$ is given by

$$V_{\mathbf{T}}(h) = 1 \otimes h, \quad h \in \mathcal{H},$$

where 1 denotes the constant function in H_K which maps every element of Ω to 1.

Let H_K^c denote the set of all constant functions in H_K and E_0^\perp denote the orthogonal projection of H_K onto $(H_K^c)^\perp$. A direct computation verifies that $\ker(I - V_{\mathbf{T}}) = (H_K^c)^\perp \otimes \mathcal{H}$. Due to Theorem 3.14, it follows that $\mathbf{B}_{\mathbf{T}}$, defined in Equation (3.25), is a $\frac{1}{K}$ -contraction. A straightforward computation implies that $\mathbf{B}_{\mathbf{T}}$ is a $\frac{1}{K}$ -contraction if and only if

$$E_0^\perp - \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_\alpha^{\underline{s}}(\mathbf{M}) E_0^\perp \psi_\alpha^{\underline{s}}(\mathbf{M})^* \geq 0, \quad (3.27)$$

where the sequence $\{b_{\underline{s}}\}_{\underline{s} > \underline{0}}$ of real numbers is given by Corollary 2.5. To prove that K is a CNP kernel, it suffices to show $b_{\underline{s}} \geq 0$ for each signature $\underline{s} > \underline{0}$, thanks to Corollary 2.5.

Let $N \geq 1$ be an arbitrary natural number, \underline{s}' be a signature of length $N + 1$ and $\psi_{\alpha'}^{\underline{s}'}$ be an arbitrary element of the orthonormal basis of $\mathcal{P}_{\underline{s}'}$. From [19, Proposition 4.11.36], it follows that

$$\psi_\alpha^{\underline{s}}(\mathbf{M})^* \psi_{\alpha'}^{\underline{s}'} = \begin{cases} 0, & \text{if } |\underline{s}'| < |\underline{s}| \\ \sum_{|\tilde{\underline{s}}|=|\underline{s}'|-|\underline{s}|} \frac{a_{\tilde{\underline{s}}'}}{a_{\underline{s}'}} \left(\psi_\alpha^{\underline{s}}(\partial) \psi_{\alpha'}^{\underline{s}'} \right)_{\tilde{\underline{s}}}, & \text{if } |\underline{s}'| \geq |\underline{s}| \end{cases} \quad (3.28)$$

where the summation is taken over all signatures $\tilde{\underline{s}}$ such that $|\tilde{\underline{s}}| = |\underline{s}'| - |\underline{s}|$, ∂ denotes the tuple of partial differentials $(\partial_{z_1}, \dots, \partial_{z_d})$ and $\left(\psi_\alpha^{\underline{s}}(\partial) \psi_{\alpha'}^{\underline{s}'} \right)_{\tilde{\underline{s}}}$ denotes the component of $\psi_\alpha^{\underline{s}}(\partial) \psi_{\alpha'}^{\underline{s}'}$ in $\mathcal{P}_{\tilde{\underline{s}}}$ (see also [18, Lemma 15]). Let \mathbf{w} be an arbitrary element of Ω . Therefore, from Equation (3.28), we have

$$\sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_\alpha^{\underline{s}}(\mathbf{M}) E_0^\perp \psi_\alpha^{\underline{s}}(\mathbf{M})^* K_{\underline{s}'}(\cdot, \mathbf{w}) = \sum_{|\underline{s}| \leq N} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_\alpha^{\underline{s}}(\mathbf{M}) \psi_\alpha^{\underline{s}}(\mathbf{M})^* K_{\underline{s}'}(\cdot, \mathbf{w}).$$

From [18, Page 5], it follows that there exists a scalar $\gamma(\underline{s}')$ such that

$$\sum_{|\underline{s}| \leq N} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_\alpha^{\underline{s}}(\mathbf{M}) \psi_\alpha^{\underline{s}}(\mathbf{M})^* K_{\underline{s}'}(\cdot, \mathbf{w}) = \gamma(\underline{s}') K_{\underline{s}'}(\cdot, \mathbf{w}). \quad (3.29)$$

