

An excursion into Berezin-Toeplitz quantization and related topics

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Abstract. We present an introduction to the Berezin and Berezin-Toeplitz quantizations, starting from their historical origins and relationships with other quantization methods, discussing various instructive examples like the Segal-Bargmann-Fock space, and culminating by highlights of proofs of the existence of these quantizations using both the Boutet de Monvel theory and the approach via Fefferman’s expansion and Forelli-Rudin construction. The exposition strives to be reasonably self-contained and accessible to non-experts.

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Quantization has traditionally been understood as a recipe in physics for passing from a classical system — which, loosely speaking, is something that concerns macroscopic objects and that we are familiar with from everyday’s life — to the “corresponding” quantum system, which pertains to microscopic objects where things are subject to more complicated rules. The latter should reduce to the former as the size of the objects gets large, that is, as the “Planck constant”, which, heuristically, corresponds to the magnitude where the quantum phenomena become relevant, tends to zero. (This is the so-called “correspondence principle”, or “classical limit”.)

Over the time, it became apparent that such a concept is not totally appropriate, both mathematically and physically. From the point of view of physics, it is more appropriate to understand quantization just as a correspondence between classical and quantum systems; that is, there may be quantum systems which have no classical counterpart, as well as different quantum systems corresponding to the same classical system. From the mathematical point of view, one even encounters

obstacles of a different kind — namely, various “no-go” theorems show that there can exist no mathematical recipe that would fulfill all the axioms required by the physical interpretation.

As a result, nowadays we face the existence of many different quantization theories, ranging from geometric quantization, deformation quantization and various related operator-theoretic quantizations to Feynman path integrals, asymptotic quantization, or stochastic quantization, to mention just a few. No one of the existing approaches solves the quantization problem completely; on the other hand, on the mathematics side all these have evolved into rich theories of their own right, and with results of great depth and beauty.

The aim of this paper is to give a flavour of two of the approaches that belong to the list above, namely the Berezin and the Berezin-Toeplitz quantizations. Compared to other similar surveys like [1] or [40], we have tried to intersperse the exposition with simple examples that illustrate the main ideas, thus keeping it — we hope — accessible even to students or newcomers to the area.

The paper is organized as follows. In Section 1, we present in some more detail what has been mentioned in the first two paragraphs above, namely, the original aspirations of the quantization theory and the various ramifications that the subsequent developments have led to. Section 2 discusses what turns out to be the simplest example of Berezin-Toeplitz quantization, namely the Toeplitz operators on the Fock space. The basic principles of the Berezin-Toeplitz and Berezin quantizations in curved (i.e. non-Euclidean) spaces and the necessary tools for them are discussed in Sections 4 and 3, respectively, while the full account of these theories appears in Section 5 and 6. The last Section 7 contains miscellaneous additional comments, bibliographic remarks, and the like.

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1. The problem of quantization

1.1. The canonical quantization

The original concept of quantization, going back to Weyl, von Neumann, and Dirac, consists in assigning operators to functions:

$$f \longmapsto Q_f.$$

Here the functions f are supposed to live on some manifold, called the *classical phase space*; for reasons going back to classical mechanics, the manifold is taken to be *symplectic*, meaning it is equipped with a differential form of a certain kind. (We will be more specific about this later.) The operators live on some

fixed, separable infinite-dimensional Hilbert space H , and are assumed to be self-adjoint if f is real-valued. (They need not be bounded in general.) One calls the functions f *classical observables*, while the corresponding operators Q_f are the associated *quantum observables*. The physical interpretation is that upon performing some experiment to measure a quantity (position, velocity, momentum, energy, ...) represented by f , the possible outcomes will have the probability distribution $\langle \Pi(Q_f)u, u \rangle$, where $\Pi(Q_f)$ is the spectral measure of the operator Q_f , while $u \in H$ is a unit vector characterizing the “state” of the given quantum system. In particular, if Q_f has pure point spectrum consisting of eigenvalues λ_j with eigenvectors u_j , $\|u_j\| = 1$, then the possible outcomes of measuring f will be λ_j with probability $|\langle u, u_j \rangle|^2$; if $u = u_j$ for some j , the measurement will be deterministic and will always return λ_j . Noncommutativity of operators corresponds to the impossibility of measuring simultaneously the corresponding observables.

The simplest example of a quantization rule as above is for $M = \mathbf{R}^{2n}$, the real $2n$ -space, with elements written as $(p, q) \in \mathbf{R}^n \times \mathbf{R}^n$; one thinks of $q_1, \dots, q_n$ as the coordinates of a particle in $\mathbf{R}^n$, and of $p_1, \dots, p_n$ as the velocities (or, more precisely, momenta) of the particle; in other words, M is the *phase space* of a single particle moving in $\mathbf{R}^n$. We take $H = L^2(\mathbf{R}^n)$ for the Hilbert space, viewed as L^2 -functions in the position variables q ; and define the quantum observables Q_f , for f one of the coordinate functions on $\mathbf{R}^{2n}$, by

$$\begin{aligned} Q_{q_j} : f(q) &\longmapsto q_j f(q), \\ Q_{p_j} : f(q) &\longmapsto \frac{h}{2\pi i} \frac{\partial f(q)}{\partial q_j} \end{aligned} \tag{1}$$

(the *Schrödinger representation*). These operators satisfy the *canonical commutation relations* (or just *CCR* for short)

$$\begin{aligned} [Q_{q_j}, Q_{q_k}] &= [Q_{p_j}, Q_{p_k}] = 0, \quad \forall j, k, \\ [Q_{q_j}, Q_{p_k}] &= 0 \quad \text{for } j \neq k, \\ [Q_{q_j}, Q_{p_j}] &= \frac{ih}{2\pi} I, \end{aligned} \tag{2}$$

where $[A, B] := AB - BA$ denotes the commutator of two operators. The parameter h , on which this map Q also depends, is the *Planck constant*; this should be thought of as a small positive number, and the *classical limit* $h \searrow 0$ should somehow recover the classical system from the quantum one, as already mentioned.

Note that under the physical interpretation just explained, (1) implies, in particular, that it is possible to measure simultaneously the position variables q (in fact, the joint spectral distribution of the $Q_{q_1}, \dots, Q_{q_n}$ is just the Lebesgue measure on $\mathbf{R}^n$, so the probability of finding the particle in a state given by $u \in L^2(\mathbf{R}^n)$ to be present in some set $\Omega \subset \mathbf{R}^n$ in an experiment is equal to the integral of $|u|^2$ over Ω), or the momentum variables p , or even p_j and q_k for $j \neq k$, but not q_j and p_j ; the last is a reflection of the celebrated Heisenberg uncertainty

principle. As h tends to zero, even the operators Q_{q_j} and Q_{p_j} become commutative, and the problems with simultaneous non-measurability thus disappear.

Of course, it remains to say how to assign the operators Q_f to more general functions f than the coordinate functions. There are some requirements which such an assignment should satisfy, coming from the physical interpretation:

(A1) The map $f \mapsto Q_f$ should be linear.

(A2) (The *von Neumann rule*.) For any polynomial $\phi : \mathbf{R} \rightarrow \mathbf{R}$, we should have

$$Q_{\phi \circ f} = \phi(Q_f).$$

(In particular, $Q_1 = I$.)

(A3) $[Q_f, Q_g] = -\frac{i\hbar}{2\pi} Q_{\{f,g\}}$, where

$$\{f, g\} = \sum_{j=1}^n \left(\frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} \right)$$

is the *Poisson bracket* of f and g .

Here the axiom (A2) just means that if our experiment yields λ as an outcome for measuring f with some probability, then it should yield λ^2 with the same probability when measuring f^2 , or, more generally, $\phi(\lambda)$ with the same probability when measuring $\phi(f)$. Similarly, the linearity axiom (A1) is quite natural. Finally, the last axiom (A3) has to do with the time evolution of the system, as described by the Hamiltonian formalism in classical mechanics (we will not go into details about that here). (The last axiom also extends in an obvious way to any other manifold M on which we have an analogue of the Poisson bracket defined — these are precisely the symplectic manifolds that we have already hinted at.) Note that for f, g the coordinate functions on $M = \mathbf{R}^{2n}$, the last axiom reduces precisely to the canonical commutation relations (2).

We are thus lead to the problem of extending the rules (1) in such a way that the axioms (A1)–(A3) above are satisfied. So, what are the solutions to this extension problem? (And, more generally, what would be the solutions for some more general symplectic manifold M ?)

1.2. Inconsistencies

Unfortunately, here bad news come. Namely, the above axioms are inconsistent (even in the simplest case of $M = \mathbf{R}^{2n}$).

To see that, denote for brevity $P = Q_{p_1}$, $Q = Q_{q_1}$, $p = p_1$, $q = q_1$; then

$$pq = \frac{(p+q)^2 - p^2 - q^2}{2}$$

implies, using (A1) and (A2), that

$$Q_{pq} = \frac{(P+Q)^2 - P^2 - Q^2}{2} = \frac{PQ + QP}{2}.$$

On the other hand, by (A2) $Q_{q^2} = Q^2$ and $Q_{p^2} = P^2$, so we can apply the same argument to p^2, q^2 in the place of p, q :

$$p^2 q^2 = \frac{(p^2 + q^2)^2 - p^4 - q^4}{2}$$

implies, using (A1) and (A2), that

$$Q_{p^2 q^2} = \frac{P^2 Q^2 + Q^2 P^2}{2}.$$

Finally, as $p^2 q^2 = (pq)^2$, (A2) requires that we should have $Q_{p^2 q^2} = Q_{pq}^2$. However, an easy computation, using the canonical commutation relation for P and Q , shows that

$$\frac{P^2 Q^2 + Q^2 P^2}{2} \neq \left(\frac{PQ + QP}{2} \right)^2$$

(the two sides differ by a nonzero multiple of the identity). Thus we have arrived at a contradiction.

Note that our argument above used just (A1) and (A2), so even these two axioms alone are inconsistent. It was shown by Groenewold in 1946 (with an improvement by van Hove in 1951) that, likewise, (A1) and (A3) alone are inconsistent. Finally, the present author noticed (much later) that also (A2) and (A3) by themselves lead to contradiction. In other words, not only the three axioms (A1)–(A3) all together — although quite innocuous and very natural from the point of view of physics — but even any two of them are already inconsistent!

The contradiction deduced above used polynomial classical observables f , i.e. very nice functions; if we allow some “wilder” functions f as observables, then it can, in fact, be shown that already the von Neumann rule (A2) *alone* and the canonical commutation relations (2) lead to a contradiction. Namely, recall that there exists a continuous function f (Peáno curve) which maps $\mathbf{R}$ continuously and surjectively onto $\mathbf{R}^{2n}$. Let g be a right inverse for f , so that $g : \mathbf{R}^{2n} \rightarrow \mathbf{R}$ and $f \circ g = \text{id}$; such g exists owing to the surjectivity of f , and can be chosen to be measurable and locally bounded. Denote, for brevity, $T = Q_g$ and consider the functions $\phi = p_1 \circ f$, $\psi = q_1 \circ f$. Then by the axiom (A2),

$$\phi(T) = Q_{p_1 \circ f \circ g} = Q_{p_1}, \quad \psi(T) = Q_{q_1 \circ f \circ g} = Q_{q_1},$$

and

$$0 = (\phi\psi - \psi\phi)(T) = \phi(T)\psi(T) - \psi(T)\phi(T) = [Q_{p_1}, Q_{q_1}] = -\frac{i\hbar}{2\pi}I,$$

a contradiction.

What should we do to resolve this disappointing situation? First of all, we will work solely with continuous or, still better, smooth (infinitely differentiable) functions; these are anyway the only ones that we really meet in the physical realm, and it rules out the pathologies we saw in the preceding paragraph. Next, we discard the von Neumann rule, except for $\phi = \mathbf{1}$, i.e.

$$Q_1 = I.$$

The only discrepancy left there is thus the one between the linearity axiom (A1) and the Poisson brackets axiom (A3). There are two established approaches how to deal with that.

The first approach is to actually insist on both axioms, but restrict even further the set of quantizable observables, i.e. the domain of the map $f \mapsto Q_f$ (we have already restricted it to smooth functions a few lines above). For instance, for our quantization on $M = \mathbf{R}^{2n}$, if we allow only functions f at most linear in the momentum variables p_j , then the recipe

$$Q_f : \psi \longmapsto -\frac{i\hbar}{2\pi} \left(\sum_j \frac{\partial f}{\partial p_j} \frac{\partial \psi}{\partial q_j} \right) + \left(f - \sum_j p_j \frac{\partial f}{\partial p_j} \right) \psi,$$

where $\psi = \psi(q) \in L^2(\mathbf{R}^n)$, does the job we need: it extends the Schrödinger representation (1) and satisfies (A1) and (A3). (Note that the last makes sense, since the Poisson bracket of two functions at most linear in p is again at most linear in p .) In the case of a general symplectic manifold M in the place of $\mathbf{R}^{2n}$, one can similarly make things work by restricting, in an appropriate sense, to functions at most linear in “half of the variables”. In technical terms, choosing this “half of the variables” requires the concept of the so-called *polarizations* of the manifold; by definition, a polarization is a smooth choice of subspaces of dimension n in each fiber $T_x M$, $x \in M$, of the tangent bundle TM of M . The whole approach leads to particularly appealing results in the context of manifolds M with nice group actions (symmetries), when methods of representation theory apply, and is known as the *geometric quantization* (Kostant [34], Souriau [41]).

