WIGNER TRANSFORM AND PSEUDODIFFERENTIAL OPERATORS ON SYMMETRIC SPACES OF NON-COMPACT TYPE

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Abstract. We obtain a general expression for a Wigner transform (Wigner function) on symmetric spaces of non-compact type and study the Weyl calculus of pseudodifferential operators on them.

1. Introduction

The Wigner transform and the Weyl calculus of pseudodifferential operators have long played prominent roles in PDE theory [11] [15], time-frequency analysis [7] [21] [5] and mathematical physics [19]. As their definition relies on the Fourier transform, it is not surprising that they have been studied most extensively in the context of the Euclidean n-space. The aim of this paper is to extend these notions to a more general context where a version of the Fourier transform is available: namely, to symmetric spaces of non-compact type, with the Fourier-Helgason transform.

There have been several efforts in this direction before in the literature. First of all, there is an extensive theory of Weyl calculi for which the symmetric domains are the phase spaces; these are special cases of the so-called "invariant operator calculi" developed recently by Arazy and Upmeier [4]. (It should be noted that these calculi seem not to involve any analogue of the Wigner transform.) Our goal here is different in that we have the symmetric domains only as the configuration space, i.e. the Wigner transform and symbols of the Weyl operators are functions on the cotangent bundles of the symmetric domains (or, more precisely, on the products $\Omega \times \Omega^*$, where Ω^* is the Fourier-Helgason dual of the symmetric space Ω; the latter product is essentially isomorphic to the cotangent bundle $T^*\Omega$). In this direction, Tate [16] studied the situation for the simplest complex bounded symmetric domain, the unit disc; generalization to the unit ball of $\mathbb{R}^n$ (realized as one-sheeted hyperboloid in $\mathbb{R}^{n+1}$) has then been carried out by Bertola and the first author [1]. We also mention that apparently yet another kind of the Weyl calculus for the disc, for which the symbols also live on the tangent bundle of the disc and which ultimately leads to the occurrence of Bessel functions, was introduced by Terras [17] and studied by Trimeche [18] or Peng and Zhao [14]; it seems unclear whether this calculus is in any way related to Tate's and ours. (We pause to remark that the Bessel-function Weyl calculus, however, seems to have rather complicated behaviour under holomorphic transformations of the unit disc.)

In the physical literature there have been several different generalizations of the original Wigner function [20] to non-flat configuration spaces and their phase

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spaces. One approach towards a generalization exploits the fact that that the original Wigner function lives on a coadjoint orbit of the Weyl-Heisenberg group and can be obtained using the square-integrability property of its representations. A general description of this method, exploiting square-integrable group representations, may be found in [13] and earlier references cited therein. An approach that is very close to the one adopted in the present paper has been used in [2, 3] to obtain Wigner functions on hyperboloids and spheres. However, the results obtained there were on a case by case basis, while we present here a general theory. Another suggestion for a generaliztion, using the entire dual space of the Weyl-Heisenberg group has been given in [12]. The virtue of our present approach lies in its generality and the fact that our construction preserves both the marginality and unitarity properties that allowed the original Wigner function to be interpreted as a pseudo-probability distribution on phase space.

The Wigner transform is constructed in Section 3 below, after reviewing the necessary prerequisites on symmetric spaces in Section 2. The non-Euclidean Weyl calculus of pseudodifferential operators is introduced in Section 4. The invertibility of the Wigner transform and its unitarity are discussed in Sections 5 and 6, respectively. The final Section 7 contains miscellaneous concluding remarks, open problems, etc. For the most part, our approach parallels fairly directly that of Tate's in [16]; however, Theorem 12 and Corollary 10 seem to be new even for his situation of the unit disc.

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2. Bounded symmetric domains

Recall that a connected Riemannian manifold Ω of dimension d is called a *symmetric space* if for any $x \in \Omega$ there exists a (necessarily unique) element $s_x \in G$, the group of isometries of Ω, which is involutive (i.e. $s_x \circ s_x = \mathrm{id}$) and has x as an isolated fixed-point. One calls s_x the geodesic symmetry at x. The symmetric space is called *irreducible* if it is not isomorphic to a Cartesian product of another two symmetric spaces. Irreducible symmetric spaces come in three types: *Euclidean* (these are just $\mathbb{R}^d$ and its quotients), of *compact type* (the compact ones) and of *non-compact type.* Any symmetric space of non-compact type can be realized as (i.e. is isomorphic to) a domain in $\mathbb{R}^d$ which is circular with respect to the origin and convex (the so-called Harish-Chandra realization). Throughout the rest of this paper, we will thus assume that Ω is of the latter form, i.e. a symmetric space of non-compact type in its Harish-Chandra realization.

It turns out that the geodesic symmetries s_x in fact act transitively on Ω, i.e. for any $y, z \in \Omega$ there exists an $x \in \Omega$ such that $s_x y = z$; denoting by $K = \{g \in G : g(0) = 0\}$ the stabilizer in G of the origin $0 \in \Omega$, it therefore follows that Ω is isomorphic to the coset space G/K. (It is also true that elements of K are orthogonal maps on $\mathbb{R}^d$ that preserve Ω, and that K is a maximal compact subgroup of G.) There exists a unique (up to constant multiples) G-invariant measure on Ω (obtained as the projection of the Haar measure on G); we will denote it by $d\mu(z)$. (Thus $d\mu(z) = d\mu(g(z))$ for any $g \in G$.)

For $x \in \Omega$, there exists a unique geodesic symmetry $\phi_x \in G$ which interchanges x and the origin, i.e.

$$\phi_x \circ \phi_x = \mathrm{id},\ \phi_x(0) = x,\ \phi_x(x) = 0, \tag{1}$$

and ϕ_x has only isolated fixed-points. In fact, ϕ_x has only one fixed point, namely the geodesic mid-point between 0 and x; we will denote, quite generally, the geodesic mid-point between some given $x, y \in \Omega$ by $m_{x,y}$ or m_{xy}. (Thus the fixed point of ϕ_x is precisely $m_{x,0}$, and $\phi_x = s_{m_{x,0}}$.)

Employing the standard notation, let $G = NAK$ be the Iwasawa decomposition of G, $\mathfrak{a}$ the Lie algebra of the maximal Abelian part A, $\mathfrak{a}^*$ its dual, $r = \dim_{\mathbb{R}} \mathfrak{a}$ its dimension (known as the *rank* of Ω), $\rho = (\rho_1, \dots, \rho_r) \in \mathfrak{a}^*$ the sum of positive roots, M and M' the centralizer and the normalizer of A in K, respectively, and $W = M'/M$ the Weyl group. For any $\lambda \in \mathfrak{a}^* \cong \mathbb{R}^r$ and b in the coset space $B := K/M = G/MAN$, one defines the "plane waves" on Ω by

$$e_{\lambda,b}(x) := e^{(i\lambda+\rho)(A(x,b))}, \qquad x \in \Omega,$$

where $A(x, b)$ is the unique element of $\mathfrak{a}$ satisfying, if $b = kM$ and $x = gK$,

$$k^{-1}g \in N \exp A(x, b)\, K$$

under the Iwasawa decomposition $G = NAK$.