To determine the value of $\gamma(\underline{s}')$, take inner product with $K(\cdot, \mathbf{w})$ to the both sides of Equation (3.29) and observe that

$$\begin{aligned} \gamma(\underline{s}') K_{\underline{s}'}(\mathbf{w}, \mathbf{w}) &= \left\langle \sum_{|\underline{s}| \leq N} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_\alpha^{\underline{s}}(\mathbf{M}) \psi_\alpha^{\underline{s}}(\mathbf{M})^* K_{\underline{s}'}(\cdot, \mathbf{w}), K(\cdot, \mathbf{w}) \right\rangle \\ &= \sum_{|\underline{s}| \leq N} b_{\underline{s}} \left\langle K_{\underline{s}'}(\cdot, \mathbf{w}), \sum_{\alpha=1}^{d_{\underline{s}}} \psi_\alpha^{\underline{s}}(\mathbf{M}) \psi_\alpha^{\underline{s}}(\mathbf{M})^* K(\cdot, \mathbf{w}) \right\rangle \\ &= \sum_{|\underline{s}| \leq N} b_{\underline{s}} \langle K_{\underline{s}'}(\cdot, \mathbf{w}), K_{\underline{s}}(\cdot, \mathbf{w}) K(\cdot, \mathbf{w}) \rangle \\ &= \sum_{|\underline{s}| \leq N} b_{\underline{s}} \left\langle K_{\underline{s}'}(\cdot, \mathbf{w}), K_{\underline{s}}(\cdot, \mathbf{w}) \sum_{\tilde{\underline{s}} \geq \underline{0}} a_{\tilde{\underline{s}}} K_{\tilde{\underline{s}}}(\cdot, \mathbf{w}) \right\rangle \end{aligned}$$

$$\begin{aligned}
&= \sum_{|\underline{s}| \leq N} b_{\underline{s}} \left\langle K_{\underline{s}'}(\cdot, \mathbf{w}), \sum_{\tilde{\underline{s}} \geq \underline{0}} a_{\tilde{\underline{s}}} \sum_{|p|=|\underline{s}|+|\tilde{\underline{s}}|} c_{\underline{s}, \tilde{\underline{s}}}^p K_p(\cdot, \mathbf{w}) \right\rangle \\
&= \left(\sum_{\substack{|\underline{s}| \leq N, \tilde{\underline{s}} \geq \underline{0}, \\ |\underline{s}|+|\tilde{\underline{s}}|=|\underline{s}'|}} a_{\tilde{\underline{s}}} b_{\underline{s}} c_{\underline{s}, \tilde{\underline{s}}}^{s'} \right) \frac{K_{\underline{s}'}(\mathbf{w}, \mathbf{w})}{a_{\underline{s}'}}.
\end{aligned}$$

Here, the fifth equality holds because of Equation (2.14). Therefore, we have

$$\gamma(\underline{s}') = \frac{1}{a_{\underline{s}'}} \left(\sum_{\substack{|\underline{s}| \leq N, \tilde{\underline{s}} \geq \underline{0}, \\ |\underline{s}|+|\tilde{\underline{s}}|=|\underline{s}'|}} a_{\tilde{\underline{s}}} b_{\underline{s}} c_{\underline{s}, \tilde{\underline{s}}}^{s'} \right). \quad (3.30)$$

Finally, evaluating the left hand side of Equation (3.27) at $K_{\underline{s}'}(\cdot, \mathbf{w})$ and then, taking inner product with $K_{\underline{s}'}(\cdot, \mathbf{w})$, we obtain