The second approach, on the other hand, starts by relaxing the Poisson brackets axiom (A3) to hold only asymptotically as $\hbar \rightarrow 0$:

$$[Q_f, Q_g] = -\frac{i\hbar}{2\pi} Q_{\{f,g\}} + O(\hbar^2). \quad (3)$$

This is the basic idea behind the *deformation quantization*. Before spelling out the precise definition of the latter in detail, let us look at a simple example on $\mathbf{R}^{2n}$, which we now describe.

1.3. Weyl quantization

An “arbitrary” function $f(p, q)$ on $\mathbf{R}^{2n}$ can be expanded into exponentials via the Fourier transform:

$$f(p, q) = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \hat{f}(\xi, \eta) e^{2\pi i(\xi \cdot p + \eta \cdot q)} d\xi d\eta. \quad (4)$$

From the Schrödinger representation (1) and the Taylor series for the exponential, is it easy to interpret the exponentials $e^{2\pi i \xi \cdot Q_p}$ and $e^{2\pi i \eta \cdot Q_q}$:

$$e^{2\pi i \xi \cdot Q_p} u(q) = u(q + \hbar \xi), \quad e^{2\pi i \eta \cdot Q_q} u(q) = e^{2\pi i \eta \cdot q} u(q).$$

With a bit of effort, one can also take a good guess what $e^{2\pi i(\xi \cdot Q_p + \eta \cdot Q_q)}$ should be. Indeed, given an $u \in L^2(\mathbf{R}^n)$, the function

$$g(q, t) = [e^{2\pi i t(\xi \cdot Q_p + \eta \cdot Q_q)} u](q), \quad t \in \mathbf{R},$$

should be a solution to $\partial g / \partial t = 2\pi i(\xi \cdot Q_p + \eta \cdot Q_q)g$ subject to the initial condition $g(q, 0) = u(q)$; in other words,

$$\frac{\partial g}{\partial t} - \sum_{j=1}^n h \xi_j \frac{\partial g}{\partial q_j} = 2\pi i \eta \cdot q g, \quad g(q, 0) = u(q).$$

Fixing q for a moment and setting $G(t) = g(q - th\xi, t)$, this becomes

$$G'(t) = 2\pi i \eta \cdot (q - th\xi) G(t), \quad G(0) = u(q),$$

with the solution $G(t) = e^{2\pi i t \eta \cdot q - \pi i t^2 h \eta \cdot \xi} u(q)$, or

$$g(q, t) = e^{2\pi i t \eta \cdot (q + th\xi) - \pi i t^2 h \eta \cdot \xi} u(q + th\xi) = e^{2\pi i t \eta \cdot q + \pi i t^2 h \eta \cdot \xi} u(q + th\xi).$$

Taking $t = 1$ we are thus lead to

$$e^{2\pi i(\xi \cdot Q_p + \eta \cdot Q_q)} u(q) = e^{2\pi i \eta \cdot q + \pi i h \eta \cdot \xi} u(q + h\xi).$$

Returning to (4), let us now postulate that

$$Q_f = \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \hat{f}(\xi, \eta) e^{2\pi i(\xi \cdot Q_p + \eta \cdot Q_q)} d\xi d\eta =: W_f.$$

In other words, using the previous formula,

$$\begin{aligned} W_f u(q) &= \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \hat{f}(\xi, \eta) e^{2\pi i \eta \cdot q + \pi i h \eta \cdot \xi} u(q + h\xi) d\xi d\eta \\ &= h^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} \hat{f}\left(\frac{\xi - q}{h}, \eta\right) e^{\pi i \eta \cdot (q + \xi)} u(\xi) d\xi d\eta \\ &= h^{-n} \int_{\mathbf{R}^n} \int_{\mathbf{R}^n} f\left(p, \frac{q + y}{2}\right) e^{2\pi i(q - y) \cdot p / h} u(y) dy dp \end{aligned}$$

by Plancherel's theorem. This is the celebrated *Weyl calculus* of pseudodifferential operators; a beautiful reference for it is Folland's book [27]. It can be shown that, appropriately interpreted, W_f makes sense even for any tempered distribution f on $\mathbf{R}^{2n}$, being then a continuous operator from the Schwartz space $\mathcal{S}(\mathbf{R}^n)$ into the tempered distributions $\mathcal{S}'(\mathbf{R}^n)$ on $\mathbf{R}^n$. If f is sufficiently nice — for instance, if $f \in \mathcal{S}(\mathbf{R}^{2n})$ — then W_f is continuous even from $\mathcal{S}(\mathbf{R}^n)$ into itself. For such f and g , the product $W_f W_g$ therefore makes sense, and it turns out that

$$W_f W_g = W_{fg} + h W_{C_1(f, g)} + O(h^2) \quad (5)$$

as $h \searrow 0$, where

$$C_1(f, g) = \frac{i}{4\pi} \sum_{j=1}^n \left(\frac{\partial f}{\partial q_j} \frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j} \frac{\partial g}{\partial q_j} \right)$$

satisfies

$$C_1(f, g) - C_1(g, f) = -\frac{i}{2\pi} \{f, g\}.$$

Hence

$$[W_f, W_g] = -\frac{ih}{2\pi} W_{\{f, g\}} + O(h^2)$$

and so that the Weyl calculus satisfies (3).

One can even do slightly better than that. Namely, the product formula (5) can even be improved to higher order: there exist $C_2, C_3, \dots$ such that

$$\begin{aligned} W_f W_g &= W_{fg} + h W_{C_1(f,g)} + h^2 W_{C_2(f,g)} + O(h^3), \\ W_f W_g &= W_{fg} + h W_{C_1(f,g)} + h^2 W_{C_2(f,g)} + h^3 W_{C_3(f,g)} + O(h^4), \end{aligned}$$

and so on. Symbolically,

$$W_f W_g = W_{f * g} \quad (6)$$

where

$$f * g := fg + h C_1(f, g) + h^2 C_2(f, g) + h^3 C_3(f, g) + \dots$$

The last expression should be viewed just as a formal power series in h (no convergence is asserted!), and (6) should just be understood as above, i.e.

$$W_f W_g = \sum_{j=0}^{N-1} h^j W_{C_j(f,g)} + O(h^N),$$

for any $N = 0, 1, 2, \dots$

Ultimately, one is even led to the idea that for the quantization it not really necessary to have the operators Q_f , but it suffices to have a noncommutative product like $*$. This is the essence of the second approach to resolving the inconsistency of the axioms (A1)–(A3), called the *deformation quantization*.

1.4. Deformation quantization

The precise definition runs as follows. Given our manifold M , consider the ring $C^\infty(M)[[h]]$ of all formal power series in h over $C^\infty(M)$. That is, the elements of $C^\infty(M)[[h]]$ are formal power series

$$f = \sum_{j=0}^{\infty} h^j f_j(x) \quad (7)$$

with $f_j \in C^\infty(M)$, and addition and multiplication defined in the usual way.

A *star product* is an associative $\mathbf{C}[[h]]$ -bilinear mapping $*$ such that

$$f * g = \sum_{j=0}^{\infty} h^j C_j(f, g), \quad \forall f, g \in C^\infty(M), \quad (8)$$

where the bilinear operators C_j satisfy

$$\begin{aligned} C_0(f, g) &= fg, & C_1(f, g) - C_1(g, f) &= -\frac{i}{2\pi} \{f, g\}, \\ C_j(f, \mathbf{1}) &= C_j(\mathbf{1}, f) = 0 & \forall j \geq 1. \end{aligned}$$

(The $\mathbf{C}[[h]]$ -bilinearity means that $f * g$ is linear in each argument and $(hf) * g = f * (hg) = h(f * g)$; consequently, for any f, g as in (7),

$$\left(\sum_{j=0}^{\infty} h^j f_j(x) \right) * \left(\sum_{k=0}^{\infty} h^k g_k(x) \right) = \sum_{j,k,m=0}^{\infty} h^{j+k+m} C_m(f_j, g_k)(x),$$

where the last sum should, of course, be re-arranged by combining together the terms with the same power h^{j+k+m} of h .)

We have seen at the end of §1.3 that the Weyl calculus, with the star product defined by (6), satisfies (8) (in fact, that is exactly how the Weyl star-product was defined). From (6) and the fact that multiplication of operators is associative, i.e. $(W_f W_g) W_k = W_f (W_g W_k)$, it is also immediate that the Weyl star-product (6) is associative. Thus the Weyl calculus from §1.3 is an example of deformation quantization on $\mathbf{R}^{2n}$.

The drawback of the Weyl quantization is, however, that it does not readily extend to more general phase spaces than $\mathbf{R}^{2n}$. Indeed, its definition used heavily the Fourier transform, and the Fourier transform is something which is specific only for the Euclidean spaces and a few of other situations.

Although the definition of deformation quantization, together with its physics interpretation etc., goes back to 1977 (it was introduced by Bayen, Flato, Fronsdal, Lichnerowicz and Sternheimer in [4]), its existence on a general symplectic manifolds was established only years later. The first proof was given by DeWilde and Lecomte in 1983 [17], followed by different proofs by Fedosov in 1985 [25] and Omori, Maeda and Yoshioka in 1991 [38]; finally, in 1997 Kontsevich established its existence even on any Poisson (i.e. more general than symplectic) manifold [33]. These constructions also allow to describe all possible deformation quantizations of a given manifold, and it turns out that they can be bijectively classified, up to a natural “equivalence”, by the elements of the formal power series ring $H^2(\Omega, \mathbf{R})[[h]]$ over the second cohomology group $H^2(\Omega, \mathbf{R})$. For wealth of further information on deformation quantization, the reader is referred e.g. to the survey by Gutt [29].

One disadvantage of the deformation quantization is that it works with formal power series: no convergence is assumed, nor — it turns out — can be guaranteed in general, which makes the whole thing somewhat awkward when it comes to performing some concrete calculations. It is therefore of interest to have deformation quantizations that would be induced by some operators behind, as was the case of the Weyl quantization and the formula (6), and it would be even nicer if these operators were somehow naturally related to the geometry and analysis on the manifold in question — as was, again, the case for the Weyl transform and its relationship to the Fourier transform.

In the rest of this paper, we will discuss two instances of such deformation quantizations, which exist on domains in $\mathbf{C}^n$ (or, more generally, on nice Kähler manifolds). Before plunging into the formal definitions and technicalities, let us show how things work in the simplest example when the domain in question is the entire complex space $\mathbf{C}^n$.

2. The Fock space

2.1. Fock space on $\mathbf{C}$

The *Fock*, or *Segal-Bargmann*, space on $\mathbf{C}$ is, by definition,

$$\mathcal{F}(\mathbf{C}) = \mathcal{F} := L^2_{\text{hol}}(\mathbf{C}, \pi^{-1} e^{-|z|^2} dz),$$

the subspace of all entire functions in $L^2(\mathbf{C}, \pi^{-1} e^{-|z|^2} dz)$. Given a function $f \in \mathcal{F}$, its Taylor series $f(z) = \sum_{j=0}^{\infty} f_j z^j$ converges on all of $\mathbf{C}$, and uniformly on any compact subset. In particular, for any $R \in (0, +\infty)$ we have

$$\begin{aligned} \int_{|z| < R} |f(z)|^2 e^{-|z|^2} \frac{dz}{\pi} &= \int_{|z| < R} \sum_{j,k=0}^{\infty} f_j z^j \overline{f_k z^k} e^{-|z|^2} \frac{dz}{\pi} \\ &= \int_0^{2\pi} \int_0^R \sum_{j,k=0}^{\infty} f_j \overline{f_k} r^{j+k} e^{(j-k)i\theta} e^{-r^2} \frac{r dr d\theta}{\pi} \\ &= \sum_{j=0}^{\infty} |f_j|^2 \int_0^R r^{2j} e^{-r^2} 2r dr \\ &= \sum_{j=0}^{\infty} |f_j|^2 \int_0^{\sqrt{R}} t^j e^{-t} dt, \end{aligned}$$

where we have used the polar coordinates $z = r e^{i\theta}$, and the interchange of integration and summation in the third equality is justified by the uniform convergence. Letting $R \rightarrow +\infty$ yields

$$\|f\|^2 = \sum_{j=0}^{\infty} |f_j|^2 \int_0^{\infty} t^j e^{-t} dt = \sum_{j=0}^{\infty} |f_j|^2 j!. \quad (9)$$

Thus an entire function f belongs to $\mathcal{F}$ if and only if its Taylor coefficients satisfy $\sum_j |f_j|^2 j! < \infty$.

A similar computation (using the Cauchy-Schwarz inequality, Fubini's theorem and (9) to justify some interchanges of integration and summation signs) gives a formula for the scalar product of two functions $f, g \in \mathcal{F}$ in terms of their Taylor coefficients:

$$\langle f, g \rangle = \sum_{j=0}^{\infty} f_j \overline{g_j} j!. \quad (10)$$

In particular, the monomials z^n , $n = 0, 1, 2, \dots$, form an orthogonal basis of $\mathcal{F}$, and

$$\frac{z^n}{\sqrt{n!}}, \quad n = 0, 1, 2, \dots, \quad (11)$$

is an orthonormal basis.

2.2. Reproducing kernels for $\mathcal{F}$

For any $z \in \mathbf{C}$ we have, by the preceding computations,

$$\begin{aligned} |f(z)| &= \left| \sum_j f_j z^j \right| \leq \sum_j |f_j| |z|^j = \sum_j |f_j| \sqrt{j!} \frac{|z|^j}{\sqrt{j!}} \\ &\leq \left(\sum_j |f_j|^2 j! \right)^{1/2} \left(\sum_j \frac{|z|^{2j}}{j!} \right)^{1/2} = \|f\| e^{|z|^2/2}. \end{aligned}$$

Thus, first, $f \mapsto f(z)$ is a bounded linear functional on $\mathcal{F}$; and second, it is in fact uniformly bounded for z in a bounded set in $\mathbf{C}$.