The Helgason-Fourier transform of $f \in C_0^\infty(\Omega)$ is a function on $\Omega^* := \mathfrak{a}^* \times B$ ($\cong \mathbb{R}^r \times K/M$) given by

$$\tilde{f}(\lambda, b) := \int_\Omega f(x) e_{-\lambda,b}(x)\, d\mu(x).$$

For any $f \in C_0^\infty(\Omega)$ we then have the Fourier inversion formula

$$f(x) = \int_{\mathfrak{a}^*} \int_B \tilde{f}(\lambda, b) e_{\lambda,b}(x)\, d\rho(\lambda, b)$$

and the Plancherel theorem

$$\int_\Omega |f(x)|^2\, d\mu(x) = \int_{\mathfrak{a}^*} \int_B |\tilde{f}(\lambda, b)|^2 d\rho(\lambda, b).$$

Here

$$d\rho(\lambda, b) := |c(\lambda)|^{-2}\, db\, d\lambda,$$

where db is the unique K-invariant probability measure on K/M, $d\lambda$ is a suitably normalized Lebesgue measure on $\mathfrak{a}^* \cong \mathbb{R}^r$, and $c(\lambda)$ is a certain meromorphic function on the complexification $\mathfrak{a}^{*\mathbb{C}} \cong \mathbb{C}^r$ of $\mathfrak{a}^*$ (the Harish-Chandra c-function). From the Plancherel theorem it can be deduced, in particular, that $f \mapsto \tilde{f}$ extends to a Hilbert space isomorphism of $L^2(d\mu)$ into $L^2(\Omega^*, d\rho)$ whose image consists of functions $F(\lambda, b)$ which satisfy a certain symmetry condition (relating the values $F(\lambda, b)$ and $F(s\lambda, b)$ for s in the Weyl group; see Corollary VI.3.9 in [10].)

A (linear) differential operator L on Ω is called G-invariant if

$$L(f \circ g) = (Lf) \circ g$$

for any $f \in C^\infty(\Omega)$ and any $g \in G$. For any such L, it is known that the "plane waves" are eigenfunctions of L:

$$Le_{\lambda,b} = \tilde{L}(\lambda) e_{\lambda,b}$$

where $\tilde{L}(\lambda)$ is a polynomial in r variables; that is, each such L is a Fourier multiplier with respect to the Helgason-Fourier transform. The correspondence $L \mapsto \tilde{L}$ sets up an isomorphism between the ring of all G-invariant differential operators on Ω and the ring of all polynomials on $\mathbb{R}^r \cong \mathfrak{a}^*$ invariant under the Weyl group W.

The "plane waves" $e_{\lambda,b}$ obey the following transformation rule under composition with elements of G:

$$e_{\lambda,b} \circ g = e_{\lambda,b}(g0)\, e_{\lambda,g^{-1}b}. \tag{2}$$

(We will often write $g0, gz$, etc. instead of $g(0), g(z)$ etc.) It follows from here that

$$e_{\lambda,gb}(g0) e_{\lambda,b}(g^{-1}0) = 1$$

and

$$\begin{aligned} d\rho(\lambda, gb) &= |e_{\lambda,b}(g^{-1}0)|^2\, d\rho(\lambda, b), \\ d\rho(\lambda, b) &= |e_{\lambda,gb}(g0)|^2\, d\rho(\lambda, gb). \end{aligned} \tag{3}$$

Indeed, from the formula for the Helgason-Fourier transform and (2) we have

$$\begin{aligned} \tilde f(\lambda, gb) &= \int_\Omega f(z)\, e_{-\lambda,gb}(z)\, d\mu(z) \\ &= \int_\Omega f(z)\, \frac{e_{-\lambda,b}(g^{-1}z)}{e_{-\lambda,b}(g^{-1}0)}\, d\mu(z) \\ &= \frac{1}{e_{-\lambda,b}(g^{-1}0)} \int_\Omega f(gz)\, e_{-\lambda,b}(z)\, d\mu(z) \\ &= \frac{(f\circ g)^\sim(\lambda, b)}{e_{-\lambda,b}(g^{-1}0)}, \end{aligned}$$

whence from

$$\begin{aligned} f(z) &= \int_{\Omega^*} \tilde f(\lambda, b)\, e_{\lambda,b}(z)\, d\rho(\lambda, b) \\ &= \int_{\Omega^*} \tilde f(\lambda, gb)\, e_{\lambda,gb}(z)\, d\rho(\lambda, gb) \\ &= \int_{\Omega^*} \frac{(f\circ g)^\sim(\lambda, b)}{e_{-\lambda,b}(g^{-1}0)}\, \frac{e_{\lambda,b}(g^{-1}z)}{e_{\lambda,b}(g^{-1}0)}\, d\rho(\lambda, gb) \end{aligned}$$

we get, upon replacing f by $f \circ g^{-1}$ and z by gz,

$$f(z) = \int_{\Omega^*} \frac{\tilde f(\lambda, b)}{e_{-\lambda,b}(g^{-1}0)}\, \frac{e_{\lambda,b}(z)}{e_{\lambda,b}(g^{-1}0)}\, d\rho(\lambda, gb),$$

proving the claim. (Note that $e_{-\lambda,b} = \overline{e_{\lambda,b}}$.)

Since $|e_{\lambda,b}(x)|^2 = e^{2\rho(A(x,b))}$ does not depend on λ, (3) in fact implies that

$$\begin{aligned} d(gb) &= |e_{\lambda,b}(g^{-1}0)|^2\, db, \\ db &= |e_{\lambda,gb}(g0)|^2\, d(gb). \end{aligned} \tag{4}$$

A function f on Ω is called K-*invariant* if $f(kx) = f(x)$ for all $x \in \Omega$ and $k \in K$. For such functions, the Helgason-Fourier transform $\tilde f(\lambda, b)$ does not depend on b, and reduces to the *spherical transform*

$$\tilde f(\lambda) = \int_\Omega f(z)\, \Phi_{-\lambda}(z)\, d\mu(z),$$

where Φ_λ are the *spherical functions*

$$\Phi_\lambda(z) := \int_K e_{\lambda,b}(kz)\, dk = \int_K e_{\lambda,kb}(z)\, dk.$$

One has $\Phi_\lambda = \Phi_{s\lambda}$ for all s in the Weyl group, i.e. $\tilde f$ is W-invariant. The Fourier inversion formula and the Plancherel theorem assume the form

$$f(z) = \int_{\mathfrak{a}^*} \tilde f(\lambda)\, \Phi_\lambda(z)\, d\rho(\lambda), \tag{5}$$

$$\int_\Omega |f(z)|^2\, d\mu(z) = \int_{\mathfrak{a}^*} |\tilde f(\lambda)|^2\, d\rho(\lambda),$$

respectively, where (abusing notation a little)

$$d\rho(\lambda) := |c(\lambda)|^{-2}\, d\lambda.$$

Some examples. 1. The absolutely simplest example of the type of symmetric space studied here could be the unit interval $\Omega = (-1,1) \subset \mathbb{R}$, on which $G = O(1,1)/\mathbb{R}$ acts by

$$gx = \frac{ax+b}{cx+d}, \qquad x \in \Omega,\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in O(1,1),$$

that is,

$$gx = \epsilon \frac{x\cosh t + \sinh t}{x \sinh t + \cosh t}, \qquad x \in \Omega,\ t \in \mathbb{R},\ \epsilon \in \{\pm 1\}. \tag{6}$$

In particular,

$$\phi_a x = \frac{a-x}{1-ax}, \qquad x, a \in \Omega.$$

The stabilizer of the origin is $K = O(1) = \{\pm 1\}$, the invariant measure is $d\mu(x) = \frac{dx}{1-x^2}$, and $N = \{1\}$, $A = G$, $M = K$, $B = \{1\}$. The Lie algebra $\mathfrak{g}$ can be identified with $\mathbb{R}$, and the exponential map $\mathfrak{g} \to G$ is

$$\xi \longmapsto \tanh \xi. \tag{7}$$

It follows that

$$e_{\lambda,b}(x) = \Big(\frac{1+x}{1-x}\Big)^{i\lambda/2}, \qquad \lambda \in \mathbb{R}.$$