$$\begin{aligned}
0 &\leq \left\langle E_0^\perp K_{\underline{s}'}(\cdot, \mathbf{w}) - \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_\alpha^{\underline{s}}(\mathbf{M}) E_0^\perp \psi_\alpha^{\underline{s}}(\mathbf{M})^* K_{\underline{s}'}(\cdot, \mathbf{w}), K_{\underline{s}'}(\cdot, \mathbf{w}) \right\rangle \\
&= \left\langle K_{\underline{s}'}(\cdot, \mathbf{w}) - \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_\alpha^{\underline{s}}(\mathbf{M}) E_0^\perp \psi_\alpha^{\underline{s}}(\mathbf{M})^* K_{\underline{s}'}(\cdot, \mathbf{w}), K_{\underline{s}'}(\cdot, \mathbf{w}) \right\rangle \\
&= \|K_{\underline{s}'}(\cdot, \mathbf{w})\|^2 (1 - \gamma(\underline{s}')) \\
&= \frac{\|K_{\underline{s}'}(\cdot, \mathbf{w})\|^2}{a_{\underline{s}'}} \left(a_{\underline{s}'} - \sum_{\substack{|\underline{s}| \leq N, \tilde{\underline{s}} \geq \underline{0}, \\ |\underline{s}|+|\tilde{\underline{s}}|=|\underline{s}'|}} a_{\tilde{\underline{s}}} b_{\underline{s}} c_{\underline{s}, \tilde{\underline{s}}}^{s'} \right) \\
&= \frac{\|K_{\underline{s}'}(\cdot, \mathbf{w})\|^2}{a_{\underline{s}'}} \left(\sum_{\substack{\underline{s}, \tilde{\underline{s}} \geq \underline{0}, \\ |\underline{s}|+|\tilde{\underline{s}}|=|\underline{s}'|}} a_{\tilde{\underline{s}}} b_{\underline{s}} c_{\underline{s}, \tilde{\underline{s}}}^{s'} - \sum_{\substack{|\underline{s}| \leq N, \tilde{\underline{s}} \geq \underline{0}, \\ |\underline{s}|+|\tilde{\underline{s}}|=|\underline{s}'|}} a_{\tilde{\underline{s}}} b_{\underline{s}} c_{\underline{s}, \tilde{\underline{s}}}^{s'} \right) \\
&= \frac{\|K_{\underline{s}'}(\cdot, \mathbf{w})\|^2}{a_{\underline{s}'}} b_{\underline{s}'} c_{\underline{s}', \underline{0}}^{s'}.
\end{aligned}$$

This implies that $b_{\underline{s}'} \geq 0$. Note that, in the above series of equalities, the third equality holds because of Equation (3.30) and the fourth equality holds due to Equation (3.20). This proves that K is a CNP kernel. \square

Remark 3.16. Let Ω be a Cartan domain of rank $r > 1$ and for $\nu \in \mathcal{W}_\Omega$, let $K^{(\nu)}$ be the weighted Bergman kernel on Ω . A direct consequence of Proposition 2.9 and Theorem 3.15 imply that there exists a $\frac{1}{K^{(\nu)}}$ -contraction $\mathbf{T} = (T_1, \dots, T_d)$ such that \mathbf{T} does not admit a characteristic function.

For $\nu > \frac{d}{r}$, the tuple of multiplication operators $\mathbf{M}^{(\nu)}$ on the weighted Bergman space $\mathbb{A}^{(\nu)}(\Omega)$ is a $\frac{1}{K^{(\nu)}}$ -contraction, [2, Theorem 3.3]. Lemma 3.8 yields that $\mathbf{M}^{(\nu)}$ is pure and $\Delta_{\mathbf{M}^{(\nu)}} = E_0$. Recall that $\{\psi_\alpha^{\underline{s}} : \underline{s} \geq \underline{0}, 1 \leq \alpha \leq d_{\underline{s}}\}$ is a complete orthogonal set of $\mathbb{A}^{(\nu)}(\Omega)$. For any $N \geq 0$, let \mathcal{M}_N be the closed subspace of $\mathbb{A}^{(\nu)}(\Omega)$ spanned by the set $\{\psi_\alpha^{\underline{s}} : |\underline{s}| \leq N, 1 \leq \alpha \leq d_{\underline{s}}\}$ and $\mathbf{T}_N := P_N \mathbf{M}^{(\nu)}|_{\mathcal{M}_N}$, where P_N is the orthogonal

projection of $\mathbb{A}^{(\nu)}(\Omega)$ onto \mathcal{M}_N . It is easy to verify that \mathbf{T}_N is a pure $\frac{1}{K^{(\nu)}}$ -contraction and $\Delta_{\mathbf{T}_N} = E_0$. Furthermore, an adaption of the proof of [7, Proposition 3.3] establishes that \mathbf{T}_N does not admit any characteristic function for $\nu \geq \frac{d}{r} + 1$.

4. CHARACTERISTIC FUNCTION: CONSTRUCTION

In this section, we explicitly construct the characteristic function of a d -tuple of $\frac{1}{K}$ -contraction for certain \mathbb{K} -invariant kernels K . Let $K = \sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} K_{\underline{s}}$ be a \mathbb{K} -invariant kernel on Ω and $\mathbf{T} = (T_1, \dots, T_d)$ be a $\frac{1}{K}$ -contraction. Then, recall that, the series

$$\sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$$

is convergent in strong operator topology and

$$\sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* \leq I.$$

The positive square root of $I - \sum_{\underline{s} > \underline{0}} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$ is denoted by $\Delta_{\mathbf{T}}$. Also, recall