The latter implies (since locally uniform limits of holomorphic functions are holomorphic) that $\mathcal{F}$ is a closed subspace in $L^2(\mathbf{C}, e^{-|z|^2} dz)$, hence a Hilbert space on its own right.

The former implies that there exist $K_z \in \mathcal{F}$ such that

$$f(z) = \langle f, K_z \rangle \quad \forall f \in \mathcal{F}.$$

In fact, it is not difficult to compute what K_z is explicitly. Indeed, for any $f \in \mathcal{F}$ and $z \in \mathbf{C}$,

$$f(z) = \sum_j f_j z^j = \sum_j f_j \frac{z^j}{j!} j! = \langle f, K_z \rangle,$$

by (10), where

$$K_z(w) = \sum_j \frac{\overline{z^j}}{j!} w^j = e^{\overline{z}w}.$$

Thus $K_z(w) = e^{\overline{z}w}$. The function of two variables

$$K(w, z) := K_z(w) = e^{\overline{z}w}$$

is called the *reproducing kernel* of $\mathcal{F}$, and will play an important role throughout this section.

2.3. Toeplitz operators on $\mathcal{F}$

For $f \in L^\infty(\mathbf{C})$, the *Toeplitz operator* with *symbol* f is, by definition, the operator $T_f : \mathcal{F} \rightarrow \mathcal{F}$ given by

$$T_f u = P(fu)$$

where $P : L^2(\mathbf{C}, \pi^{-1} e^{-|z|^2} dz) \rightarrow \mathcal{F}$ is the orthogonal projection. In other words,

$$T_f = PM_f|_{\mathcal{F}}$$

where $M_f : u \mapsto fu$ is the operator of “multiplication by f ”. There is still other way of expressing T_f , using the reproducing kernel:

$$\begin{aligned} T_f u(z) &= \langle T_f u, K_z \rangle = \langle P(fu), K_z \rangle = \langle fu, PK_z \rangle \\ &= \langle fu, K_z \rangle \quad (\text{since } K_z \in \mathcal{F}, \text{ so } PK_z = K_z) \\ &= \int_{\mathbf{C}^n} f(w) u(w) K(z, w) e^{-|w|^2} \frac{dw}{\pi}, \end{aligned}$$

showing that T_f is an integral operator with integral kernel equal to $f(w)K(z, w)$ (with respect to the weight $e^{-|z|^2}\pi^{-1}$).

Several properties of Toeplitz operators are immediate from their definition:

- The map $f \mapsto T_f$ is linear.
- $\|T_f\| \leq \|M_f\| = \|f\|_\infty$; in particular, T_f is bounded for $f \in L^\infty$.
- $T_1 = I$, the identity operator on $\mathcal{F}$.
- Toeplitz operators behave nicely under taking adjoints: $T_f^* = T_{\bar{f}}$.

It is frequently convenient to consider T_f even for unbounded f , when it often makes sense as a densely defined operator. For instance, since a product of two holomorphic functions is again holomorphic,

$$T_z u = P(zu) = zu$$

if $zu \in L^2$; so T_z is just “multiplication by z ” on $\mathcal{F}$ (defined on the domain $\{u \in \mathcal{F} : zu \in \mathcal{F}\}$, which is dense in $\mathcal{F}$ since it contains the basis elements (11)). Similarly, T_{z^m} for any $m = 0, 1, 2, \dots$, is just the operator of “multiplication by z^m ”, defined again on a dense domain in $\mathcal{F}$ (containing the algebraic linear span of the basis elements (11), i.e. all polynomials).

More generally, for any $f \in L^\infty$,

$$T_{zf} u = P(zfu) = P(fP(zu)) = T_f T_z u$$

if $zu \in L^2$; thus T_{zf} again makes sense as a densely defined operator, whose domain contains that of T_z , and $T_{zf} = T_f T_z$ on $\text{dom } T_z$. Similarly,

$$T_{z^m f} = T_f T_{z^m} = T_f z^m \quad (12)$$

for any $m = 0, 1, 2, \dots$.

Taking adjoints gives:

$$T_{\bar{z}^m f} = T_{\bar{z}^m} T_f. \quad (13)$$

(It is possible to give examples, however, that in general $T_f T_g \neq T_{fg}$.)

Let us now compute the adjoint $T_z^* = T_{\bar{z}}$. By (10), the definition of the reproducing kernel, and (13),

$$\begin{aligned} (T_z^* z^m)(w) &= \langle T_z^* z^m, K_w \rangle = \langle z^m, T_z K_w \rangle = \langle z^m, z K_w \rangle \\ &= \langle z^m, z \sum_j z^j \frac{\bar{w}^j}{j!} \rangle \\ &= \langle z^m, \sum_j z^{j+1} \frac{\bar{w}^j}{j!} \rangle \\ &= \frac{w^{m-1}}{(m-1)!} \langle z^m, z^m \rangle = \frac{m!}{(m-1)!} w^{m-1} \\ &= m w^{m-1}. \end{aligned}$$

Thus $T_z^* z^m = m z^{m-1}$, or

$$T_z^* = \frac{\partial}{\partial z} \equiv \partial.$$

Similarly $T_{z^m}^* = \partial^m$.

2.4. Scaled Fock spaces

From these findings, we get the commutation relation

$$[T_z, T_{\bar{z}}]u = [z, \partial]u = z\partial u - \partial(zu) = -(\partial zu) = -u,$$

or $[T_z, T_{\bar{z}}] = -I$. Setting $z = p + iq$ for the real and imaginary parts, this means

$$[T_p, T_q] = \frac{1}{2i}I,$$

which agrees with the CCR for the Schrödinger representation, except for the constant factor of $\hbar/2$.

It is easy to make even this constant factor come out right. Let us replace the Gaussian weight $\pi^{-1}e^{-|z|^2}$, which we have been using so far, by the scaled version:

$$\mathcal{F}_\alpha(\mathbf{C}) = \mathcal{F}_\alpha := L_{\text{hol}}^2(\mathbf{C}, \frac{\alpha}{\pi}e^{-\alpha|z|^2} dz),$$

where $\alpha > 0$ is a positive parameter. The same calculations as above reveal that an entire function $f(z) = \sum_j f_j z^j$ belongs to $\mathcal{F}_\alpha$ if and only if

$$\sum_{j=0}^{\infty} |f_j|^2 \frac{j!}{\alpha^j} < \infty,$$

that the inner product of $f, g \in \mathcal{F}_\alpha$ is given in terms of their Taylor coefficients by

$$\langle f, g \rangle_{\mathcal{F}_\alpha} = \sum_{j=0}^{\infty} \frac{\alpha^j}{j!} f_j \bar{g}_j,$$

and that $\mathcal{F}_\alpha$ has the reproducing kernel

$$K_\alpha(z, w) = e^{\alpha \bar{w} z}.$$

We have also the Toeplitz operators on $\mathcal{F}_\alpha$,

$$T_f u = P_\alpha(fu),$$

where $P_\alpha : L^2(\mathbf{C}, \frac{\alpha}{\pi}e^{-\alpha|z|^2} dz) \rightarrow \mathcal{F}_\alpha$ is the orthogonal projection. (Thus T_f now depends also on the parameter α , although this is not reflected by the notation.) Finally, all the formulas from the end of §2.3 remain valid, except that a factor of α appears in T_z^* :

$$\begin{aligned} T_{zf} &= T_f T_z, & T_{z^m} u &= T_z^m u = z^m u, \\ T_{\bar{z}f} &= T_{\bar{z}} T_f, & T_{\bar{z}^m} &= T_{\bar{z}}^m = T_z^{*m}, \end{aligned}$$

and

$$T_z^* = \frac{1}{\alpha} \partial.$$

Of course, all these reduce to our previous formulas for $\mathcal{F}$ when $\alpha = 1$.

The commutation relations for T_p, T_q , $z = p + iq \in \mathbf{C} \cong \mathbf{R}^2$, now become

$$[T_q, T_p] = \frac{1}{2\alpha i} I.$$

Taking $\alpha = \pi/h$ thus exactly recovers the CCR for the Schrödinger representation (1) we have started with.

Let us now explore what are the commutation relations for Toeplitz operators T_f, T_g when f, g are polynomials in z and $\bar{z}$ (or, equivalently, in q and p).

Recall $T_{\bar{z}} = \frac{1}{\alpha} \partial$. By the Leibniz rule,

$$T_{\bar{z}z^m} u = T_{\bar{z}} T_{z^m} u = \frac{1}{\alpha} \partial(z^m u) = \frac{mz^{m-1}}{\alpha} u + z^m \frac{1}{\alpha} \partial u,$$

or $T_{\bar{z}z^m} = T_{z^m} T_{\bar{z}} + \frac{1}{\alpha} T_{mz^{m-1}}$. Thus

$$T_{z^m} T_{\bar{z}} = T[\bar{z}z^m - \frac{1}{\alpha}(z^m)'] = T[(\bar{z} - \frac{1}{\alpha}\partial)z^m],$$

where, for typographical reasons, we have started writing $T[f]$ instead of T_f when needed. Multiplying both sides by $T_{\bar{z}^k}$ from the left, and remembering that $T_{\bar{z}^k} f = T_{\bar{z}^k} T_f$ for any f , while ∂ commutes with $\bar{z}$, we obtain

$$T_{\bar{z}^k} T_{z^m} T_{\bar{z}} = T_{\bar{z}^k} T_{z^m} T_{\bar{z}} = T_{\bar{z}^k} T[(\bar{z} - \frac{1}{\alpha}\partial)z^m] = T[\bar{z}^k (\bar{z} - \frac{1}{\alpha}\partial)z^m] = T[(\bar{z} - \frac{1}{\alpha}\partial)\bar{z}^k z^m].$$

It follows by linearity that

$$T_f T_{\bar{z}} = T[(\bar{z} - \frac{1}{\alpha}\partial)f]$$

for any polynomial f in $z, \bar{z}$.

Iterating this m times yields

$$T_f T_{\bar{z}^m} = T[(\bar{z} - \frac{1}{\alpha}\partial)^m f],$$

which by the binomial theorem (note that $\bar{z}$ and ∂ commute!) equals

$$\sum_{j=0}^m \frac{m!}{j!(m-j)!} \frac{(-1)^j}{\alpha^j} \bar{z}^{m-j} \partial^j f = \sum_j \frac{(-1)^j}{j! \alpha^j} (\bar{\partial}^j \bar{z}^m) \partial^j f,$$

so

$$T_f T_{\bar{z}^m} = T\left[\sum_j \frac{(-1)^j}{j! \alpha^j} (\bar{\partial}^j \bar{z}^m) \partial^j f\right].$$

Multiplying both sides by T_{z^k} from the right, and remembering that $T_f T_{z^k} = T_{z^k} f$ for any f , while $\bar{\partial}$ commutes with z , we obtain

$$\begin{aligned} T_f T_{\bar{z}^m} T_{z^k} &= T_f T_{\bar{z}^m} T_{z^k} = T\left[\sum_j \frac{(-1)^j}{j! \alpha^j} (\bar{\partial}^j \bar{z}^m) \partial^j f\right] T_{z^k} \\ &= T\left[\sum_j \frac{(-1)^j}{j! \alpha^j} z^k (\bar{\partial}^j \bar{z}^m) \partial^j f\right] \\ &= T\left[\sum_j \frac{(-1)^j}{j! \alpha^j} (\bar{\partial}^j \bar{z}^m z^k) \partial^j f\right]. \end{aligned}$$

By linearity again, we thus get

$$T_f T_g = T\left[\sum_j \frac{(-1)^j}{j! \alpha^j} (\bar{\partial}^j g) \partial^j f\right] = \sum_j \alpha^{-j} T_{(-1)^j (\bar{\partial}^j g) (\partial^j f) / j!}$$

for any polynomials f, g in $z, \bar{z}$. (Note that the sum has only finitely many nonzero terms.)

The beginning of the last expansion reads

$$T_f T_g = T_{fg} - \frac{1}{\alpha} T_{(\partial f)(\bar{\partial} g)} + O(\alpha^{-2}).$$

Interchanging f, g and subtracting, we thus arrive at

$$[T_f, T_g] = \frac{1}{\alpha} T_{(\partial g)(\bar{\partial} f) - (\partial f)(\bar{\partial} g)} + O(\alpha^{-2}).$$

For $\alpha = \pi/h$, this becomes

$$[T_f, T_g] = \frac{h}{\pi} T_{(\partial g)(\bar{\partial} f) - (\partial f)(\bar{\partial} g)} + O(\alpha^{-2}).$$

Upon passing from $z, \bar{z}$ to the real and imaginary parts $z = p + iq$ (and from the holomorphic and antiholomorphic derivatives $\partial, \bar{\partial}$ to the real derivatives $\partial/\partial p, \partial/\partial q$), this turns out to exactly recover our Poisson bracket axiom (A3).

In conclusion, we see that the map

$$f \longmapsto T_f \quad \text{on } \mathcal{F}_\alpha, \quad \alpha = \frac{\pi}{h},$$

produces a deformation quantization on $\mathbf{C}$, with star-product given by the formula

$$f * g = \sum_j \frac{(-1)^j h^j}{j! \pi^j} (\bar{\partial}^j g) \partial^j f$$

(at least for f, g polynomials in $z, \bar{z}$).