The invariant differential operators on Ω are precisely the polynomials in $\widetilde\Delta := ((1-x^2)\frac{\partial}{\partial x})^2$, and

$$\widetilde\Delta e_{\lambda,b} = -\lambda^2 e_{\lambda,b}.$$

However, this example is not really a symmetric space of noncompact type, since, by dimensional reasons, the Lie algebra $\mathfrak{g}$ is necessarily abelian and thus Ω is actually a Euclidean space. In fact, the exponential map (7) gives an isomorphism of $\mathbb{R}$ onto Ω under which the action (6) becomes just the Euclidean motion $\xi \mapsto \epsilon(\xi + t)$, $d\mu(x)$ becomes the Lebesgue measure $d\xi$, $\widetilde\Delta$ becomes $\partial^2/\partial\xi^2$, and $e_{\lambda,b}(x)$ reduces to the ordinary exponential $e^{i\lambda\xi}$. Since the Weyl group is just $W = \{\pm 1\}$ while $\rho = 0$ and $c(\lambda) \equiv 1$, the Helgason-Fourier transform on Ω thus reduces just to the ordinary Fourier transform on $\mathbb{R}$.

2. The simplest genuine example is thus the unit disc $\Omega = \{z \in \mathbb{C} \cong \mathbb{R}^2 : |z| < 1\}$, considered by Tate [16]. In this case $\Omega = G/K$ with $G = U(1,1)/\mathbb{C}$ acting again by

$$gz = \frac{az+b}{cz+d}, \qquad z \in \Omega,\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(1,1),$$

and $K = U(1)$, $A = \{\begin{pmatrix}\cosh t & \sinh t\\ \sinh t & \cosh t\end{pmatrix} : t \in \mathbb{R}\}$ is the same as in the preceding example, $M = \{1\}$, $W = \{\pm 1\}$ and $\rho = 1$. The geodesic symmetries are given by

$$\phi_a z = \frac{a - z}{1 - \overline{a} z}.$$

The quotient space $B = K/M$ can be identified with the unit circle $\mathbf{T}$, and

$$e_{\lambda,b}(z) = \Big(\frac{1-|z|^2}{|z-b|^2}\Big)^{\frac{1+i\lambda}{2}}, \qquad \lambda \in \mathbb{R},\ b \in \mathbf{T},\ z \in \Omega.$$

The invariant measure is $d\mu(z) = (1-|z|^2)^{-2}\, dz \wedge d\overline{z}$, the invariant differential operators are precisely the polynomials in $\widetilde{\Delta} := (1-|z|^2)^2\Delta$, where Δ is the ordinary Laplace operator, and

$$\widetilde{\Delta} e_{\lambda,b} = -(\lambda^2+1) e_{\lambda,b}.$$

The Plancherel measure $d\rho$ is given by $d\rho(\lambda) = \frac{\lambda}{4\pi} \tanh \frac{\pi\lambda}{2}\, d\lambda$, yielding the simplest nontrivial example of the Helgason-Fourier transform.

3. The real hyperbolic n-space, modelled in [1] as one-sheeted hyperboloid, can be realized as the unit ball $\Omega = \{x \in \mathbb{R}^n : |x| < 1\} = G/K$ with $G = O(n,1)/\mathbb{R}$, $K = O(n)$. The geodesic symmetries are the Moebius maps

$$\phi_a x = \frac{(1 - 2\langle a,x\rangle + |x|^2)a - (1-|a|^2)x}{1 - 2\langle a,x\rangle + |a|^2|x|^2}, \qquad x, a \in \Omega;$$

the maximal abelian subgroup A can be identified with $\{\tau_a : a = re_1,\ -1 < r < 1\}$ where $\tau_a(x) := \phi_a(-x)$ and $e_1 = (1,0,0,\dots,0)$; and $M = \{k \in K : ke_1 = e_1\} \cong O(n-1)$, so that $B = K/M$ can again be identified with the unit sphere $\partial\Omega = \mathbf{S}^{n-1}$. The Weyl group W is again just $\{\pm 1\}$, the sum of positive roots is $\rho = n-1$, and the "plane waves" are

$$e_{\lambda,b}(x) = \Big(\frac{1-|x|^2}{|x-b|^2}\Big)^{\frac{n-1+i\lambda}{2}}, \qquad x \in \Omega,\ b \in \partial\Omega,\ \lambda \in \mathbb{R}.$$

The invariant differential operators are precisely the polynomials in

$$\widetilde{\Delta} := (1-|x|^2)\Big[(1-|x|^2)\sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + (2n-4)\sum_{j=1}^n x_j \frac{\partial}{\partial x_j}\Big],$$

and $\widetilde{\Delta} e_{\lambda,b} = -(\lambda^2 + (n-1)^2) e_{\lambda,b}$. Note that for $n = 1$ and $n = 2$, this example recovers the previous two as special cases.

4. All three examples above are in turn special cases of the unit ball of real $n \times m$ matrices

$$\Omega = \{z \in \mathbb{R}^{n\times m} : I - z^t z \text{ is positive definite}\}$$

(or, equivalently, $\|z\| < 1$ when z is viewed as an operator $z : \mathbb{R}^m \to \mathbb{R}^n$). One has $\Omega = G/K$ with $G = O(n,m)/\mathbb{R}$ acting by

$$gz = (az+b)(cz+d)^{-1}, \qquad z \in \Omega,\ g = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in O(n,m)$$

(with $a \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^{n\times m}$, etc.). The stabilizer subgroup K consists of all block-diagonal ($b = c = 0$) elements in G, while A can be taken as $\{\tau_a : a = \sum_j r_j e_j,\ -1 < r_j < 1\}$, where $\tau_a(x) := \phi_a(-x)$ and e_j is the $n \times m$ matrix with 1 on the (j,j)-position and 0 everywhere else, $1 \le j \le \min(m,n)$. In particular, the rank of Ω is $r = \min(m, n)$. (The previous three examples, corresponding to $m = 1$, were thus of rank 1.)

5. General symmetric spaces of non-compact type include, in addition to analogous unit balls of symmetric or anti-symmetric matrices, also some other infinite series of matrix domains, as well as so-called "exceptional" symmetric domains related to (some) exceptional Lie groups.

For more details and the proofs of all the above, as well as for the complete classification (up to isomorphism) of all symmetric spaces, we refer e.g. to Helgason's books [10], [9], [8].

3. Wigner transform

Recall that $m_{x,y}$ stands for the geodesic midpoint between two points x, y of Ω. We begin by establishing a few properties of the Jacobian $J(x, y)$ of this map, defined by the following equality

$$\int_\Omega f(m_{z,y})\, d\mu(z) = \int_\Omega f(x)\ J(x, y)\, d\mu(x). \tag{8}$$

Proposition 1. *For any $g \in G$, $J(gx, gy) = J(x, y)$.*

Proof. From the definition of J and invariance of $d\mu$ we get

$$\begin{aligned}
\int_\Omega f(x)\, J(gx, gy)\, d\mu(x) &= \int_\Omega f(x)\, J(gx, gy)\, d\mu(gx) \\
&= \int_\Omega f \circ g^{-1}(x)\, J(x, gy)\, d\mu(x) \\
&= \int_\Omega f \circ g^{-1}(m_{z,gy})\, d\mu(z) \\
&= \int_\Omega f \circ g^{-1}(g m_{g^{-1}z,y})\, d\mu(z) \\
&= \int_\Omega f(m_{g^{-1}z,y})\, d\mu(z) \\
&= \int_\Omega f(m_{z,y})\, d\mu(z) \\
&= \int_\Omega f(x)\, J(x, y)\, d\mu(x),
\end{aligned}$$

where the fourth equality follows from the fact that $m_{gx,gy} = g m_{x,y}$. □

Corollary 2. $J(x, y) = J(y, x)$.