that a $\frac{1}{K}$ -contraction $\mathbf{T} = (T_1, \dots, T_d)$ is said to admit a characteristic function if there exists a Hilbert space \mathcal{E} and an analytic function $\Theta_{\mathbf{T}} : \Omega \rightarrow \mathcal{B}(\mathcal{E}, \overline{\text{Ran}} \Delta_{\mathbf{T}})$ such that the corresponding multiplication operator $M_{\Theta_{\mathbf{T}}} : H_K \otimes \mathcal{E} \rightarrow H_K \otimes \overline{\text{Ran}} \Delta_{\mathbf{T}}$ satisfies $I - V_{\mathbf{T}} V_{\mathbf{T}}^* = M_{\Theta_{\mathbf{T}}} M_{\Theta_{\mathbf{T}}}^*$.

4.1. Functional Calculus. To explicitly compute the characteristic function of a $\frac{1}{K}$ -contraction $\mathbf{T} = (T_1, \dots, T_d)$ on a Hilbert space \mathcal{H} , we need the existence of the functional calculus $K(\mathbf{T}, \mathbf{w})$ for each $\mathbf{w} \in \Omega$. In order to define the operator $K(\mathbf{T}, \mathbf{w})$ for each $\mathbf{w} \in \Omega$, we assume the followings.

- (A) For each $\mathbf{w} \in \Omega$, $\sum_{\underline{s}} a_{\underline{s}} K_{\underline{s}}(\mathbf{M}, \mathbf{w})$ and $\sum_{\underline{s} > \underline{0}} b_{\underline{s}} K_{\underline{s}}(\mathbf{M}, \mathbf{w})$ converges in the strong operator topology.
- (B) For some $c > 0$, $\sum_{\underline{s} > \underline{0}} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \Delta_{\mathbf{T}}^2 \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* \geq cI$.

For the rest of this section, we assume Conditions (A) and (B). *Note that if \mathbf{T} is a pure $\frac{1}{K}$ -contraction, then \mathbf{T} satisfies Condition (B).*

We observe that if H_K is the subspace consisting of holomorphic functions in some L^2 space, then Condition (A) automatically holds.

Lemma 4.1. *If H_K is the subspace consisting of holomorphic functions in some L^2 space (with the inherited norm), then any bounded holomorphic function f on Ω is a bounded multiplier of H_K , and the corresponding multiplication operator M_f satisfies $\|M_f\| \leq \|f\|_{\infty}$.*

Proof. Immediate from $\|M_f u\|^2 = \int |f u|^2 \leq \|f\|_{\infty}^2 \int |u|^2 = \|f\|_{\infty}^2 \|u\|^2$. □

Corollary 4.2. *Under the hypothesis of Lemma 4.1, for any F as in Proposition 2.2 and $\mathbf{w} \in \Omega$, $F(\cdot, \mathbf{w})$ is a bounded multiplier on H_K . In particular, $K(\cdot, \mathbf{w})$ is a bounded multiplier, and if K is zero-free, then so is $1/K(\cdot, \mathbf{w})$.*

Proof. Immediate from Corollary 2.3. □

Corollary 4.3. *Again under the hypothesis of Lemma 4.1, for any F as in Proposition 2.2 and $\mathbf{w} \in \Omega$, the series*

$$\sum_{\underline{s}} c_{\underline{s}} K_{\underline{s}}(\mathbf{M}, \mathbf{w}),$$

where \mathbf{M} denotes the commuting tuple of multiplications by the coordinate functions, converges to the operator of multiplication by $F(\cdot, \mathbf{w})$ on \mathcal{H}_K in operator norm. In particular, this holds for $F = K$, and if K is zero-free, then also for $F = 1/K$.

Proof. By the lemma 4.1, $\|f_m - f\|_{\infty} \rightarrow 0$ implies $\|M_{f_m} - M_f\| \rightarrow 0$. Take $f_m := \sum_{|\underline{s}| \leq m} c_{\underline{s}} K_{\underline{s}}(\cdot, \mathbf{w})$ and use Proposition 2.2. \square

The corollary above proves that if the norm of the reproducing kernel Hilbert space H_K is given by an integral, then Condition (A) is always satisfied. Now, we provide a few necessary lemmas.