2.5. Fock spaces on $\mathbf{C}^n$

Everything we have done for the Fock space on $\mathbf{C}$ extends also to the analogous spaces

$$\mathcal{F}_\alpha(\mathbf{C}^n) := L^2_{\text{hol}}(\mathbf{C}^n, e^{-\alpha\|z\|^2} (\alpha/\pi)^n dz)$$

on any $\mathbf{C}^n$, $n \geq 1$. Namely, the inner product in $\mathcal{F}_\alpha$ is still given by the formula (10), except that now $j \in \mathbf{N}^n$, $\mathbf{N} = \{0, 1, 2, \dots\}$, is a multiindex. The reproducing kernel is

$$K_\alpha(z, w) = e^{\alpha\langle z, w \rangle},$$

and the Toeplitz operators satisfy

$$T_{z_j} = z_j, \quad T_{z_j}^* = \frac{1}{\alpha} \frac{\partial}{\partial z_j} \equiv \frac{1}{\alpha} \partial_j.$$

The product of Toeplitz operators is given by the formula

$$T_f T_g = \sum_{j \text{ multiindex}} \frac{(-1)^{|j|}}{j! \alpha^{|j|}} T[(\partial^j f)(\bar{\partial}^j g)],$$

at least for f, g polynomials in $z_j, \bar{z}_j$, $j = 1, \dots, n$. Finally, setting $\alpha = \pi/h$, we again arrive at a deformation quantization on $\mathbf{C}^n$, with star-product

$$f * g = \sum_{m=0}^{\infty} h^m C_m(f, g),$$

$$C_m(f, g) = \frac{(-1)^m}{\pi^m} \sum_{j \in \mathbf{N}^n, |j|=m} \frac{1}{j!} T[(\partial^j f)(\bar{\partial}^j g)]$$

(at least for f, g polynomials in $z, \bar{z}$).

We remark that there is actually an isomorphism, the *Bargmann transform*, mapping $L^2(\mathbf{R}^n)$ unitarily onto $\mathcal{F}_\alpha(\mathbf{C}^n)$. Transferring the Weyl operators W_f from §1.3 to $\mathcal{F}_\alpha$ via this isomorphism, W_f actually becomes precisely T_f for f a first-degree polynomial in $z_j, \bar{z}_j$; but this is no longer true for more general f . Thus $f \mapsto W_f$ and $f \mapsto T_f$ are actually two *different* deformation quantizations of $\mathbf{C}^n$. We will meet yet another quantization later on in Section 5.

Even though our “Toeplitz quantization” on $\mathbf{C}^n$ using Toeplitz operators on Fock spaces is simple and nice, as yet it has several shortcomings. First of all, the operators $T_z, T_{\bar{z}}$ above are unbounded operators; although they have a common dense domain (the polynomials in $\mathcal{F}_\alpha$), extra care would be needed to deal with all the computations above on a rigorous level. Furthermore, it is not completely apparent to what extent the formula

$$T_f T_g = \sum_{j \text{ multiindex}} \frac{(-1)^{|j|}}{j! \alpha^{|j|}} T[(\partial^j f)(\bar{\partial}^j g)]$$

remains valid when f, g are not polynomials. Finally, we would need to see what to do to quantize other domains than $\mathbf{C}^n$.

There are tools to handle all this, which we now introduce.

3. Bergman spaces and their operators

3.1. The Bergman kernel

Let Ω a bounded domain in $\mathbf{C}^n$, and let us keep the notation dz for the Lebesgue measure on Ω . The subspace $L^2_{\text{hol}}(\Omega)$ of all holomorphic functions in $L^2(\Omega, dz)$ is known as the *Bergman space*. By the mean-value property of holomorphic functions, if $z \in \Omega$ and $r > 0$ is such that the polydisc $D_{z,r} := \{w \in \mathbf{C}^n : |w_j - z_j| < r \forall j = 1, \dots, n\}$ lies wholly in Ω , then

$$f(z) = (\pi r^2)^{-n} \int_{D_{z,r}} f(w) dw,$$

so

$$|f(z)| \leq (\pi r^2)^{-n} \left(\int_{D_{z,r}} dw \right)^{1/2} \left(\int_{D_{z,r}} |f(w)|^2 dw \right)^{1/2} \leq (\pi r^2)^{-n/2} \|f\|.$$

Consequently, the evaluation functional $f \mapsto f(z)$ is bounded on $L_{\text{hol}}^2(\Omega)$, and uniformly for z in compact subsets of Ω . From the latter it follows, first of all, that L_{hol}^2 is a closed subspace of L^2 , hence a Hilbert space in its own right; while the former again implies that there exists a unique $K_z \in L_{\text{hol}}^2(\Omega)$ such that

$$f(z) = \langle f, K_z \rangle \quad \forall f \in L_{\text{hol}}^2(\Omega).$$

The function

$$K(x, y) \equiv K_y(x) = \langle K_y, K_x \rangle = \overline{K(y, x)} \quad (14)$$

is thus the *reproducing kernel* of $L_{\text{hol}}^2(\Omega)$, called the *Bergman kernel*; note that from (14) it is immediate that it is holomorphic in x and anti-holomorphic in y . Furthermore, since Ω was assumed to be bounded, hence of finite Lebesgue measure, the function constant on belongs to $L_{\text{hol}}^2(\Omega)$, and, consequently,

$$1 = \mathbf{1}(x) = \langle \mathbf{1}, K_x \rangle \leq \|\mathbf{1}\| \|K_x\|, \quad (15)$$

implying that $\|K_x\| > 0$ for all $x \in \Omega$.

3.2. Berezin symbols

While quantization is a recipe for associating operators to functions, here we come across an assignment going in the other direction, i.e. mapping operators on some Hilbert space into functions on some domain. These functions are commonly called the *symbol* of the corresponding operator, and the whole process is often called a *symbol calculus*, or *dequantization*. (Similarly, quantization is sometimes called an *operator calculus* in various contexts.) Here is an instance of such process, which is characteristic for the Bergman spaces.

For an operator T on the Bergman space $L_{\text{hol}}^2(\Omega)$, the *Berezin symbol* $\tilde{T}$ of T is the function on Ω given by

$$\tilde{T}(x) = \frac{\langle TK_x, K_x \rangle}{\langle K_x, K_x \rangle} = \langle Tk_x, k_x \rangle, \quad k_x := \frac{K_x}{\|K_x\|}.$$

Note that this definition makes sense, since the denominator is positive by (15).

There are a number of properties of the symbol map $T \mapsto \tilde{T}$ immediate from its definition:

- The mapping $T \mapsto \tilde{T}$ is linear.
- $\tilde{I} = \mathbf{1}$, i.e. the symbol of the identity operator is the function constant one.
- $\widetilde{T^*} = \overline{\tilde{T}}$.
- If T is bounded, then $\tilde{T}$ is a bounded function; in fact, $\|\tilde{T}\|_\infty \leq \|T\|$.

Moreover, the functions $\tilde{T}$ is smooth (in fact, even real-analytic), because it is the restriction to the diagonal $x = y$ of the function of two variables

$$\tilde{T}(x, y) := \frac{\langle TK_y, K_x \rangle}{\langle K_y, K_x \rangle} = \frac{\langle TK_y, K_x \rangle}{K(x, y)}$$

holomorphic in $x, \bar{y}$ on the set where $K(x, y) \neq 0$. (Since we know that $K(x, x) = \|K_x\|^2 > 0$ by (15), by continuity $K(x, y)$ is nonzero in some neighbourhood of the diagonal.)

However, the most important property of the symbol map is that

$$T \mapsto \tilde{T} \quad \text{is one-to-one.} \quad (16)$$

Indeed, suppose $\tilde{T}(x) = \tilde{T}(x, x) = 0 \ \forall x$. Setting $x = u + iv$, $\bar{y} = u - iv$, it follows that $G(u, v) := \tilde{T}(u + iv, \bar{u} + i\bar{v})$ is a holomorphic function of u, v which vanishes for all u, v real. By uniqueness principle for holomorphic functions, G must vanish identically, so $\tilde{T}(x, y) = 0 \ \forall x, y$, hence $\langle TK_x, K_y \rangle = TK_x(y) = 0 \ \forall x, y$. However,

$$\tilde{T}^* f(x) = \langle T^* f, K_x \rangle = \langle f, TK_x \rangle = \int_{\Omega} f(y) \overline{TK_x(y)} dy,$$

so $T^* f(x) = 0$ for all f and x . Hence, $T^* = 0$ and $T = 0$, proving the injectivity of the map $T \mapsto \tilde{T}$.

3.3. Toeplitz operators on the Bergman space

As before, the Toeplitz operator on $L^2_{\text{hol}}(\Omega)$ with symbol $\phi \in L^\infty(\Omega)$ is defined as

$$T_\phi f = P(\phi f)$$

where $P : L^2 \rightarrow L^2_{\text{hol}}$ is the orthogonal projection (called the *Bergman projection*). All the properties familiar from the Fock space setting remain in force here:

- $f \mapsto T_f$ is linear;
- $T_{\mathbf{1}} = I$;
- $T_f^* = T_{\bar{f}}$;
- $\|T_f\| \leq \|f\|_\infty$.

Furthermore, for ϕ bounded holomorphic, T_ϕ is just the operator of “multiplication by ϕ ” on the Bergman space; and for ϕ bounded holomorphic and f arbitrary,

$$T_{f\phi} = T_f T_\phi, \quad T_{\bar{\phi}f} = T_{\bar{\phi}} T_f.$$

The difference against the Fock space is that now, since Ω is bounded, there are plenty of bounded holomorphic functions on Ω (not just the constants), e.g. all holomorphic polynomials.

We finally remark — although this is not needed, unlike the corresponding property of the Berezin symbol map from §3.2, anywhere in the sequel — that the map $f \mapsto T_f$ is also one-to-one. Indeed, assume that $T_f = 0$; then $\langle T_f u, v \rangle = \langle fu, v \rangle = 0$ for any holomorphic polynomials u, v , in particular, $\langle fz^j, z^m \rangle = 0$, or

$$\int_{\Omega} f(z) z^j \bar{z}^m dz = 0$$

for any multiindices j, m . By the Stone-Weierstrass theorem, this implies that

$$\int f(z) g(z) dz = 0$$

for any function g continuous on the closure $\bar{\Omega}$ of Ω . By the Riesz representation theorem, this means that $f(z) dz$ is the zero measure, and, consequently, that $f = 0$ almost everywhere, as claimed.

3.4. Berezin transform

The Toeplitz correspondence assigns the operator T_f to a function f , while the Berezin symbol map assigns the function $\tilde{T}$ to an operator T . The *Berezin transform* is the composition of these two maps; that is, it assigns to a function f on Ω again a function on Ω , denoted Bf or $\tilde{f}$, and given by

$$Bf := \tilde{f} := \widetilde{T_f}.$$

Chasing through the definitions shows that B is in fact an integral operator:

$$\tilde{f}(x) = \frac{\langle fK_x, K_x \rangle}{\langle K_x, K_x \rangle} = \int_{\Omega} f(y) \frac{|K(x, y)|^2}{K(x, x)} dy.$$

One also checks easily that B has the following properties, which can either be derived from those of the Toeplitz operators and the Berezin symbols, or verified directly.

- $f \mapsto Bf$ is linear;
- $B\mathbf{1} = \mathbf{1}$;
- $B\bar{f} = \overline{Bf}$;
- $\|Bf\|_{\infty} \leq \|f\|_{\infty}$.

Also, Bf is always a real-analytic function on Ω , and the operator B is one-to-one.

3.5. Weighted variants

In an obvious manner, all the objects described in §§3.1–3.4 generalize also to the case of weighted L^2 spaces. Namely, let $w > 0$ be a positive continuous weight on Ω , integrable there with respect to the Lebesgue measure. The associated *weighted Bergman space* on Ω with respect to w is the subspace $L^2_{\text{hol}}(\Omega, w)$ of all holomorphic functions in $L^2(\Omega, w)$. Using the mean-value property of harmonic functions, one again shows that the point evaluations $f \mapsto f(z)$ are continuous on $L^2_{\text{hol}}(\Omega, w)$, uniformly on compact subsets (the continuity and positivity of w is needed here); implying as before that $L^2_{\text{hol}}(\Omega, w)$ is a closed subspace of $L^2(\Omega, w)$ — hence a Hilbert space on its own — and that it possesses a reproducing kernel, the *weighted Bergman kernel* $K_w(x, y) \equiv K_{w,y}(x)$. The Berezin symbol $\tilde{T}$ of an operator T on $L^2_{\text{hol}}(\Omega, w)$ is the function on Ω

$$\tilde{T}(x) = \frac{\langle TK_{w,x}, K_{w,x} \rangle}{\langle K_{w,x}, K_{w,x} \rangle} = \langle Tk_{w,x}, k_{w,x} \rangle, \quad k_{w,x} := \frac{K_{w,x}}{\|K_{w,x}\|}.$$

(Naturally, $\tilde{T}$ depends also on the weight w , although this is not reflected in the notation.) Here one needs that $K_w(x, x) = \|K_{w,x}\|^2 > 0$ for all $x \in \Omega$, which again follows as in (15) (and the hypothesis of the integrability of w ensures that the function constant one belongs to $L^2_{\text{hol}}(\Omega, w)$). Importantly, the Berezin symbol map $T \mapsto \tilde{T}$ is still one-to-one (with the same proof as in the unweighted case).

The Toeplitz operator on $L^2_{\text{hol}}(\Omega, w)$ with symbol $\phi \in L^{\infty}(\Omega)$ is defined as

$$T_{\phi}f = P_w(\phi f)$$

where $P_w : L^2(\Omega, w) \rightarrow L^2_{\text{hol}}(\Omega, w)$ is the orthogonal projection (the *weighted Bergman projection*). Finally, the *weighted Berezin transform* of a function f on Ω is another function on Ω , given by

$$B_w f := \tilde{f} := \widetilde{T_f}$$

(again, the simpler notation $\tilde{f}$ does not reflect that fact that $\tilde{f}$ depends also on the weight w); and B_w is in fact an integral operator

$$B_w f(x) = \frac{\langle f K_{w,x}, K_{w,x} \rangle}{\langle K_{w,x}, K_{w,x} \rangle} = \int_{\Omega} f(y) \frac{|K_w(x, y)|^2}{K_w(x, x)} w(y) dy.$$

Let us now (at last!) describe how all these concepts can be utilized for the construction of the special deformation quantizations on Ω mentioned in the previous sections.