Proof. Take for g the geodesic symmetry interchanging x and y. □

Now let F be a function on $\Omega \times \Omega$. The *Wigner transform* $\mathcal{W}_F : \Omega \times \Omega^* \to \mathbb{C}$ of F is defined by

$$\begin{aligned}
\mathcal{W}_F(x; \lambda, b) :&= |e_{\lambda,b}(x)|^{-2} \int_\Omega e_{\lambda,b}(y)\ e_{-\lambda,b}(s_x y)\ F(s_x y, y)\ J(x, y)\, d\mu(y) \\
&= |e_{\lambda,b}(x)|^{-2} \int_\Omega e_{\lambda,b}(s_x y)\ e_{-\lambda,b}(y)\ F(y, s_x y)\ J(x, y)\, d\mu(y).
\end{aligned}$$

The second expression follows from the first upon the change of variable $y \mapsto s_x y$ and noting that $J(x, y) = J(s_x x, s_x y) = J(x, s_x y)$ by the preceding proposition.

Note that the quantity $|e_{\lambda,b}(x)|^{-2}$ is, in fact, independent of λ.

The next three theorems show that our Wigner transform retains the properties we expect from the Euclidean case.

Theorem 3. (Invariance) *For any* $g \in G$,

$$\mathcal{W}_{F\circ g}(x;\lambda,b) = \mathcal{W}_F(gx;\lambda,gb),$$

where $F \circ g(x,y) := F(gx, gy)$.

Proof. Note that for any $x, y \in \Omega$ and $g \in G$,

$$s_{gx}gy = gs_x y.$$

Using the definition of $\mathcal{W}$, the invariance of $d\mu$ and J, and (2), we therefore have

$$\begin{aligned}
&\mathcal{W}_{F\circ g}(g^{-1}x;\lambda,b)\\
&\quad= |e_{\lambda,b}(g^{-1}x)|^{-2}\int_\Omega e_{\lambda,b}(y)e_{-\lambda,b}(s_{g^{-1}x}y)F(gs_{g^{-1}x}y, gy)J(g^{-1}x,y)\,d\mu(y)\\
&\quad= |e_{\lambda,b}(g^{-1}x)|^{-2}\int_\Omega e_{\lambda,b}(g^{-1}y)e_{-\lambda,b}(s_{g^{-1}x}g^{-1}y)F(gs_{g^{-1}x}g^{-1}y, y)\\
&\qquad\qquad J(g^{-1}x, g^{-1}y)\,d\mu(y)\\
&\quad= |e_{\lambda,b}(g^{-1}x)|^{-2}\int_\Omega e_{\lambda,b}(g^{-1}y)e_{-\lambda,b}(g^{-1}s_x y)F(s_x y, y)J(x,y)\,d\mu(y)\\
&\quad= |e_{\lambda,b}(g^{-1}0)e_{\lambda,gb}(x)|^{-2}\int_\Omega e_{\lambda,b}(g^{-1}0)e_{\lambda,gb}(y)e_{-\lambda,b}(g^{-1}0)e_{-\lambda,gb}(s_x y)\\
&\qquad\qquad F(s_x y, y)J(x,y)\,d\mu(y)\\
&\quad= |e_{\lambda,gb}(x)|^{-2}\int_\Omega e_{\lambda,gb}(y)e_{-\lambda,gb}(s_x y)F(s_x y,y)J(x,y)\,d\mu(y)\\
&\quad= \mathcal{W}_F(x;\lambda,gb),
\end{aligned}$$

as asserted. □

Theorem 4. (Marginality) *For* F *of the form* $F(x,y) = f(x)\overline{g(y)}$, *with* $f, g \in L^2(\Omega, d\mu)$, *we have the marginality relations*

$$\int_\Omega \mathcal{W}_F(x;\lambda,b)\,|e_{\lambda,b}(x)|^2\,d\mu(x) = \tilde{f}(\lambda,b)\,\overline{\tilde{g}(\lambda,b)};$$

$$\int_{\Omega^*} \mathcal{W}_F(x;\lambda,b)\,|e_{\lambda,b}(x)|^2\,d\rho(\lambda,b) = f(x)\,\overline{g(x)}.$$

Proof. For the first, use the defining property (8) of the Jacobian:

$$\begin{aligned}
&\int_\Omega \mathcal{W}_F(x;\lambda,b)\,|e_{\lambda,b}(x)|^2\,d\mu(x)\\
&\qquad= \int_\Omega\int_\Omega e_{\lambda,b}(y)\;e_{-\lambda,b}(s_x y)\;f(s_x y)\;\overline{g(y)}\;J(x,y)\,d\mu(x)\,d\mu(y)\\
&\qquad= \int_\Omega\int_\Omega e_{\lambda,b}(y)\;e_{-\lambda,b}(s_{m_{z,y}}y)\;f(s_{m_{z,y}}y)\;\overline{g(y)}\,d\mu(z)\,d\mu(y)\\
&\qquad= \int_\Omega\int_\Omega e_{\lambda,b}(y)\;e_{-\lambda,b}(z)\;f(z)\;\overline{g(y)}\,d\mu(z)\,d\mu(y) \qquad (\text{since } s_{m_{z,y}}y = z)\\
&\qquad= \tilde{f}(\lambda,b)\,\overline{\tilde{g}(\lambda,b)}.
\end{aligned}$$

For the second, note that by Plancherel

$$\int_{\Omega^*} e_{\lambda,b}(y)\, e_{-\lambda,b}(z)\, d\rho(\lambda,b) = \delta_{yz}. \tag{9}$$

Thus

$$\begin{aligned}
&\int_{\Omega^*} \mathcal{W}_F(x;\lambda,b)\, |e_{\lambda,b}(x)|^2\, d\rho(\lambda,b) \\
&\qquad = \int_\Omega \int_{\Omega^*} e_{\lambda,b}(y)\, e_{-\lambda,b}(s_x y)\, f(s_x y)\, \overline{g(y)}\, J(x,y)\, d\rho(\lambda,b)\, d\mu(y) \\
&\qquad = \int_\Omega \delta_{y,s_x y}\, f(s_x y)\, \overline{g(y)}\, J(x,y)\, d\mu(y) \\
&\qquad = f(s_x x)\, \overline{g(x)}\, J(x,x) \\
&\qquad = f(x)\, \overline{g(x)}\, J(x,x).
\end{aligned}$$

On the other hand, by the invariance of J we have $J(x,x) = J(\phi_x x, \phi_x x) = J(0,0)$, and taking in the defining property for J

$$\int_\Omega f(x)\, J(x,0)\, d\mu(x) = \int_\Omega f(m_{z,0})\, d\mu(z)$$

for f an approximate identity (i.e. letting f tend to the delta function at the origin), we get $J(0,0) = 1$. Thus the second part of the theorem follows. □