Lemma 4.4. *The operator $V_{\mathbf{T}} : \mathcal{H} \rightarrow H_K \otimes \overline{\text{Ran}} \Delta_{\mathbf{T}}$, given by Equation (3.22), is bounded below.*

Proof. Let h be an element in \mathcal{H} . Then, we have

$$\langle V_{\mathbf{T}} h, V_{\mathbf{T}} h \rangle = \left\langle \sum_{\underline{s} \geq 0} a_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \Delta_{\mathbf{T}}^2 \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* h, h \right\rangle \geq c \langle h, h \rangle.$$

This shows that $\|V_{\mathbf{T}} h\| \geq \sqrt{c} \|h\|$ holds for every $h \in \mathcal{H}$. Therefore, $V_{\mathbf{T}}$ is bounded below. \square

Remark 4.5. *In particular, $V_{\mathbf{T}}$ has closed range, hence (by Banach's Closed Range Theorem) so has $V_{\mathbf{T}}^*$; thus $\text{Ran } V_{\mathbf{T}}^*$ is all of \mathcal{H} .*

Lemma 4.6. *For each $\mathbf{w} \in \Omega$, both the series*

$$\sum_{\underline{s} \geq 0} a_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \quad \text{and} \quad \sum_{\underline{s} > 0} b_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{T})$$

converge in strong operator topology.

Proof. Condition (A) implies that the series $\sum_{\underline{s} \geq 0} a_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I_{\overline{\text{Ran}} \Delta_{\mathbf{T}}}$ converges in strong operator topology. Let $\varepsilon > 0$ and $f \in H_K \otimes \overline{\text{Ran}} \Delta_{\mathbf{T}}$. Then there exists $N \geq 1$ such that

$$\left\| \sum_{|\underline{s}|=N_1}^{N_2} a_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I_{\overline{\text{Ran}} \Delta_{\mathbf{T}}} f \right\| < \varepsilon$$

holds for every $N_1, N_2 \geq N$. This implies that

$$\begin{aligned} & \left\| \sum_{|\underline{s}|=N_1}^{N_2} a_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) V_{\mathbf{T}}^* f \right\| \\ &= \left\| V_{\mathbf{T}}^* \sum_{|\underline{s}|=N_1}^{N_2} a_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I_{\overline{\text{Ran}} \Delta_{\mathbf{T}}} f \right\| \\ &\leq \left\| \sum_{|\underline{s}|=N_1}^{N_2} a_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{M}) \otimes I_{\overline{\text{Ran}} \Delta_{\mathbf{T}}} f \right\| < \varepsilon, \quad N_1, N_2 \geq N. \end{aligned}$$

Here, the first inequality occurs since $V_{\mathbf{T}}$ is a contraction. This proves that the series $\sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{T})$ converges in strong operator topology on $\text{Ran} V_{\mathbf{T}}^*$. Remark 4.5 yields that $\text{Ran} V_{\mathbf{T}}^*$ is \mathcal{H} . This proves that $\sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{T})$ converges in strong operator topology on \mathcal{H} .

A similar argument shows that the series $\sum_{\underline{s} \geq \underline{0}} b_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{T})$ converges in the strong operator topology. \square

Recall that if K is a \mathbb{K} -invariant kernel, then there exists a sequence $\{b_{\underline{s}}\}$ of real numbers such that $1 - \frac{1}{K} = \sum_{\underline{s} \geq \underline{0}} b_{\underline{s}} K_{\underline{s}}$ holds. This implies that

$$\left(\sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \right) \left(I - \sum_{\underline{s} \geq \underline{0}} b_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \right) = I. \quad (4.31)$$

For any $\mathbf{w} \in \Omega$, let

$$K_{\mathbf{w}}(\mathbf{T}) = \sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha}^{d_{\underline{s}}} \overline{\psi_{\alpha}^{\underline{s}}(\mathbf{w})} \psi_{\alpha}^{\underline{s}}(\mathbf{T}).$$

Then, for any $\mathbf{z} \in \Omega$, we have

$$K_{\mathbf{z}}(\mathbf{T})^* = \sum_{\underline{s} \geq \underline{0}} a_{\underline{s}} \sum_{\alpha}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{z}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*.$$

Equation (4.31) implies that

$$K_{\mathbf{z}}(\mathbf{T})^* = \left(I - \sum_{\underline{s} \geq \underline{0}} b_{\underline{s}} \sum_{\alpha=0}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{z}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* \right)^{-1}.$$