4. Basic ideas of Berezin(-Toeplitz) quantization(s)

4.1. Berezin-Toeplitz quantization

For the Fock spaces $\mathcal{F}_\alpha$, $\alpha = \pi/h$, we have seen that the Toeplitz calculus assigning to a function f on $\mathbf{C}^n$ the Toeplitz operator T_f on $\mathcal{F}_\alpha$ yields a deformation quantization of $\mathbf{C}^n$. The main idea of *Berezin-Toeplitz* quantization is to use the Toeplitz operators in the same way also on a general domain Ω . Of course, what is unclear is the right substitute for the Gaussian measures $e^{-\pi|z|^2/h}$ on $\mathbf{C}^n$.

The main problem in the Berezin-Toeplitz quantization is thus to find a family of weights ρ_h , $h > 0$, on the domain Ω such that the corresponding Toeplitz operators on $L^2_{\text{hol}}(\Omega, \rho_h)$ satisfy

$$T_f T_g = \sum_{j=0}^{\infty} h^j T[C_j(f, g)] \quad (17)$$

in some sense, where C_j are some bidifferential operators such that $C_0(f, g) = fg$ and

$$C_1(f, g) - C_1(g, f) = \frac{i}{2\pi} \{f, g\}$$

for some given Poisson bracket $\{\cdot, \cdot\}$ on Ω .

Recall that for $\Omega = \mathbf{C}$ and $\rho_h(z) = e^{-\pi|z|^2/h} h^{-1} dz$, this was fulfilled with $C_j(f, g) = \frac{1}{j!} (\partial^j f)(\bar{\partial}^j g)$. (And similarly for $\mathbf{C}^n$.)

The operators $C_j \equiv C_j^{BT}$ then define a star-product

$$f *_{BT} g := \sum_{j=0}^{\infty} h^j C_j^{BT}(f, g), \quad f, g \in C^\infty(\Omega),$$

called *Berezin-Toeplitz star-product* (and denoted $*_{BT}$ to distinguish it from the various other star-products around).

4.2. Berezin quantization

This method is not based on Toeplitz operators, but rather on the Berezin symbols.

Consider, quite generally, any weight w on Ω of the kind discussed in §3.5. Since the Berezin symbol map $T \mapsto \tilde{T}$ is one-to-one, we can introduce a noncommutative product $*_w$ on (some) functions on Ω by

$$\tilde{S} *_w \tilde{T} := \widetilde{ST}.$$

The product $f *_w g$ is thus defined only for f, g in the set

$$\mathcal{A}_w := \{\tilde{T} : T \text{ is a bounded linear operator on } L^2_{\text{hol}}(\Omega, w)\}$$

(which also depends on w). The product $f *_w g$ then also belongs to $\mathcal{A}_w$, and $*_w$ is associative (since the multiplication of operators is).

The idea is to glue these non-commutative products $*_w$, as w is let to vary with the Planck constant h , into a star product.

More precisely, the *Berezin quantization* amounts to finding a family of weights ρ_h , $h > 0$, such that, first of all, the intersection

$$\mathcal{A} := \bigcap_{h>0} \mathcal{A}_{\rho_h}$$

is sufficiently large; and, second, that for $f, g \in \mathcal{A}$,

$$f *_h g = \sum_{j=0}^{\infty} h^j C_j(f, g)$$

asymptotically as $h \searrow 0$, where C_j are some bidifferential operators such that $C_0(f, g) = fg$ and

$$C_1(f, g) - C_1(g, f) = \frac{i}{2\pi} \{f, g\}$$

for a given Poisson bracket $\{\cdot, \cdot\}$ on Ω .

Here “sufficiently large” means, basically, that $\mathcal{A}$ should be so large that the bilinear operators $C_j(f, g)$ are uniquely determined by their values on $f, g \in \mathcal{A}$. Since C_j are differential operators in each argument, this will be the case, for instance, whenever for any point x , any finite set J of multiindices, and any set of complex numbers c_j , $j \in J$, we can find an element $f \in \mathcal{A}$ such that $\partial^j f(x) = c_j \forall j \in J$. In particular, it is enough if $\mathcal{A}$ contains all polynomials (in z and $\bar{z}$) on Ω .

The resulting bidifferential operators $C_j \equiv C_j^B$ then, of course, define the desired star-product

$$f *_B g := \sum_{j=0}^{\infty} h^j C_j^B(f, g), \quad f, g \in C^\infty(\Omega),$$

called the *Berezin star-product* (and denoted $*_B$ to distinguish it from the Berezin-Toeplitz star-product of §4.1).

So far, we have not exhibited any example of the Berezin quantization, even on $\mathbf{C}^n$. We will do that by showing that it is in fact related to another problem, which has a very familiar answer on $\mathbf{C}^n$.

4.3. Berezin quantization via the Berezin transform

In fact, the problem described in §4.2 can be reduced to one concerning the asymptotic behaviour of the weighted Berezin transforms B_w with the appropriate weights w . More precisely, the following holds.

Suppose we can find a family of weights ρ_h , $h > 0$, on Ω , such that as $h \rightarrow 0$, the corresponding weighted Berezin transforms $B_{\rho_h} \equiv B_h$ have an asymptotic expansion

$$B_h = Q_0 + hQ_1 + h^2Q_2 + \dots, \quad (18)$$

with some differential operators Q_j , where $Q_0 = I$. Let $c_{j\alpha\beta}$ be the coefficients of Q_j , i.e.

$$Q_j f =: \sum_{\alpha, \beta \text{ multiindices}} c_{j\alpha\beta} \partial^\alpha \bar{\partial}^\beta f;$$

and set

$$f *_B g := \sum_{j=0}^{\infty} h^j C_j(f, g),$$

where

$$C_j(f, g) \equiv C_j^{Bt}(f, g) := \sum_{\alpha, \beta} c_{j\alpha\beta} (\bar{\partial}^\beta f)(\partial^\alpha g). \quad (19)$$

If it happens that

$$C_1(f, g) - C_1(g, f) = \frac{i}{2\pi} \{f, g\},$$

then $*_{Bt}$ is a star product and

$$f *_B g = f *_B g \quad \forall f, g, \quad (20)$$

i.e. $*_{Bt}$ coincides with the Berezin star-product from §4.2.

The rest of this subsection is devoted to the proof of this assertion. Once this has been done, the construction of the Berezin quantization reduces to constructing a family of weights for which the associated Berezin transforms have the nice asymptotics (18); this will be done in Section 5. Furthermore, the assertion also yields immediately an easy example of a Berezin quantization on $\mathbf{C}^n$; this, as well as some other examples, will be presented in §4.6 below.

So let us prove (20). Suppose we have a family of weights ρ_h such that (18) holds. Denote, for brevity, by $Z_j = T_{z_j}$, $j = 1, \dots, n$, the Toeplitz operator on $L^2_{\text{hol}}(\Omega, \rho_h)$ whose symbol is the coordinate function z_j ; we have seen that Z_j are actually just the multiplication operators

$$Z_j : f(z) \mapsto z_j f(z).$$

Let Z_j^* be the adjoint of Z_j on $L^2_{\text{hol}}(\Omega, \rho_h)$. (Thus Z_j^* depends also on h , although it is not visible in the notation.) For $p(z, \bar{z}) = \sum_{\alpha, \beta} p_{\alpha\beta} z^\alpha \bar{z}^\beta$ a polynomial in z and $\bar{z}$, define the operators

$$V_p := \sum_{\alpha, \beta} p_{\alpha\beta} Z^\alpha Z^{*\beta}$$

on each $L^2_{\text{hol}}(\Omega, \rho_h)$, $h > 0$ (where we are using the obvious multiindex conventions $Z^\alpha = Z_1^{\alpha_1} \dots Z_n^{\alpha_n}$ etc.). Note that owing to the hypothesis that the domain Ω is bounded, Z_j and, hence, V_p are bounded linear operators.

Recall now our notation $K_y = K_{\rho_h}(\cdot, y)$ for the reproducing kernels, and the notation for the “two-variable Berezin symbol” of an operator T on $L^2_{\text{hol}}(\Omega, \rho_h)$,

$$\tilde{T}(x, y) := \frac{\langle TK_y, K_x \rangle}{\langle K_y, K_x \rangle} = \frac{TK_y(x)}{K(x, y)} = \frac{\overline{T^* K_x(y)}}{K(x, y)},$$

which is defined in some neighbourhood of the diagonal in $\Omega \times \Omega$ (where $K(x, y) \neq 0$) and whose restriction to the diagonal $x = y$ coincides with the Berezin symbol $\tilde{T}(x)$ of T . Applying this in particular to the operator V_p , we get

$$\begin{aligned} \tilde{V}_p(x, y) &= \frac{V_p K_y(x)}{K(x, y)} = \frac{\sum_{\alpha, \beta} p_{\alpha\beta} (Z^\alpha Z^{*\beta} K_y)(x)}{K(x, y)} \\ &= \frac{\sum_{\alpha, \beta} p_{\alpha\beta} x^\alpha (Z^{*\beta} K_y)(x)}{K(x, y)} = \frac{\sum_{\alpha, \beta} p_{\alpha\beta} x^\alpha \langle Z^{*\beta} K_y, K_x \rangle}{K(x, y)} \\ &= \frac{\sum_{\alpha, \beta} p_{\alpha\beta} x^\alpha \langle K_y, Z^\beta K_x \rangle}{K(x, y)} = \frac{\sum_{\alpha, \beta} p_{\alpha\beta} x^\alpha \overline{y^\beta K_x(y)}}{K(x, y)} \\ &= \sum_{\alpha, \beta} p_{\alpha\beta} x^\alpha \overline{y^\beta} = p(x, \bar{y}) \quad \text{for any } h. \end{aligned}$$

In particular, $\tilde{V}_p(x) = \tilde{V}_p(x, x) = p(x, \bar{x})$ for any h . Consequently, $p \in \mathcal{A}_{\rho_h}$ for all h , that is, $p \in \mathcal{A}$; thus $\mathcal{A}$ contains all polynomials, settling the first requirement for the Berezin quantization from §4.2.

Next, for any two operators T_1, T_2 on $L^2_{\text{hol}}(\Omega, \rho_h)$,

$$\begin{aligned} \widetilde{(T_1 T_2)}(x, y) &= \frac{\langle T_2 K_y, T_1^* K_x \rangle}{\langle K_y, K_x \rangle} = \frac{\int T_2 K_y(z) \overline{T_1^* K_x(z)} \rho_h(z) dz}{\langle K_y, K_x \rangle} \\ &= \int \frac{\tilde{T}_2(z, y) K_h(z, y) \cdot \tilde{T}_1(x, z) K_h(x, z)}{\langle K_y, K_x \rangle} \rho_h(z) dz. \end{aligned}$$

In particular,

$$\begin{aligned} \widetilde{(T_1 T_2)}(x, x) &= \int \tilde{T}_1(x, z) \tilde{T}_2(z, x) \frac{|K_h(x, z)|^2}{K_h(x, x)} \rho_h(x) dx \\ &= (B_h[\tilde{T}_1(x, \cdot) \tilde{T}_2(\cdot, x)])(x). \end{aligned}$$

Thus if (18) holds, i.e.

$$B_h = \sum_{j=0}^{\infty} h^j Q_j \quad \text{as } h \rightarrow 0,$$

with some differential operators $Q_j f = \sum_{\alpha, \beta} c_{j\alpha\beta} \partial^\alpha \bar{\partial}^\beta f$, and C_j are defined by $C_j(f, g) := \sum_{\alpha, \beta} c_{j\alpha\beta} (\bar{\partial}^\beta f)(\partial^\alpha g)$, then we get for $h \rightarrow 0$

$$\begin{aligned} \widetilde{(T_1 T_2)}(x, x) &= \sum_{j=0}^{\infty} h^j Q_j [\widetilde{T_1}(x, \cdot) \widetilde{T_2}(\cdot, x)](x) \\ &= \sum_{j, \alpha, \beta} h^j c_{j\alpha\beta} \bar{\partial}^\beta \widetilde{T_1}(x, \cdot) \partial^\alpha \widetilde{T_2}(\cdot, x) \big|_x. \end{aligned}$$

Now since $\widetilde{T}(x) = \widetilde{T}(x, x)$ and $\widetilde{T}(x, y)$ is holomorphic in x and anti-holomorphic in y , we have

$$\bar{\partial}^\beta \widetilde{T_1}(x, \cdot) \big|_x = \bar{\partial}^\beta \widetilde{T_1}(x)$$

(the $\widetilde{T}$ on the left-hand side is the $\widetilde{T}(x, y)$, and the $\widetilde{T}$ on the right-hand side is the $\widetilde{T}(x)$). Similarly,

$$\partial^\alpha \widetilde{T_2}(\cdot, x) \big|_x = \partial^\alpha \widetilde{T_2}(x).$$

Thus

$$\begin{aligned} \widetilde{T_1 T_2} &= \sum_{j, \alpha, \beta} h^j c_{j\alpha\beta} (\bar{\partial}^\beta \widetilde{T_1}) (\partial^\alpha \widetilde{T_2}) \\ &= \sum_j h^j C_j(\widetilde{T_1}, \widetilde{T_2}) = \widetilde{T_1} *_{Bt} \widetilde{T_2}, \end{aligned}$$

by the definition of $*_{Bt}$. On the other hand, $\widetilde{T_1 T_2} = \widetilde{T_1} *_{\rho_h} \widetilde{T_2}$, by the definition of $*_w$ (with $w = \rho_h$) in §4.2; so

$$\widetilde{T_1} *_{Bt} \widetilde{T_2} = \widetilde{T_1} *_{\rho_h} \widetilde{T_2}.$$

Applying this to $T_1 = V_p, T_2 = V_q$ with some polynomials p, q in $z, \bar{z}$, and recalling that $\widetilde{V_p} = p$, this means that

$$p *_{Bt} q = p *_{\rho_h} q$$

for any polynomials p, q in $z, \bar{z}$. Since any $f \in C^\infty(\Omega)$ can be approximated, at any given point, to any finite order by polynomials, and the $C_j(\cdot, \cdot)$ for both $*_{Bt}$ and $*_B$ are differential operators in each argument, necessarily $C_j^{Bt}(f, g)(x) = C_j^B(f, g)(x)$ for all $f, g \in C^\infty(\Omega)$ and $x \in \Omega$; that is, $*_{Bt} = *_B$, completing our proof.