Remark. In addition to the Iwasawa decomposition $G = NAK$, one also has the Bruhat decomposition $G = \overline{KA^+K}$, where A^+ is a certain "positive" subset of A and the bar stands for closure. It can be deduced from the latter that the ambient space $\mathbb{R}^d \supset \Omega = G/K$ admits a "polar decomposition" as $\mathbb{R}^d \cong \overline{K/M \times \mathfrak{a}^+}$ — more precisely, any $x \in \mathbb{R}^d$ can be written in the form $x = ka$ with a lying in a fixed subspace isomorphic to $\mathfrak{a} \cong \mathfrak{a}^* \cong \mathbb{R}^r$; and if we set $\mathfrak{a}^+ = \{t_1 e_1 + \dots + t_r e_r : t_1 > t_2 > \dots > t_r > 0\}$, where $e_1, \dots, e_r$ is an appropriate basis for $\mathfrak{a}$, then the correspondence $\mathbb{R}^d \ni x \longleftrightarrow (kM, a) \in K/M \times \overline{\mathfrak{a}^+}$ is one-to-one except for the set of measure zero where $t_j = t_{j+1}$ or $t_j = 0$ for some j (then the $t_1, \dots, t_r$ are still determined uniquely, but kM is not). (The r-tuple $\mathbf{d}(x) := (t_1, \dots, t_r)$ is called the "complex distance" of x from the origin.) In this way, the cotangent space $T_x^*\Omega \cong \mathbb{R}^d$ at any point $x \in \Omega$ can essentially be identified with $K/M \times \mathfrak{a}^*$, and we can thus think of the Fourier-Helgason transform $\tilde{f} : \Omega^* \to \mathbb{C}$ as living on the cotangent space $T_x^*\Omega$. Similarly, the Wigner transform $\mathcal{W}_F : \Omega \times \Omega^* \to \mathbb{C}$ can be envisaged as living in fact on the cotangent bundle $T^*\Omega$. In a way, this is reminiscent of viewing the ordinary Fourier transform $\tilde{f}(\xi)$ on $\mathbb{R}^2 \cong \mathbb{C}$ in the polar coordinates as $\tilde{f}(\xi) \equiv \tilde{f}(r,\theta)$ where $\xi = re^{i\theta}$; the subtle difference is that instead of the simple symmetry relation $\tilde{f}(r,\theta) = \tilde{f}(-r, \theta+\pi)$, for the Fourier-Helgason transform one has the more complicated symmetry relations, mentioned in Section 2, relating $\tilde{f}(\lambda,b)$ and $\tilde{f}(s\lambda,b)$ for s in the Weyl group. □

4. Pseudodifferential operators

In analogy with the Euclidean case, the Wigner function can be used to define the Weyl calculus of pseudodifferential operators by assigning to a "symbol" function

a on $\Omega \times \Omega^*$ the operator Ψ_a on $L^2(\Omega, d\mu)$ defined by

$$\langle \Psi_a u, v \rangle = \int_\Omega \int_{\Omega^*} \mathcal{W}_{u\otimes\overline{v}}(x;\lambda,b)\; a(x,\lambda,b)\; |e_{\lambda,b}(x)|^2 \, d\rho(\lambda,b)\, d\mu(x),$$

where $(u \otimes \overline{v})(x,y) := u(x)\overline{v(y)}$. In other words,

$$\begin{aligned}\Psi_a u(y) &= \int_{\Omega^*} \int_\Omega a(m_{z,y};\lambda,b)\; e_{\lambda,b}(y)\; e_{-\lambda,b}(z)\; u(z)\, d\mu(z)\, d\rho(\lambda,b) \\ &= \int_\Omega \widetilde{a}(y,z)\; u(z)\, d\mu(z),\end{aligned}$$

where $\widetilde{a}$ is the integral kernel

$$\widetilde{a}(y,z) := \int_{\Omega^*} a(m_{z,y};\lambda,b)\; e_{\lambda,b}(y)\; e_{-\lambda,b}(z)\, d\rho(\lambda,b). \tag{10}$$

Note that for $a(x;\lambda,b) = a(x)$ depending only on the space variable, Ψ_a reduces just to a multiplication operator: indeed, by Plancherel's formula (9),

$$\Psi_a u(y) = \int_\Omega a(m_{z,y})\; \delta_{y,z}\; u(z)\, d\mu(z) = a(m_{y,y})u(y) = a(y)u(y).$$

Similarly, for $a(x;\lambda,b) = a(\lambda)$ depending only on λ the operator Ψ_a reduces to the corresponding Fourier multiplier:

$$\Psi_a u(y) = \int_{\Omega^*} a(\lambda)\; e_{\lambda,b}(y)\; \tilde{u}(\lambda,b)\, d\rho(\lambda,b) = \Big(a(\lambda)\tilde{u}(\lambda,b)\Big)^\wedge,$$

where $^\wedge$ stands for the inverse Fourier-Helgason transform. This shows, in particular, that all invariant differential operators on Ω arise as Ψ_a for $a = a(\lambda)$ an appropriate W-invariant polynomial on $\mathfrak{a}^*$.

The invariance properties of the Wigner transform are reflected in the corresponding invariance properties for the Weyl pseudodifferential operators Ψ_a and their integral kernels $\widetilde{a}$.

Theorem 5. *For any $g \in G$, we have*

$$\widetilde{a}(gy, gz) = \widetilde{a^g}(y,z),$$

i.e. $\widetilde{a} \circ g = \widetilde{a^g}$, where $a^g(x;\lambda,b) := a(gx;\lambda,gb)$.

Proof. From (2),

$$\begin{aligned}\widetilde{a}(gy,gz) &= \int_{\Omega^*} a(gm_{z,y};\lambda,b)\; e_{\lambda,b}(gy)\; e_{-\lambda,b}(gz)\, d\rho(\lambda,b) \\ &= \int_{\Omega^*} |e_{\lambda,b}(g0)|^2 a(gm_{z,y};\lambda,b)\; e_{\lambda,g^{-1}b}(y)\; e_{-\lambda,g^{-1}b}(z)\, d\rho(\lambda,b) \\ &= \int_{\Omega^*} a(gm_{z,y};\lambda,gb)\; e_{\lambda,b}(y)\; e_{-\lambda,b}(z)\; |e_{\lambda,gb}(g0)|^2\, d\rho(\lambda,gb) \\ &= \int_{\Omega^*} a(gm_{z,y};\lambda,gb)\; e_{\lambda,b}(y)\; e_{-\lambda,b}(z)\, d\rho(\lambda,b) \\ &= \widetilde{a^g}(y,z),\end{aligned}$$

where the penultimate equality used (3). ∎

For $g \in G$, let U_g denote the unitary operator on $L^2(\Omega, d\mu)$ of composition with g^{-1}:

$$U_g f(z) := f(g^{-1}z).$$

Theorem 6. $U_g^* \Psi_a U_g = \Psi_{a^g}$.

Proof. Using the invariance of $d\mu$ and the preceding theorem, we get

$$\begin{aligned}\langle \Psi_a U_g u, U_g v\rangle &= \int_\Omega \int_\Omega \widetilde{a}(y,z)\, u(g^{-1}y)\, \overline{v(g^{-1}z)}\, d\mu(y)\, d\mu(z) \\ &= \int_\Omega \int_\Omega \widetilde{a}(gy,gz)\, u(y)\, \overline{v(z)}\, d\mu(y)\, d\mu(z) \\ &= \int_\Omega \int_\Omega \widetilde{a^g}(y,z)\, u(y)\, \overline{v(z)}\, d\mu(y)\, d\mu(z) \\ &= \langle \Psi_{a^g} u, v\rangle,\end{aligned}$$

completing the proof. □

5. Invertibility

We proceed by showing that, to a certain extent, the assignments $a \mapsto \widetilde{a}$ and $F \mapsto \mathcal{W}_F$ are inverses of each other. While in the Euclidean case this is true without any restrictions, for symmetric domains this turns out to hold, in general, only for functions F which are of a special form. On the level of the Wigner transform, this corresponds to symbols a on $\Omega \times \Omega^*$ which are independent of the variable b:

$$a(x; \lambda, b) = a(x; \lambda).$$

Theorem 7. *Let a be a function on $\Omega \times \Omega^*$ which is independent of the variable b. Then*

$$\mathcal{W}_{\widetilde{a}} = a.$$

We begin with a lemma.