4.2. Characteristic Function. In this subsection, we give an explicit construction of the characteristic function of \mathbf{T} with the assumption that K is a CNP kernel. The CNP property of K implies that each $b_{\underline{s}}$ must be a non-negative real number. Let $\tilde{\mathcal{H}} = \bigoplus_{\underline{s} \geq \underline{0}} \bigoplus_{\alpha}^{d_{\underline{s}}} \mathcal{H}$. Note that $\tilde{\mathcal{H}}$ is a direct sum of countably many copies of the Hilbert space \mathcal{H} . Every $\mathbf{z} \in \Omega$ gives rise to an operator

$$\mathbf{Z} = \left(\sqrt{b_{\underline{s}}} \psi_{\alpha}^{d_{\underline{s}}}(\mathbf{z}) I \right)_{\underline{s} \geq \underline{0}, \alpha=1}^{d_{\underline{s}}} : \tilde{\mathcal{H}} \rightarrow \mathcal{H}, \quad (4.32)$$

mapping $(h_{\alpha}^{\underline{s}})_{\underline{s} \geq \underline{0}, \alpha=1}^{d_{\underline{s}}}$ to $\sum_{\underline{s} \geq \underline{0}} \sqrt{b_{\underline{s}}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{z}) h_{\alpha}^{\underline{s}}$ and

$$\begin{aligned} \|\mathbf{Z} (h_{\alpha}^{\underline{s}})_{\underline{s} \geq \underline{0}, \alpha=1}^{d_{\underline{s}}}\|^2 &= \left\| \sum_{\underline{s} \geq \underline{0}} \sqrt{b_{\underline{s}}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{z}) h_{\alpha}^{\underline{s}} \right\|^2 \\ &\leq \left(\sum_{\underline{s} \geq \underline{0}} \sum_{\alpha=1}^{d_{\underline{s}}} b_{\underline{s}} |\psi_{\alpha}^{\underline{s}}(\mathbf{z})|^2 \right) \left(\sum_{\underline{s} \geq \underline{0}} \sum_{\alpha=1}^{d_{\underline{s}}} \|h_{\alpha}^{\underline{s}}\|^2 \right) \\ &= \left(1 - \frac{1}{K(\mathbf{z}, \mathbf{z})} \right) \left(\sum_{\underline{s} \geq \underline{0}} \sum_{\alpha=1}^{d_{\underline{s}}} \|h_{\alpha}^{\underline{s}}\|^2 \right). \end{aligned}$$

This proves that $\|\mathbf{Z}\| \leq \left(1 - \frac{1}{K(\mathbf{z}, \mathbf{z})}\right)^{1/2} < 1$. Consider the operator

$$\tilde{\mathbf{T}} := \left(\sqrt{b_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) \right)_{\underline{s} > 0, \alpha=1}^{d_{\underline{s}}} : \tilde{\mathcal{H}} \rightarrow \mathcal{H},$$

defined by,

$$\tilde{\mathbf{T}} \left((h_{\alpha}^{\underline{s}})_{\underline{s} > 0, \alpha=1}^{d_{\underline{s}}} \right) = \sum_{\underline{s} > 0} \sum_{\alpha=1}^{d_{\underline{s}}} \sqrt{b_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T}) h_{\alpha}^{\underline{s}}.$$

Then, it is easily verified that $\tilde{\mathbf{T}}^* : \mathcal{H} \rightarrow \tilde{\mathcal{H}}$ is given by $\tilde{\mathbf{T}}^* h = \left(\sqrt{b_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{T})^* h \right)_{\underline{s} > 0, \alpha=1}^{d_{\underline{s}}}$, $h \in \mathcal{H}$. Using the fact that \mathbf{T} is a $\frac{1}{K}$ -contraction, it is straight forward to verify that $\tilde{\mathbf{T}}^*$ is a contraction. Also, we have $\Delta_{\tilde{\mathcal{H}}}^2 = I_{\tilde{\mathcal{H}}} - \tilde{\mathbf{T}}\tilde{\mathbf{T}}^*$. Let $D_{\tilde{\mathcal{H}}} := \left(I_{\tilde{\mathcal{H}}} - \tilde{\mathbf{T}}\tilde{\mathbf{T}}^* \right)^{\frac{1}{2}}$ and $\mathcal{D}_{\tilde{\mathcal{H}}} := \overline{\text{Ran} D_{\tilde{\mathcal{H}}}}$. The identity

$$\tilde{\mathbf{T}} D_{\tilde{\mathcal{H}}} = \Delta_{\mathcal{H}} \tilde{\mathbf{T}} \tag{4.33}$$

follows from a direct computation. For each $\mathbf{z} \in \Omega$, since the operator \mathbf{Z} , defined by Equation (4.32), is a strict contraction, the operator $I_{\mathcal{H}} - \mathbf{Z}\tilde{\mathbf{T}}^*$ is invertible. Now, we are prepared to provide the definition of the characteristic function of \mathbf{T} .