4.4. Berezin-Toeplitz quantization via the Berezin transform

On a slightly more heuristic level, it is possible to derive not only the Berezin, but also the Berezin-Toeplitz quantization (§4.1) from the asymptotics (18) of the Berezin transform; that is, to show that if (18) holds, then

$$[T_f, T_g] \approx h T_{\{f, g\}} \tag{21}$$

as the Planck constant $h \searrow 0$. While this will not be directly needed anywhere in the sequel, we believe it is worth mentioning here.

Assume first that $f, \bar{g}$ are holomorphic. Then for any $\phi \in L_{\text{hol}}^2$

$$\langle T_f \phi, K_x \rangle = \langle f \phi, K_x \rangle = f(x) \phi(x) = f(x) \langle \phi, K_x \rangle.$$

It follows that $T_f^* K_x = \overline{f(x)} K_x$. Similarly $T_g K_x = g(x) K_x$. Hence

$$\begin{aligned} \widetilde{T_f T_g}(x) &= \frac{\langle T_f T_g K_x, K_x \rangle}{\langle K_x, K_x \rangle} = \frac{\langle T_g K_x, T_f^* K_x \rangle}{\langle K_x, K_x \rangle} \\ &= \frac{\langle g(x) K_x, \overline{f(x)} K_x \rangle}{\langle K_x, K_x \rangle} = f(x)g(x); \end{aligned}$$

that is, $\widetilde{T_f T_g} = fg$.

On the other hand, by definition the Berezin transform and (18),

$$\widetilde{T}_{fg} = B_h(fg) = fg + hQ_1(fg) + O(h^2).$$

Subtracting this from $\widetilde{T_f T_g} = fg$ gives

$$\begin{aligned} (T_f T_g - T_{fg})^\sim &= -hQ_1(fg) + O(h^2) \\ &= -h\widetilde{T_{Q_1(fg)}} + O(h^2). \end{aligned}$$

“Removing the tilde” (yes, this is the heuristic part) we get, for $f, \bar{g}$ holomorphic,

$$T_f T_g - T_{fg} = -hT_{C_1^B(g, f)} + O(h^2), \quad (22)$$

where C_1^B is the C_1 from the Berezin quantization. Note that, as we have seen in §4.3, $C_1^B(g, f)$ involves only holomorphic derivatives of f and anti-holomorphic derivatives of g (i.e. only $\partial^\alpha f$ and $\bar{\partial}^\beta g$). This also means, in particular, that for any holomorphic functions u, v ,

$$C_1^B(ug, \bar{v}f) = uC_1^B(g, f)\bar{v}.$$

On the other hand, we have seen in §3.3 that for u, v as above and arbitrary F and G ,

$$T_G T_u = T_{uG}, \quad T_{\bar{v}} T_F = T_{\bar{v}F}.$$

Multiplying (22) by $T_{\bar{v}}$ from the left and T_u from the right, we therefore obtain

$$\begin{aligned} T_{\bar{v}f} T_{gu} - T_{\bar{v}fg} u &= T_{\bar{v}} [T_f T_g - T_{fg}] T_u \\ &= -hT_{\bar{v}} T_{C_1^B(g, f)} T_u + O(h^2) \\ &= -hT_{\bar{v}C_1^B(g, f)u} + O(h^2) \\ &= -hT_{C_1^B(ug, \bar{v}f)} + O(h^2). \end{aligned}$$

That is, (22) holds not only for $f, \bar{g}$ holomorphic, but for any f, g of the form $u\bar{v}$ with holomorphic u, v . By the same approximation argument as in the end of §4.3, we conclude that actually

$$T_f T_g - T_{fg} = -hT_{C_1^B(g, f)} + O(h^2)$$

for any $f, g \in C^\infty(\Omega)$. That is, we have obtained the first two terms

$$T_f T_g = T_{C_0^{BT}(f, g)} + hT_{C_1^{BT}(f, g)} + O(h^2)$$

of the Berezin-Toeplitz star-product (17), showing, incidentally, that $(C_0^{BT}(f, g) = fg$ and)

$$C_1^{BT}(f, g) = -C_1^B(g, f). \quad (23)$$

It is clear how to continue this argument to obtain also the higher-order terms C_j^{BT} and, hence, the entire Berezin-Toeplitz star-product.

4.5. Connection between Berezin and Toeplitz quantizations

The relationship (23) between the Berezin and the Berezin-Toeplitz operator C_1 can actually be put into a rather neat form. Recall that we have our three mappings $f \mapsto T_f$ (the Toeplitz operators), $T \mapsto \widetilde{T}$ (the Berezin symbol), and their composition $f \mapsto \widetilde{T}_f = B_h f$ (the Berezin transform). In terms of these, the Berezin-Toeplitz star product was defined by

$$T_f T_g = T_{f *_{BT} g}, \quad (24)$$

while the Berezin star product was, essentially, defined by

$$\widetilde{T} *_B \widetilde{S} = \widetilde{TS}.$$

Applying the last formula to $T = T_f$, $S = T_g$, and using (24), gives

$$\widetilde{T}_f *_B \widetilde{T}_g = \widetilde{T_f T_g} = \widetilde{T_{f *_{BT} g}},$$

or

$$Bf *_B Bg = B(f *_{BT} g).$$

In other words, the Berezin and the Berezin-Toeplitz star-products are intertwined (conjugate) by the Berezin transform. From this, one easily gets also the higher-order analogues of the relation (23), i.e. involving C_j^B and C_j^{BT} (and the operators Q_j) for $j \geq 1$.

4.6. Some examples of Berezin and Berezin-Toeplitz quantizations

We have already worked out the Berezin-Toeplitz quantization on $\mathbf{C}^n$ in some detail in Section 2¹; let us see how the other approaches discussed in this section work out in this case.

Thus, let $\Omega = \mathbf{C}^n$ and $\rho_h(z) = e^{-\alpha|z|^2} (\alpha/\pi)^n dz$, with $\alpha = \pi/h > 0$; note that the “classical limit” $h \searrow 0$ now corresponds to $\alpha \rightarrow +\infty$. Since we know the reproducing kernel to be given by $K_\alpha(x, y) = e^{\alpha\langle x, y \rangle}$, the formula for the Berezin transform becomes

$$\begin{aligned} B_\alpha f(x) &= \int_{\mathbf{C}^n} f(y) \frac{|K_h(x, y)|^2}{K_h(x, x)} \rho_h(y) dy \\ &= \left(\frac{\alpha}{\pi}\right)^n \int_{\mathbf{C}^n} f(y) e^{-\alpha\|x-y\|^2} dy. \end{aligned}$$

¹Strictly speaking, the Berezin-Toeplitz quantization as defined in §4.1 does not apply to $\mathbf{C}^n$, since our domain Ω throughout this whole section is assumed to be bounded (in order to have nontrivial bounded holomorphic functions on Ω , such as the polynomials); however, it is still illustrative to include also the case of $\Omega = \mathbf{C}^n$ here, albeit with the caveats about dealing with unbounded operators like T_z etc. in general.

This is precisely the heat solution operator at the time $t = 1/4\alpha$:

$$B_\alpha f = e^{\Delta/4\alpha} f.$$

In particular, as $\alpha \rightarrow +\infty$, we get $B_\alpha f \rightarrow f$, more precisely there is even an asymptotic expansion

$$B_\alpha f(x) = e^{\Delta/4\alpha} f(x) = f(x) + \frac{\Delta f(x)}{4\alpha} + \frac{\Delta^2 f(x)}{2!(4\alpha)^2} + \dots,$$

or more briefly

$$B_\alpha = e^{\Delta/4\alpha} = \sum_{j=0}^{\infty} \alpha^{-j} \frac{\Delta^j}{j!4^j}.$$

From §4.3, we conclude that the Berezin quantization works for the above choice of weights ρ_h on $\mathbf{C}^n$, with

$$C_j(f, g) = C_j^B(f, g) := \frac{1}{j!} \sum_{|\alpha|=j} (\bar{\partial}^\alpha f)(\partial^\alpha g).$$

This can be compared with the Berezin-Toeplitz quantization formula for the same choice of weights from Section 2:

$$C_j(f, g) = C_j^{BT}(f, g) := \frac{(-1)^j}{j!} \sum_{|\alpha|=j} (\partial^\alpha f)(\bar{\partial}^\alpha g).$$

Both quantize the Euclidean Poisson bracket on $\mathbf{C}^n$ (spelled out in the axiom (A3) in Section 1).

The second example which can be worked out explicitly to some level is the unit disc $\Omega = \mathbf{D} := \{z \in \mathbf{C} : |z| < 1\}$ in $\mathbf{C}$, with weights $\rho_h(z) = \frac{\alpha+1}{\pi}(1-|z|^2)^\alpha$, $\alpha > -1$; the parameter α again plays the role of the reciprocal of h , so that $h \searrow 0$ corresponds to $\alpha \rightarrow +\infty$. A standard calculation in polar coordinates, similar to the one we did for the Fock space, shows that the reproducing kernels are

$$K_\alpha(x, y) = \frac{1}{(1-x\bar{y})^{\alpha+2}}.$$

This gives the formula for the Berezin transform:

$$B_\alpha f(x) = \frac{\alpha+1}{\pi} \int_{\mathbf{D}} f(y) \frac{(1-|x|^2)^{\alpha+2}}{|1-x\bar{y}|^{2\alpha+4}} (1-|y|^2)^\alpha dy.$$

With some work, it can again be shown that as $\alpha \rightarrow +\infty$,

$$B_\alpha f = f + \frac{\tilde{\Delta} f}{4\alpha} + \dots$$

where

$$\tilde{\Delta} f = (1-|z|^2)^2 \Delta$$

is the invariant Laplacian on $\mathbf{D}$. (The Q_j for $j > 1$ are already a bit complicated and involve Bernoulli numbers; an explicit expression for general j is not known.)

The results of §4.3 thus again tell us that the Berezin quantization on $\mathbf{D}$ works for the above choice of weights, with

$$C_0^B(f, g) = fg, \quad C_1^B(f, g) = (1 - |z|^2) \bar{\partial} f \partial g.$$

Similarly, the Berezin-Toeplitz quantization works, with

$$C_0^{BT}(f, g) = fg, \quad C_1^{BT}(f, g) = -(1 - |z|^2) \partial f \bar{\partial} g.$$

Explicit expressions for C_j^B and C_j^{BT} for general $j \geq 2$ are again unknown.

Both methods quantize the Poisson bracket

$$\{f, g\} = (1 - |z|^2)^2 (\bar{\partial} f \partial g - \partial g \bar{\partial} f)$$

associated to the invariant (=Poincaré, Lobachevsky) metric on $\mathbf{D}$.

Our third and final example concerns the unit ball $\Omega = \mathbf{B}^n := \{z \in \mathbf{C}^n : |z| < 1\}$ in $\mathbf{C}^n$, with weights $\rho_h(z) = c_\alpha (1 - |z|^2)^\alpha$, where $\alpha = 1/h \rightarrow +\infty$ and c_α is a normalizing constant making ρ_h to be of total mass 1. The reproducing kernel equals

$$K_\alpha(x, y) = \frac{1}{(1 - \langle x, y \rangle)^{\alpha+n+1}},$$

yielding the expression for the Berezin transform

$$B_\alpha f(x) = c_\alpha \int_{\mathbf{B}^n} f(y) \frac{(1 - |x|^2)^{\alpha+n+1}}{|1 - \langle x, y \rangle|^{2\alpha+2n+2}} (1 - |y|^2)^\alpha dy.$$

Again,

$$B_\alpha f = f + \frac{\tilde{\Delta} f}{4\alpha} + \dots$$

as $\alpha \rightarrow +\infty$, with $\tilde{\Delta}$ the invariant Laplacian on $\mathbf{B}^n$. Both the Berezin and the Berezin-Toeplitz quantizations work for the above choice of weights, and their coefficients C_j are given by formulas of a similar nature as for the disc.

For a later occasion, it is instructive to summarize some observations from these examples here. Looking at the weights and the corresponding reproducing kernels in the three cases, namely,

$$\rho_\alpha(z) = \left(\frac{\alpha}{\pi}\right)^n e^{-\alpha|z|^2}, \quad K_\alpha(x, y) = e^{\alpha\langle x, y \rangle}$$

for the Fock space on $\mathbf{C}^n$;

$$\rho_\alpha(z) = \frac{\alpha+1}{\pi} (1 - |z|^2)^\alpha, \quad K_\alpha(x, y) = (1 - x\bar{y})^{-\alpha-2}$$

for the disc; and

$$\rho_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha, \quad K_\alpha(x, y) = (1 - \langle x, y \rangle)^{-\alpha-n-1}$$

for the ball, we observe that $K_\alpha(x, x)$ is just the reciprocal of the weight $\rho_h(x)$, up to the normalization constants and possibly a shift in the exponent α .