Lemma 8. *For $x, y \in \Omega$ and $\lambda \in \mathfrak{a}^*$,*

$$\int_B e_{\lambda,b}(x)\, e_{-\lambda,b}(y)\, db = \Phi_\lambda(\phi_y x) = \Phi_\lambda(\phi_x y).$$

Proof. By (2),

$$e_{\lambda,b}(x) = e_{\lambda,b}(\phi_y \phi_y x) = e_{\lambda,b}(\phi_y 0) e_{\lambda,\phi_y b}(\phi_y x) = e_{\lambda,b}(y) e_{\lambda,\phi_y b}(\phi_y x).$$

Hence

$$e_{\lambda,b}(x) e_{-\lambda,b}(y) = |e_{\lambda,b}(y)|^2\, e_{\lambda,\phi_y b}(\phi_y x).$$

But by (4), $|e_{\lambda,b}(y)|^2\, db = d(\phi_y b)$; thus

$$\begin{aligned}\int_B e_{\lambda,b}(x)\, e_{-\lambda,b}(y)\, db &= \int_B e_{\lambda,\phi_y b}(\phi_y x)\, d(\phi_y b) \\ &= \int_B e_{\lambda,b}(\phi_y x)\, db = \Phi_\lambda(\phi_y x),\end{aligned}$$

proving the first claim. For the second, note that $\phi_{\phi_x y} \phi_x \phi_y =: k$ maps 0 to 0, hence belongs to K; and from $\phi_{\phi_x y} = k \phi_y \phi_x$ we then get $\phi_x y = \phi_{\phi_x y} 0 = k \phi_y \phi_x 0 = k \phi_y x$. Since Φ_λ is K-invariant, it follows that $\Phi_\lambda(\phi_x y) = \Phi_\lambda(\phi_y x)$. □

Proof of Theorem 7. Note that from the transformation properties of $\mathcal{W}$ and $\widetilde{a}$ we have

$$\mathcal{W}_{\widetilde{a}}(x;\lambda,b) = \mathcal{W}_{\widetilde{a}}(\phi_x 0;\lambda,\phi_x\phi_x b) = \mathcal{W}_{\widetilde{a}\circ\phi_x}(0;\lambda,\phi_x b) = \mathcal{W}_{\widetilde{a^{\phi_x}}}(0;\lambda,\phi_x b),$$
$$a(x;\lambda,b) = a(\phi_x 0;\lambda,\phi_x\phi_x b) = a^{\phi_x}(0;\lambda,\phi_x b).$$

Furthermore, it is immediate from the definition that if a is independent of b, then so is a^g for any $g \in G$. Thus it is enough to prove the assertion for $x = 0$, i.e. to prove that

$$\mathcal{W}_{\widetilde{a}}(0;\lambda,b) = a(0;\lambda) \qquad \forall b \in B.$$

From the definitions we get

$$\begin{aligned}\mathcal{W}_{\widetilde{a}}(x;\lambda,b) &= |e_{\lambda,b}(x)|^{-2}\int_\Omega e_{\lambda,b}(y)\, e_{-\lambda,b}(s_x y)\, \widetilde{a}(s_x y, y)\, J(x,y)\, d\mu(y)\\ &= |e_{\lambda,b}(x)|^{-2}\int_\Omega\int_{\Omega^*} e_{\lambda,b}(y)\, e_{-\lambda,b}(s_x y)\, a(m_{y,s_x y};\lambda',b') e_{\lambda',b'}(s_x y)\\ &\qquad\qquad e_{-\lambda',b'}(y)\, J(x,y)\, d\rho(\lambda',b')\, d\mu(y)\\ &= |e_{\lambda,b}(x)|^{-2}\int_{\Omega^*} a(x;\lambda',b')\int_\Omega e_{\lambda,b}(y)\, e_{-\lambda,b}(s_x y)\, e_{\lambda',b'}(s_x y)\\ &\qquad\qquad e_{-\lambda',b'}(y)\, J(x,y)\, d\mu(y)\, d\rho(\lambda',b'),\end{aligned}$$

since $m_{y,s_x y} = x$. Thus, as $e_{\lambda,b}(0) = 1$ for any λ and b,

$$\begin{aligned}\mathcal{W}_{\widetilde{a}}(0;\lambda,b) = \int_{\Omega^*} a(0;\lambda',b')\int_\Omega & e_{\lambda,b}(y)\, e_{-\lambda,b}(s_0 y)\, e_{\lambda',b'}(s_0 y)\\ & e_{-\lambda',b'}(y)\, J(0,y)\, d\mu(y)\, d\rho(\lambda',b').\end{aligned}$$

(Here, of course, $s_0 y = -y$, but we keep s_0 in order to avoid some extra parenthesis below.) As $a(0;\lambda',b')$ is independent of b' by hypothesis, we can carry out the b' integration, the result being by the last lemma

$$\begin{aligned}\mathcal{W}_{\widetilde{a}}(0;\lambda,b) &= \int_{\mathfrak{a}^*} a(0;\lambda')\int_\Omega e_{\lambda,b}(y)\, e_{-\lambda,b}(s_0 y)\, \Phi_\lambda(\phi_y s_0 y)\, J(0,y)\, d\mu(y)\, d\rho(\lambda')\\ &= \int_\Omega e_{\lambda,b}(y)\, e_{-\lambda,b}(s_0 y)\, \check{a}(0;\phi_y s_0 y)\, J(0,y)\, d\mu(y),\end{aligned}$$

where $\check{a}$ stands for the inverse Helgason-Fourier (or, in this case, spherical) transform of $a(x;\lambda)$ with respect to λ. Applying the definition of the Jacobian, this becomes

$$\mathcal{W}_{\widetilde{a}}(0;\lambda,b) = \int_\Omega e_{\lambda,b}(m_{0y})\, e_{-\lambda,b}(s_0 m_{0y})\, \check{a}(0;\phi_{m_{0y}} s_0 m_{0y})\, d\mu(y).$$

However, $\phi_{m_{y,0}} s_0 m_{y,0} = y$, so the last expression equals

$$\mathcal{W}_{\widetilde{a}}(0;\lambda,b) = \int_\Omega e_{\lambda,b}(m_{0y})\, e_{-\lambda,b}(s_0 m_{0y})\, \check{a}(0;y)\, d\mu(y).$$

Since $\check{a}(0,\cdot)$ is a K-invariant function, we can replace y by ky, and then also integrate over k. Since $m_{0,ky} = m_{k0,ky} = km_{0,y}$ and $e_{\lambda,b}(kz) = e_{\lambda,k^{-1}b}(z)$, this gives, using again the last lemma,

$$\mathcal{W}_{\widetilde{a}}(0;\lambda,b) = \int_\Omega\int_K e_{\lambda,b}(m_{0,ky})\, e_{-\lambda,b}(s_0 m_{0,ky})\, dk\, \check{a}(0;y)\, d\mu(y)$$

$$\begin{aligned}
&= \int_\Omega \int_K e_{\lambda,k^{-1}b}(m_{0y})\, e_{-\lambda,k^{-1}b}(s_0 m_{0y})\, dk\, \check{a}(0;y)\, d\mu(y) \\
&= \int_\Omega \int_B e_{\lambda,b}(m_{0y})\, e_{-\lambda,b}(s_0 m_{0y})\, db\, \check{a}(0;y)\, d\mu(y) \\
&= \int_\Omega \Phi_\lambda(\phi_{m_{0y}} s_0 m_{0y})\, \check{a}(0;y)\, d\mu(y) \\
&= \int_\Omega \Phi_\lambda(y)\, \check{a}(0;y)\, d\mu(y) \\
&= a(0;\lambda),
\end{aligned}$$

by (5). (Note that, K being a compact group, $d(k^{-1}) = dk$.) This completes the proof. □

Remark. From the proof it is evident that the theorem in general cannot be expected to hold if the K-invariance hypothesis is dropped.