Definition 4.7. *The characteristic function $\theta_{\mathbf{T}} : \Omega \rightarrow \mathcal{B}(\mathcal{D}_{\tilde{\mathcal{H}}}, \overline{\text{Ran} \Delta_{\mathcal{H}}})$ of \mathbf{T} is defined by*

$$\theta_{\mathbf{T}}(\mathbf{z}) := \left(-\tilde{\mathbf{T}} + \Delta_{\mathcal{H}} (I_{\mathcal{H}} - \mathbf{Z}\tilde{\mathbf{T}}^*)^{-1} \mathbf{Z} D_{\tilde{\mathcal{H}}} \right) |_{\mathcal{D}_{\tilde{\mathcal{H}}}}, \mathbf{z} \in \Omega, \tag{4.34}$$

where, for each \mathbf{z} in Ω , the operator \mathbf{Z} is given in Equation (4.32).

The relation $\tilde{\mathbf{T}} D_{\tilde{\mathcal{H}}} = \Delta_{\mathcal{H}} \tilde{\mathbf{T}}$ implies that $\theta_{\mathbf{T}}(\mathbf{z}) \in \mathcal{B}(\mathcal{D}_{\tilde{\mathcal{H}}}, \overline{\text{Ran} \Delta_{\mathcal{H}}})$ for each $\mathbf{z} \in \Omega$. Therefore, the characteristic function $\theta_{\mathbf{T}}$ of \mathbf{T} is a $\mathcal{B}(\mathcal{D}_{\tilde{\mathcal{H}}}, \overline{\text{Ran} \Delta_{\mathcal{H}}})$ -valued analytic function on Ω . Also, note that if $\mathbf{z} \in \Omega$, then

$$\mathbf{Z}\tilde{\mathbf{T}}^* = \sum_{\underline{s} > 0} b_{\underline{s}} \sum_{\alpha=1}^{d_{\underline{s}}} \psi_{\alpha}^{\underline{s}}(\mathbf{z}) \psi_{\alpha}^{\underline{s}}(\mathbf{T})^*$$

and consequently, we have

$$\left(I_{\mathcal{H}} - \mathbf{Z}\tilde{\mathbf{T}}^* \right)^{-1} = K_{\mathbf{z}}(\mathbf{T})^*.$$

The proofs of the following lemmas are consequences of a straightforward computation and therefore omitted.

Lemma 4.8. *For every $\mathbf{z}, \mathbf{w} \in \Omega$, we have the following identity*

$$I - \theta_{\mathbf{T}}(\mathbf{z})\theta_{\mathbf{T}}(\mathbf{w})^* = \frac{1}{K(\mathbf{z}, \mathbf{w})} \Delta_{\mathcal{H}} (K_{\mathbf{z}}(\mathbf{T}))^* K_{\mathbf{w}}(\mathbf{T}) \Delta_{\mathcal{H}}.$$

Lemma 4.9. *For any $\mathbf{w} \in \Omega$ and $\xi \in \overline{\text{Ran} \Delta_{\mathcal{H}}}$, we have*

$$V_{\mathcal{H}}^*(K_{\mathbf{w}} \otimes \xi) = K_{\mathbf{w}}(\mathbf{T}) \Delta_{\mathcal{H}} \xi,$$

where the operator $V_{\mathcal{H}}$ is given by Equation (3.22).

The proof of the corollary below follows immediately from Lemma 4.8 and Lemma 4.9.