Furthermore, we have seen in all three cases that the Berezin transform B_α is an approximate identity as $\alpha \rightarrow +\infty$, more precisely

$$B_\alpha = I + \frac{Q_1}{\alpha} + \frac{Q_2}{\alpha^2} + \dots,$$

where Q_1 is, up to a constant factor, some kind of “invariant Laplacian” on the domain in question.

We will see in Section 5 later that both these observation, in fact, remain in force in a much more general setting.

4.7. How to choose the weights ρ_h

The main problem for carrying out both the Berezin and the Berezin-Toeplitz quantization is thus to find the weights ρ_h , $h > 0$, on Ω so that (17) and (18) hold. There is a way to see what should be the right choice, which we now describe.

It is time we gave a precise definition of the object we wish to quantize, the *Poisson bracket* on our domain (or manifold) Ω . Quite generally, a *symplectic manifold* is a real manifold equipped with a 2-form

$$\omega = \sum_{j,k=1}^m g_{jk} dx_j \wedge dx_k$$

which is non-degenerate (i.e. the matrix $\{g_{jk}\}_{j,k=1}^m$ is invertible) and closed ($d\omega = 0$). Here m is the real dimension of the manifold, which must necessarily be even. The Poisson bracket is then defined as

$$\{f, g\} = \sum_{j,k=1}^m g^{jk} \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial x_k}$$

where $\{g^{jk}\}_{j,k=1}^m$ is the inverse matrix to $\{g_{jk}\}_{j,k=1}^m$. For the case of complex manifolds that we have here, it is furthermore important that the symplectic form be compatible with the complex structure, and also it is more convenient to use the complex coordinates $z_j, \bar{z}_j$, $j = 1, \dots, n$, rather than the real coordinates x_k , $k = 1, \dots, m$, $m = 2n$. On the level of the form ω , this translates into the fact that ω is *Kähler*, meaning that (in local coordinates)

$$\omega = \sum_{j,k=1}^n g_{j\bar{k}} dz_j \wedge d\bar{z}_k$$

with some positive-definite matrix $\{g_{j\bar{k}}\}_{j,k=1}^n$ satisfying

$$\partial_l g_{j\bar{k}} = \partial_j g_{l\bar{k}}, \quad \partial_{\bar{l}} g_{j\bar{k}} = \partial_{\bar{k}} g_{j\bar{l}}. \quad (25)$$

The Poisson bracket is then given by

$$\{f, g\} = \sum_{j,k=1}^n g^{\bar{j}k} (\bar{\partial}_j f \partial_k g - \partial_j f \bar{\partial}_k g), \quad (26)$$

where $\{g^{\bar{j}k}\}_{j,k=1}^n$ is the inverse matrix to $\{g_{j\bar{k}}\}$. Finally, the 2-form ω determines (both in the symplectic and in the Kähler case) also a nonvanishing volume element ω^n on Ω .

To find the right choice of the weights ρ_h , we take guidance from group invariance.

Assume there is a group G acting on Ω by biholomorphic transformations preserving the form ω . Naturally, we would then want our quantizations to be G -invariant, i.e. to satisfy

$$(f \circ \phi) * (g \circ \phi) = (f * g) \circ \phi, \quad \forall \phi \in G.$$

On the level of the Berezin quantization, this means that the operators Q_j in (18), and, hence, B itself, should commute with the action of G . An examination of the formula defining the Berezin transform with respect to some weight ρ shows that this happens if and only if

$$\frac{|K_\rho(x, y)|^2}{K_\rho(y, y)} \rho(x) dx = \frac{|K_\rho(\phi(x), \phi(y))|^2}{K_\rho(\phi(y), \phi(y))} \rho(\phi(x)) d\phi(x).$$

In particular, the ratio

$$\frac{\rho(\phi(x)) d\phi(x)}{\rho(x) dx} = \frac{|K_\rho(x, y)|^2}{K_\rho(y, y)} \frac{K_\rho(\phi(y), \phi(y))}{|K_\rho(\phi(x), \phi(y))|^2}$$

has to be the squared modulus of a holomorphic function. Writing

$$\rho(z) dz = w(z) \cdot \omega^n(z) \tag{27}$$

with the (G -invariant) volume element ω^n and some (positive) weight function w , the last condition translates into

$$w(\phi(z)) = w(z) |f_\phi(z)|^2$$

for some holomorphic functions f_ϕ . In other words, the form $\partial\bar{\partial} \log w$ is G -invariant.

But the simplest examples of G -invariant 2-forms (and if G is sufficiently “ample”, the only ones) are clearly the constant multiples of ω . Thus we are led to

$$\partial\bar{\partial} \log w = -c\omega$$

with some constant c . It follows that

$$\omega = \partial\bar{\partial}\Phi, \quad \Phi := -\frac{1}{c} \log w,$$

i.e. that $\Phi = -\frac{1}{c} \log w$ is a real-valued *Kähler potential* for ω . This gives for the volume element

$$\omega^n(z) = \det[\partial\bar{\partial}\Phi(z)] dz,$$

and (27) gives

$$\rho(z) = e^{-c\Phi(z)} \det[\partial\bar{\partial}\Phi(z)].$$

Returning the Planck constant dependence into play, we therefore see that the sought weights ρ_h should be of the form

$$\rho_h = e^{-c\Phi} \det[\partial\bar{\partial}\Phi],$$

with some $c = c(h)$ depending only on h .

Note that the condition $\omega = \partial\bar{\partial}\Phi$ means that

$$g_{j\bar{k}}(z) = \frac{\partial^2 \Phi(z)}{\partial z_j \partial \bar{z}_k}.$$

The fact that this matrix is positive-definite, for each $z \in \Omega$, means precisely that the potential Φ is *strictly plurisubharmonic* on Ω . We will usually abbreviate “strictly plurisubharmonic” to “strictly PSH”.

Finally, the condition

$$C_1(f, g) - C_1(g, f) = -\frac{i}{2\pi} \{f, g\} \quad (28)$$

in the Berezin quantization will be satisfied if the operator Q_1 in (18) equals

$$Q_1 = \sum_{j,k=1}^n g^{\bar{j}k} \partial_k \bar{\partial}_j =: \Delta,$$

the *Laplace-Beltrami* operator associated to ω . Indeed, in that case we have by (19)

$$C_1(f, g) = \sum_{j,k=1}^n g^{\bar{j}k} (\bar{\partial}_j f)(\partial_k g),$$

and (28) follows by (26).

We have thus arrived at a final recipe for the Berezin and Berezin-Toeplitz quantizations on a domain $\Omega \subset \mathbf{C}^n$ equipped with a Kähler form ω and the corresponding Poisson bracket. Namely:

1. There must exist a Kähler potential Φ for ω , i.e. a strictly PSH function Φ such that $\omega = \partial\bar{\partial}\Phi$.
2. We take the Bergman spaces $L_{\text{hol}}^2(\Omega, e^{-c\Phi} \det[\partial\bar{\partial}\Phi])$, where $c \in \mathbf{R}$ is a parameter. Denote by $K_c(x, y)$ the reproducing kernel of this space, by B_c the associated Berezin transform, and by $T_f^{(c)}$ the Toeplitz operator on this space with symbol f .
3. See if $c = c(h)$ can be chosen so that

$$B_c = I + h\Delta + h^2 Q_2 + h^3 Q_3 + \dots \quad \text{as } h \rightarrow 0$$

with some differential operators Q_j , $Q_0 = I$, $Q_1 = \Delta$ (for the Berezin quantization); and

$$T_f^{(c)} T_g^{(c)} = \sum_{j=0}^{\infty} h^j T_{C_j(f,g)}^{(c)} \quad \text{as } h \searrow 0$$

in some sense, with $C_0(f, g) = fg$ and $C_1(f, g) - C_1(g, f) = -\frac{i}{2\pi} \{f, g\}$ (for the Berezin-Toeplitz quantization).

It turns out that under suitable hypothesis on Ω and Φ , this recipe indeed works, with

$$c(h) = 1/h.$$

For brevity, let us denote by $d\mu_h$ the corresponding measures

$$d\mu_h(z) := e^{-\Phi(z)/h} \det[g_{k\bar{j}}(z)] dz, \quad h > 0,$$

and by $L_{\text{hol},h}^2 = L_{\text{hol}}^2(\Omega, d\mu_h)$ the associated weighted Bergman spaces; also K_c, B_c and $T_f^{(c)}$ will be written as K_h, B_h and T_f , respectively. We will also sometimes use our earlier notation $\alpha = 1/h$ for $\frac{1}{h}$ rather than c .

For simplicity, we have so far really discussed only the situation when Ω is a domain in $\mathbf{C}^n$. It turns out that the whole formalism works also on arbitrary Kähler manifolds, just with some minor technical adjustments. The most conspicuous of them is that instead of considering Bergman spaces of functions on Ω , one needs to consider, more generally, spaces of sections of a holomorphic line bundle $\mathcal{L}$, equipped with a Hermitian metric (in the fibers) given locally by $e^{-\Phi}$ (more precisely: the curvature form of this Hermitian metric should coincide with the given Kähler form ω). For such $\mathcal{L}$ to exist, it is necessary that the cohomology class of ω be integral. The role of the weighted Bergman spaces $L_{\text{hol}}^2(\Omega, d\mu_h)$ is then played by the spaces of holomorphic L^2 sections of the tensor powers $\mathcal{L}^{\otimes m}$, $m = 1, 2, \dots$; in particular, the Planck constant can approach 0 only through a discrete set of values. However, the whole formalism — weighted Bergman kernels, Berezin symbols, Toeplitz operators, and Berezin transforms — still makes perfect sense, and so does the above recipe for Berezin and Berezin-Toeplitz quantizations.

Since both B_h and T_f are defined by formulas involving the weighted Bergman kernels K_h , the key to proving the viability of our recipe is obviously an understanding of the behaviour of $K_h(x, y)$ as $h \searrow 0$. Historically, there are two approaches how to handle this problem, which both appeared independently around 1997–1998. The first one was developed in the context of compact manifolds by Zelditch [44], who gave, in our language, the asymptotics of the reproducing kernels $K_h(x, x)$ on the diagonal as $h \rightarrow 0$; this was subsequently extended also away from the diagonal by Catlin [13]. These two papers did not consider B_h and T_f , but rather were inspired by certain geometric applications going back to Tian in 1990 [43] (with a follow-up by Ruan [39]). The proofs rely on a theory, due to Boutet de Monvel and Guillemin [11], of Fourier integral operators of Hermite type, which was in exactly the same way used, in fact, already in 1994 by Bordemann, Meinrenken and Schlichenmaier [9] to establish the result about T_f on compact manifolds directly without those for K_h and B_h (thus bypassing the Berezin quantization).

The second approach, due to the present author, dealt with domains in $\mathbf{C}^n$ not manifolds, and relied on somewhat simpler methods (Fefferman's expansion and $\bar{\partial}$ -techniques) to obtain the asymptotics on K_h and B_h [18] [19] [20]; naturally, some hypothesis on the behaviour of Φ at the boundary were needed. The result for T_f can, however, be established in this case only for bounded domains, and one still has to resort to the more sophisticated machinery used by Bordemann, Meinrenken and Schlichenmaier [9].

Prior to these general results, Berezin and Berezin-Toeplitz quantizations had been established only ad hoc in some special cases, such as in dimension $n = 1$ (i.e. for Riemann surfaces) with the Poincaré metric by Klimek and Lesniewski

in 1991 [32] (using uniformization), for $\Omega = \mathbf{C}^n$ with the Euclidean metric by Coburn in 1993 [14], or for bounded symmetric domains with the invariant metric by Borthwick, Lesniewski and Upmeyer in 1994 [10]. The basic idea, in any case, goes back — as the terminology rightly suggests — to Berezin in 1975 [6]. The equivalence of the Berezin quantization and the asymptotic expansion of the Berezin transform is due to Karabegov [31]. Some recent extensions and generalizations of the theory are discussed e.g. in the book [37] by Ma and Marinescu, or the paper [7] by Berndtsson, Berman and Sjöstrand.

In the rest of this paper, we will first handle the case of the Berezin quantization by the second of the above-mentioned approaches. Then we proceed to deal with the Berezin-Toeplitz quantization via the first approach, adapted to the context — to which we have also restricted ourselves hitherto in this paper — of domains in $\mathbf{C}^n$ rather than compact manifolds.

5. Berezin quantization

5.1. Basics notions of several complex variables

Recall that a smooth function $\Phi : \Omega \rightarrow \mathbf{R}$ on a domain Ω in $\mathbf{C}^n$ is called *strictly-plurisubharmonic* (strictly-PSH) if for any $z \in \Omega$ and $v \in \mathbf{C}^n$, the function of one complex variable

$$t \mapsto \Phi(z + tv), \quad t \in \mathbf{C}$$

is strictly subharmonic where defined. Equivalently, Φ is strictly-PSH if the matrix of mixed second derivatives

$$\left[\frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_k} \right]_{j,k=1}^n$$

is positive definite.

A bounded domain $\Omega \subset \mathbf{C}^n$ with smooth boundary is called *strictly pseudoconvex* if there exists a smooth function r such that

$$r > 0 \quad \text{on } \Omega, \quad r = 0, \quad \|\nabla r\| > 0 \quad \text{on } \partial\Omega,$$

$$-r \quad \text{is strictly-PSH in a neighbourhood of } \bar{\Omega}.$$

One calls r a strictly-PSH *defining function* for Ω .