To state the analogue of the last theorem in the other direction, we first need to identify the functions $\widetilde{a}(x,y)$ corresponding to symbols $a(x;\lambda,b)$ which are independent of b.

Let $\mathcal{A}$ denote the set of all functions F on $\Omega\times\Omega$ of the form

$$F(x,y) = A(m_{xy}, \phi_x y), \tag{11}$$

where $A : \Omega\times\Omega \to \mathbb{C}$ is K-invariant in the second argument, i.e. $A(u,v) = A(u,kv)$ $\forall k \in K$.

Remark. The map

$$(x,y) \longmapsto (m_{xy}, \phi_{m_{xy}} x)$$

of $\Omega\times\Omega$ into itself is a diffeomorphism onto; its inverse is given by

$$(m,u) \longmapsto (\phi_m u, \phi_m s_0 u).$$

Thus every function F on $\Omega\times\Omega$ can be written uniquely in the form $F(x,y) = G(m_{xy}, \phi_{m_{xy}} x)$ for some function G on $\Omega\times\Omega$. Functions in $\mathcal{A}$ correspond to the G which are K-invariant in the second argument.

(Indeed, recalling the notion of the complex distance $\mathbf{d}(x)$ from the origin mentioned in the end of Section 3, one can define also the complex distance $\mathbf{d}(x,y)$ of two points $x,y\in\Omega$ by $\mathbf{d}(x,y) := \mathbf{d}(\phi_x y) = \mathbf{d}(\phi_y x)$. It is then known that $\mathbf{d}(gx,gy) = \mathbf{d}(x,y)$ for any $g \in G$ (and, conversely, if $\mathbf{d}(x,y) = \mathbf{d}(x_1,y_1)$, then there is $g \in G$ with $gx = x_1$, $gy = y_1$). The condition that a function $f(x)$, $x\in\Omega$, is K-invariant means precisely that it depends only on $\mathbf{d}(x)$. Furthermore, $\mathbf{d}(x,s_0x) = \frac{2\mathbf{d}(x)}{1+\mathbf{d}(x)^2}$ (where $\frac{2\mathbf{d}}{1+\mathbf{d}^2} := (\frac{2d_1}{1+d_1^2},\dots,\frac{2d_r}{1+d_r^2})$ if $\mathbf{d} = (d_1,\dots,d_r)$), and similarly $\mathbf{d}(x,y) = \frac{2\mathbf{d}(m_{xy},x)}{1+\mathbf{d}(m_{xy},x)^2}$; that is, $\mathbf{d}(x,y)$ and $\mathbf{d}(m_{xy},x)$ are uniquely determined by each other, and similarly for $\mathbf{d}(x)$ and $\mathbf{d}(x,s_0x)$. Hence, if $F(x,y) = A(m_{xy},\phi_x y)$ where A is K-invariant in the second argument, and $G(m,u) := F(\phi_m u, \phi_m s_0 u)$, then $G(m,u)$ depends only on $m_{\phi_m u, \phi_m s_0 u} = m$ and $\mathbf{d}(\phi_m u, \phi_m s_0 u) = \mathbf{d}(u, s_0 u) = \frac{2\mathbf{d}(u)}{1+\mathbf{d}(u)^2}$, hence only on m and $\mathbf{d}(u)$, so it is K-invariant in u. Conversely, if $F(x,y) = G(m_{xy}, \phi_{m_{xy}} x)$ where G is K-invariant in the second argument, then $F(x,y)$ depends only on m_{xy} and $\mathbf{d}(m_{xy},x)$, hence only on m_{xy} and $\mathbf{d}(x,y)$, so it has the form $F(x,y) = A(m_{xy},\phi_x y)$ where $A(m,u)$ is K-invariant in u, i.e. $F\in\mathcal{A}$.)

Proposition 9. *If $a(x;\lambda,b)=a(x;\lambda)$ does not depend on b, then $\widetilde{a}\in\mathcal{A}$. Conversely, every function F in $\mathcal{A}$ arises as $\widetilde{a}$ for a unique a as above.*

Proof. For $a=a(x;\lambda)$ independent of b, we have by Lemma 8

$$\begin{aligned}\widetilde{a}(x,y)&=\int_{\Omega^*} a(m_{xy};\lambda)\, e_{\lambda,b}(x)\, e_{-\lambda,b}(y)\, d\rho(\lambda,b)\\&=\int_{\Omega^*} a(m_{xy};\lambda)\, \Phi_\lambda(\phi_x y)\, d\rho(\lambda)\\&=\breve{a}(m_{xy};\phi_x y),\end{aligned}$$

where $\breve{a}$ has the same meaning as in the proof of Theorem 7. Thus $F=\widetilde{a}$ is of the form (11) with $A=\breve{a}$, proving the first claim. The inversion formula for the spherical transform gives the second part. □

Corollary 10. *Let $F\in\mathcal{A}$. Then*

$$\widetilde{\mathcal{W}_F}=F.$$

Proof. With the a from the last proposition, we have by Theorem 7

$$\widetilde{\mathcal{W}_F}=\widetilde{\mathcal{W}_{\widetilde{a}}}=\widetilde{a}=F.$$

□

Observe that in the proof of Theorem 7, when computing $\mathcal{W}_{\widetilde{a}}(0;\lambda,b)$ we in fact never used the full hypothesis that $a(x;\lambda,b)$ is independent of b, but only that $a(0;\lambda,b)$ is independent of b. We conclude this section by recording a small corollary to this observation.

Proposition 11. *Assume that a is K-invariant, in the sense that $a=a^k$ $\forall k\in K$. Then $a(0;\lambda,b)=a(0;\lambda)$ is independent of b, and*

$$\mathcal{W}_{\widetilde{a}}(0;\lambda,b)=a(0;\lambda)\qquad\forall b\in B.$$

Proof. From $a^k=a$ we get

$$a(0;\lambda,kb)=a(k0;\lambda,kb)=a(0;\lambda,b),$$

proving that $a(0;\lambda,b)$ is independent of b, since K acts transitively on $B=K/M$. The rest is immediate from the observation preceding the proposition. □

6. Unitarity

The classical Euclidean Wigner transform is a unitary operator on $L^2(\mathbb{R}^n\times\mathbb{R}^n)$. Here is an analogue of this fact in our setting of symmetric spaces of non-compact type. This theorem seems not to have been hitherto noticed even in the simplest non-Euclidean setting of the unit disc. For brevity, let us denote by $\mathcal{B}$ the set of all functions $a(x;\lambda,b)$ on $\Omega\times\Omega^*$ which are independent of the variable b.