Corollary 4.10. *For any $\mathbf{z}, \mathbf{w} \in \Omega$ and $\xi, \eta \in \overline{\text{Ran} \Delta_{\mathcal{H}}}$, we have*

$$\langle V_{\mathcal{H}}^*(K_{\mathbf{w}} \otimes \xi), V_{\mathcal{H}}^*(K_{\mathbf{z}} \otimes \eta) \rangle = K(\mathbf{z}, \mathbf{w}) \langle (I - \theta_{\mathbf{T}}(\mathbf{z})\theta_{\mathbf{T}}(\mathbf{w})^*) \xi, \eta \rangle.$$

The characteristic function $\theta_{\mathbf{T}}$ of \mathbf{T} gives rise to the multiplication operator $M_{\theta_{\mathbf{T}}} : H_K \otimes \mathcal{D}_{\bar{\mathbf{T}}} \rightarrow H_K \otimes \overline{\text{Ran}} \Delta_{\mathbf{T}}$. The following theorem shows that the characteristic function $\theta_{\mathbf{T}}$ of \mathbf{T} , defined via Definition 4.7 satisfies the identity of Definition 3.11.

Theorem 4.11. *The multiplication operator $M_{\theta_{\mathbf{T}}} : H_K \otimes \mathcal{D}_{\bar{\mathbf{T}}} \rightarrow H_K \otimes \overline{\text{Ran}} \Delta_{\mathbf{T}}$ corresponding to the characteristic function $\theta_{\mathbf{T}}$ of \mathbf{T} satisfies the following identity*

$$I - V_{\mathbf{T}} V_{\mathbf{T}}^* = M_{\theta_{\mathbf{T}}} M_{\theta_{\mathbf{T}}}^*.$$

Proof. The proof is similar to the proof of [7, Theorem 4.11]. Therefore, the proof of the theorem is omitted. \square

As a direct consequence of the theorem above, we obtain the following corollary.

Corollary 4.12. *The characteristic function $\theta_{\mathbf{T}}$ of \mathbf{T} is a bounded analytic function on Ω . In particular, $\sup_{z \in \Omega} \|\theta_{\mathbf{T}}(z)\| \leq 1$.*

Definition 4.13. *Let \mathbf{T} and \mathbf{R} be two $\frac{1}{K}$ -contractions defined on Hilbert spaces \mathcal{H} and \mathcal{K} , respectively. The characteristic functions $\theta_{\mathbf{T}}$ and $\theta_{\mathbf{R}}$ of \mathbf{T} and \mathbf{R} , respectively, are said to coincide if there exist unitaries $\tau : \mathcal{D}_{\bar{\mathbf{T}}} \rightarrow \mathcal{D}_{\bar{\mathbf{R}}}$ and $\tau_* : \overline{\text{Ran}} \Delta_{\mathbf{T}} \rightarrow \overline{\text{Ran}} \Delta_{\mathbf{R}}$ such that*

$$\tau_* \theta_{\mathbf{T}}(z) = \theta_{\mathbf{R}}(z) \tau$$

holds for every $z \in \Omega$.

Theorem 4.14. *If \mathbf{T} is a pure $\frac{1}{K}$ -contraction, then the tuple \mathbf{T} is unitarily equivalent to the tuple $(P_{\mathbb{H}_{\mathbf{T}}}(M_{z_1} \otimes I_{\overline{\text{Ran}} \Delta_{\mathbf{T}}})|_{\mathbb{H}_{\mathbf{T}}}, \dots, P_{\mathbb{H}_{\mathbf{T}}}(M_{z_d} \otimes I_{\overline{\text{Ran}} \Delta_{\mathbf{T}}})|_{\mathbb{H}_{\mathbf{T}}})$, where $\mathbb{H}_{\mathbf{T}} := (H_K \otimes \overline{\text{Ran}} \Delta_{\mathbf{T}}) \ominus M_{\theta_{\mathbf{T}}}(H_K \otimes \mathcal{D}_{\bar{\mathbf{T}}})$ and $P_{\mathbb{H}_{\mathbf{T}}}$ denote the orthogonal projection from $H_K \otimes \overline{\text{Ran}} \Delta_{\mathbf{T}}$ onto $\mathbb{H}_{\mathbf{T}}$.*

Proof. The proof is similar to the proof of [8, Theorem 3.7] and therefore, omitted. \square

The main result of this section is the following theorem, which states that for certain \mathbb{K} -invariant kernels K , the characteristic function of a pure $\frac{1}{K}$ -contraction \mathbf{T} determines the unitary equivalence class of \mathbf{T} . The proof follows along the line of the proof of [8, Theorem 4.4]. Therefore, we leave out the proof.

Theorem 4.15. *Let K be a \mathbb{K} -invariant kernel satisfying Condition (A). Two pure $\frac{1}{K}$ -contractions are unitarily equivalent if and only if their characteristic functions coincide.*

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