Theorem 12. *The map $F\mapsto\mathcal{W}_F$ is a unitary operator from $L^2(\Omega\times\Omega,d\mu\times d\mu)\cap\mathcal{A}$ onto $L^2(\Omega\times\Omega^*,d\mu\times d\rho)\cap\mathcal{B}$.*

Proof. In view of Theorem 7 and Proposition 9, it is enough to show that the inverse map $a\mapsto\widetilde{a}$ is an isometry from $L^2(\Omega\times\Omega^*)\cap\mathcal{B}$ into $L^2(\Omega\times\Omega)\cap\mathcal{A}$ (with their respective measures); that is, that

$$\int_\Omega\int_{\mathfrak{a}^*}|a(x,\lambda)|^2\,d\rho(\lambda)\,d\mu(x)=\int_\Omega\int_\Omega|\widetilde{a}(x,y)|^2\,d\mu(y)\,d\mu(x).$$

In course of the proof of Proposition 7, we have seen that $\widetilde{a}(x,y) = \check{a}(m_{xy}, \phi_x y)$, where $\check{a}$ has again the same meaning as before. By Plancherel, the desired equality is therefore equivalent to

$$\int_\Omega \int_\Omega |\check{a}(x;y)|^2 \, d\mu(y) \, d\mu(x) = \int_\Omega \int_\Omega |\check{a}(m_{xy}, \phi_x y)|^2 \, d\mu(y) \, d\mu(x). \tag{12}$$

In view of the invariance of the measure $d\mu$, we may replace x by kx, $k \in K$, so the left-hand side of (12) equals

$$\int_\Omega \int_\Omega |\check{a}(kx,y)|^2 \, d\mu(x) \, d\mu(y), \tag{13}$$

for any $k \in K$. Similarly, on the right-hand side of (12) we can replace y by $\phi_x y$ in the inner integral, giving

$$\int_\Omega \int_\Omega |\check{a}(m_{x,\phi_x y}, y)|^2 \, d\mu(y) \, d\mu(x).$$

Replacing again x, y by kx, ky, $k \in K$, recalling that $\check{a}$ is a K-invariant function in its second argument, and using the fact that $\phi_{kx} ky = k\phi_x y$ and $m_{kx, k\phi_x y} = m_{k\phi_x 0, k\phi_x y} = k\phi_x m_{0y}$ (since ϕ_x is a Riemannian isometry), we thus see that the right-hand side of (12) is equal to

$$\int_\Omega \int_\Omega |\check{a}(k\phi_x m_{0y}, y)|^2 \, d\mu(x) \, d\mu(y), \tag{14}$$

for any $k \in K$. Since the $k \in K$ in both (13) and (14) can be taken arbitrary, the desired equality (12) is therefore actually equivalent to (as $x = \phi_x 0$)

$$\int_K \int_\Omega \int_\Omega |\check{a}(k\phi_x 0, y)|^2 \, d\mu(x) \, d\mu(y) \, dk = \int_K \int_\Omega \int_\Omega |\check{a}(k\phi_x m_{0y}, y)|^2 \, d\mu(x) \, d\mu(y) \, dk.$$

We claim that we in fact have even the equality

$$\int_K \int_\Omega |\check{a}(k\phi_x 0, y)|^2 \, d\mu(x) \, dk = \int_K \int_\Omega |\check{a}(k\phi_x m_{0y}, y)|^2 \, d\mu(x) \, dk \tag{15}$$

for any fixed $y \in \Omega$.

Indeed, denote, for brevity, $F(x) := |\check{a}(x;y)|^2$. For $z \in \Omega$, consider the integral

$$\mathcal{I}(z) := \int_K \int_\Omega F(k\phi_x z) \, dk \, d\mu(x).$$

Now any $g \in G$ can be uniquely written in the form $k\phi_x$ with $k \in K$ and $x \in \Omega$ (in fact, $x = g^{-1}0$ and $k = g\phi_x$), and the measure $dk \, d\mu(x)$ corresponds under this parameterization to the Haar measure dg on G. (Recall that G, as a semisimple Lie group, is unimodular, so dg is both the left and the right Haar measure.) Thus

$$\mathcal{I}(z) = \int_G F(gz) \, dg.$$

For any $g_1 \in G$, the invariance of the Haar measure gives

$$\mathcal{I}(g_1 z) = \int_G F(gg_1 z) \, dg = \int_G F(gg_1 z) \, d(gg_1) = \mathcal{I}(z).$$

Since G acts transitively on $\Omega = G/K$, $\mathcal{I}(z)$ is thus independent of z. In particular, $\mathcal{I}(0) = \mathcal{I}(m_{0y})$, proving (15) and completing the proof of the theorem. □

Remark. The equality (12) would follow immediately if the diffeomorphism $(x,y) \mapsto (m_{xy}, \phi_x y)$ were measure-preserving. However, a simple calculation shows that on the disc

$$\frac{d\mu(m_{xy})\, d\mu(\phi_x y)}{d\mu(x)\, d\mu(y)} = \frac{2 - \overline{x}y - \overline{y}x}{2|1-\overline{x}y|} \sqrt{\frac{1-|x|^2}{1-|y|^2}} \neq 1,$$

so this is not the case even for the unit disc.

The proof of the last theorem again indicates that $\mathcal{W}$ cannot probably be expected to act unitarily also on functions $a(x;\lambda,b)$ which are not independent of b.

Remark. Note that the class of K-invariant functions on Ω, and the corresponding class of the functions on Ω^* which are independent of b, play a distinguished role also in the properties of the Helgason-Fourier transform: for instance, the convolution $f*g$ of two functions on Ω does not in general satisfy $(f*g)^\sim = \tilde{f}\tilde{g}$ (which is notorious for the ordinary Fourier transform), however this becomes true if g is K-invariant. This makes the introduction of the two function classes $\mathcal{A}$, $\mathcal{B}$ above quite natural.

7. Concluding remarks

7.1. We have been somewhat nonspecific about what kind of functions we are dealing with: for instance, the inversion formula for the Helgason-Fourier transform holds for f smooth with compact support, and extends to $f \in L^2$ only by Plancherel. There are also analogues of the Schwartz space, one on Ω and another one on Ω^*, such that the Helgason-Fourier transform is an isomorphism of the former onto the latter; see e.g. [6], Chapter 6. The rigorously minded reader should think of the functions $a, \widetilde{a}$, etc., as belonging to the appropriate tensor products of these Schwartz spaces; in that case the convergence of all the integrals involved etc. can be verified with ease. Extensions to more general functions, or even distributions, can be achieved by the standard techniques used for handling oscillatory integrals (see e.g. [15]).

7.2. One might try introducing Hörmander classes for symbols a, and building an analogue of the usual calculus for the Weyl operators Ψ_a — composition formulas, boundedness in Sobolev spaces, etc. Some steps in this direction have been done in Tate [16] for the disc.

A related theory of pseudodifferential operators (on the disc, but very likely extending to any symmetric space of non-compact type), corresponding to the standard Kohn-Nirenberg, rather than Weyl, pseudodifferential operators in the Euclidean case, was developed by Zelditch [22]. However, expressing our operators Ψ_a as these "Kohn-Nirenberg" pseudodifferential operators (thus reducing the questions mentioned in the previous paragraph to the theory already developed by Zelditch) does not seem straightforward, even for the special case of functions independent of b.

7.3. The function class $\mathcal{A}$ is somewhat mysterious: it is totally unclear to the present authors, for instance, how to characterize the Weyl operators Ψ_a with $\widetilde{a} \in \mathcal{A}$. In the Euclidean setting, this would correspond to operators whose Schwartz kernels depend only on $\frac{x+y}{2}$ and $|x-y|$; even in this case the answer is not obvious.

7.4. Though the authors are convinced that there are no analogues of Theorem 7 and Theorem 12 for general symbols a (i.e. possibly depending on b), we are unable to provide an explicit counterexample.

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