For completeness (it will not be needed in the sequel), we remark that there are also (not necessarily strictly) plurisubharmonic (PSH) functions, for which $t \mapsto \Phi(z + tv)$ is assumed to be only subharmonic (not necessarily strictly), or, equivalently, the matrix of mixed second-order derivatives is only positive semi-definite; and (not necessarily strictly) pseudoconvex domains, which can be defined as increasing unions of strictly pseudoconvex domains. (This is *not* the same thing as having a — not necessarily strictly — PSH defining function.)

Pseudoconvex domains are the natural domains in $\mathbf{C}^n$ on which holomorphic functions live: if Ω is not pseudoconvex, then there exist a larger domain Ω' such that every holomorphic function on Ω in fact extends holomorphically to Ω' . An example of non-pseudoconvex domain is the domain $\Omega = \{z \in \mathbf{C}^n : 1 < |z| < 2\}$,

$n > 1$, for which $\Omega' = \{z \in \mathbf{C}^n : |z| < 2\}$. In dimension $n = 1$, as we all know from basic complex analysis, *all* domains are pseudoconvex.

Strictly pseudoconvex domains are those whose boundary is, additionally, in some sense “non-degenerate”, which makes it possible to establish results which have as yet no known counterparts in the non-strictly pseudoconvex case. We will come across some of these results later in this section.

The upshot of all the above is that pseudoconvex domains are the ones on which it makes sense to study holomorphic functions; strictly pseudoconvex domains are the manageable ones.

5.2. Main theorem on Berezin quantization

Theorem B. *Let $\Omega \subset \mathbf{C}^n$ be smoothly bounded and strictly pseudoconvex, and Φ a strictly-PSH function on Ω such that $e^{-\Phi} = r$ is a defining function for Ω .*

Then for the weights $w = e^{-\alpha\Phi} \det[\partial\bar{\partial}\Phi]$, we have as $\alpha \rightarrow +\infty$, $\alpha \in \mathbf{Z}$,

$$K_\alpha(x, x) \approx e^{\alpha\Phi(x)} \frac{\alpha^n}{\pi^n} \sum_{j=0}^{\infty} \frac{b_j(x)}{\alpha^j},$$

with some functions $b_j \in C^\infty(\Omega)$, $b_0 = \det[\partial\bar{\partial}\Phi]$; and

$$B_\alpha f = \sum_{j=0}^{\infty} \frac{Q_j f}{\alpha^j}$$

where Q_j are some differential operators, in particular $Q_0 = I$ and

$$Q_1 = \sum_{j,k=1}^n g^{\bar{j}k} \frac{\partial^2}{\partial z_k \partial \bar{z}_j} =: \Delta,$$

the Laplace-Beltrami operator. Here $g^{\bar{j}k}$ is the inverse matrix to $g_{j\bar{k}} := \frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_k}$.

It follows, as explained in §4.3, that denoting by $c_{j\alpha\beta}$ the coefficients of the operators Q_j ,

$$Q_j f = \sum_{\alpha, \beta \text{ multiindices}} c_{j\alpha\beta} \partial^\alpha \bar{\partial}^\beta f,$$

and setting

$$f *_{Bt} g := \sum_{j=0}^{\infty} h^j C_j(f, g),$$

where

$$C_j(f, g) := \sum_{\alpha, \beta} c_{j\alpha\beta} (\bar{\partial}^\beta f)(\partial^\alpha g),$$

we obtain a Berezin quantization on the domain Ω with the Poisson bracket associated to the Kähler form $\omega = \partial\bar{\partial}\Phi$.

It is instructive to see how Theorem B applies in the examples from §4.6. For the unit ball $\Omega = \mathbf{B}^n$ (which includes $\Omega = \mathbf{D}$ for $n = 1$), take

$$\Phi(z) = \log \frac{1}{1 - |z|^2},$$

which is a Kähler potential for the invariant metric on $\mathbf{B}^n$. Then Φ is strictly-PSH,

$$e^{-\Phi(z)} = 1 - |z|^2$$

is a strictly-PSH defining function for $\mathbf{B}^n$, and

$$b_0(z) = \det\left[\frac{\partial^2 \Phi}{\partial z_j \partial \bar{z}_k}\right] = \frac{1}{(1 - |z|^2)^{n+1}}.$$

We thus recover the formulas from §4.6 (b_0 explains the “shift in the exponent α ”). Also, we see that $c_\alpha \sim \alpha^n$.

For the Fock space on $\Omega = \mathbf{C}^n$, a Kähler potential for the Euclidean metric is $\Phi(z) = |z|^2$. In that case $b_0(z) = \det[\delta_{jk}] = 1$, so there is no “shift” this time, and Theorem B again recovers the asymptotics of K_α and B_α on the Fock space from Section 2 and §4.6.

We need to review a few prerequisites before giving a proof of the theorem.

5.3. Hartogs domains

For a domain $\Omega \subset \mathbf{C}^n$ and a real-valued smooth function ϕ on it, the *Hartogs domain* with base Ω and radius-function $e^{-\phi}$ is

$$\tilde{\Omega} := \{(z, t) \in \Omega \times \mathbf{C} : |t|^2 < e^{-\phi(z)}\}.$$

It can be shown that $\tilde{\Omega}$ is pseudoconvex if and only if Ω is pseudoconvex and ϕ is PSH; and that $\tilde{\Omega}$ is strictly pseudoconvex and smoothly bounded if Ω is strictly-pseudoconvex, ϕ is strictly-PSH and $e^{-\phi} = r$ is a defining function for Ω . Furthermore,

$$\tilde{r}(z, t) := r(z) - |t|^2 = e^{-\phi(z)} - |t|^2 \quad (29)$$

is a defining function for $\tilde{\Omega}$.

Thus the hypotheses of Theorem B guarantee precisely that taking for ϕ the Kähler potential Φ , the corresponding Hartogs domain $\tilde{\Omega}$ over Ω will be smoothly bounded and strictly pseudoconvex, with a defining function given by (29).

5.4. Hardy space

Continuing with the notations from the preceding paragraph, consider the compact manifold

$$X := \partial \tilde{\Omega}$$

equipped with the measure

$$d\sigma := \frac{J[\tilde{r}]}{\|\partial \tilde{r}\|} dS, \quad (30)$$

where dS stands for the surface measure on X and $J[\tilde{r}]$ for the *Monge-Ampère determinant*

$$J[\tilde{r}] = -\det \begin{bmatrix} \tilde{r} & \bar{\partial}\tilde{r} \\ \partial\tilde{r} & \partial\bar{\partial}\tilde{r} \end{bmatrix} > 0.$$

Let $H^2(X) = H^2$ be the subspace in $L^2(X, d\sigma)$ of functions whose Poisson extension into $\tilde{\Omega}$ is holomorphic. (Alternatively, $H^2(X)$ is the closure in $L^2(X, d\sigma)$ of functions continuous on the closure $\tilde{\bar{\Omega}}$ of $\tilde{\Omega}$ and holomorphic in its interior.)

One calls $H^2(X)$ the *Hardy space* on X .

We remark that the measure (30) — which at first sight may look a bit artificial — is actually a familiar object in differential geometry. Namely, the restriction ν of the differential form $\text{Im } \partial\tilde{r} = \frac{1}{2i}(\partial\tilde{r} - \bar{\partial}\tilde{r})$ to X is a *contact form* on X , meaning that $\nu \wedge (\partial\bar{\partial}\nu)^n$ is a non-vanishing volume element on X . Up to a constant factor, this volume element is precisely (30).

5.5. Szegő kernel

For each $(z, t) \in \tilde{\Omega}$, the evaluation functional $f \mapsto f(z, t)$ on H^2 turns out to be continuous, hence is given by the scalar product with a certain element $k_{(z,t)} \in H^2$. The function

$$K_{\text{Szegő}}((x, t), (y, s)) := \langle k_{(y,s)}, k_{(x,t)} \rangle_{H^2}$$

on $\tilde{\Omega} \times \tilde{\Omega}$ is called the *Szegő kernel*.

In other words, $K_{\text{Szegő}}$ is the reproducing kernel of the Hardy space $H^2(X)$, viewed as a space of holomorphic functions on $\tilde{\Omega}$ (rather than just their boundary values on X).

There is a simple relationship between the Hardy space $H^2(X)$ and the weighted Bergman spaces $L^2_{\text{hol},h}$ on the base Ω , as well as between the Szegő kernel $K_{\text{Szegő}}$ and the weighted Bergman kernels of $L^2_{\text{hol},h}$, which we now explain.

5.6. Ligocka's formula

The boundary X of $\tilde{\Omega}$ can be parameterized as

$$X = \{(z, e^{i\theta} e^{-\phi(z)/2}) : z \in \Omega, \theta \in [0, 2\pi]\}.$$

In these coordinates, and recalling our notations $r(z) = e^{-\phi(z)}$, $\tilde{r}(z, t) = r(z) - |t|^2$, easy computations show that

$$\begin{aligned} dS &= \sqrt{r + \|\partial r\|^2} dz d\theta, \quad \|\partial\tilde{r}\| = \sqrt{r + \|\partial r\|^2}, \\ J[\tilde{r}] &= J[r] = e^{-(n+1)\phi} \det[\partial\bar{\partial}\phi], \end{aligned} \tag{31}$$

so

$$d\sigma(z, t) = e^{-(n+1)\phi} \det[\partial\bar{\partial}\phi] dz d\theta. \tag{32}$$

Consider now a holomorphic function f on $\tilde{\Omega}$. Taking Taylor expansion in the fiber variable, we can write

$$f(z, t) = \sum_{j=0}^{\infty} f_j(z) t^j, \quad (z, t) \in \tilde{\Omega},$$

with f_j holomorphic on Ω . Expressing t in polar coordinates, one also sees immediately that

$$f(z) t^j \perp g(z) t^k \quad \forall f, g \text{ if } k \neq j$$

(orthogonality is meant in H^2). For the norm of f in $H^2(X)$, we thus get, using (32),

$$\begin{aligned} \int_X |f(z, t)|^2 d\sigma(z, t) &= \sum_{j=0}^{\infty} \int_{\Omega} |f_j(z)|^2 \left(\int_0^{2\pi} |e^{i\theta} e^{-\phi(z)/2}|^{2j} d\theta \right) e^{-(n+1)\phi(z)} \det[\partial\bar{\partial}\phi(z)] dz \\ &= \sum_{j=0}^{\infty} 2\pi \int_{\Omega} |f_j|^2 e^{-(j+n+1)\phi} \det[\partial\bar{\partial}\phi(z)] dz. \end{aligned}$$

It follows that

$$H^2(X) = \bigoplus_{j=1}^{\infty} L_{\text{hol}}^2(\Omega, 2\pi e^{-(j+n+1)\phi} \det[\partial\bar{\partial}\phi(z)] dz),$$

and

$$K_{\text{Szegö}}((x, t), (y, s)) = \frac{1}{2\pi} \sum_{k=0}^{\infty} K_{e^{-(j+n+1)\phi} \det[\partial\bar{\partial}\phi(z)]}(x, y) (t\bar{s})^j.$$

In other words, the weighted Bergman kernels of our spaces $L_{\text{hol}, h}^2$ are just the Taylor coefficients, with respect to the fiber variable, of the Szegö kernel of $H^2(X)$. This result is due to Ligocka [35]; the basic idea goes back to Forelli and Rudin [28].

5.7. Fefferman's theorem

This celebrated result of Fefferman [26] and Boutet de Monvel and Sjöstrand [12] describes the boundary behaviour of the Szegö kernel of an arbitrary (nice) domain in $\mathbf{C}^n$, thus including, in particular, the kernel $K_{\text{Szegö}}$ of our Hartogs domain $\tilde{\Omega}$. Here is the result.

Let $D \subset \mathbf{C}^n$ be a bounded strictly pseudoconvex domain with smooth boundary, and r a defining function for D . As in the special case of $D = \tilde{\Omega}$ discussed before, one defines the Hardy space $H^2(\partial D)$ as the subspace in $L^2(\partial D, d\sigma)$ (with some non-vanishing volume element σ on ∂D) of all functions whose Poisson extensions into D are not only harmonic but holomorphic; and the Szegö kernel $K_{\text{Szegö}}(z, w)$, $z, w \in D$, as the reproducing kernel of $H^2(\partial D)$, viewed as a space of functions on D (not just of their boundary values on ∂D).

Then there are functions $a, b \in C^\infty(\mathbf{C}^n)$ such that

(a) for $x \in \partial D$,

$$a(x) = \frac{n!}{\pi^n} J[r](x) > 0; \tag{33}$$

(b) the Szegő kernel on the diagonal is given by the formula

$$K_{\text{Szegő}}(x, x) = \frac{a(x)}{r(x)^n} + b(x) \log r(x).$$

This formula also extends to $K_{\text{Szegő}}(x, y)$ with $x \neq y$, namely,

$$K_{\text{Szegő}}(x, y) = \frac{a(x, y)}{r(x, y)^n} + b(x, y) \log r(x, y),$$

where $a(x, y)$, $b(x, y)$ and $r(x, y)$ are *almost-sesquiholomorphic extensions* of $a(x) = a(x, x)$, $b(x) = b(x, x)$ and $r(x) = r(x, x)$, respectively. The latter means that $\partial a(x, y)/\partial y$ and $\partial a(x, y)/\partial \bar{x}$ both vanish to infinite order on the diagonal $x = y$, and similarly for $b(x, y)$ and $r(x, y)$. Such extensions always exist, and it is a consequence of the strict pseudoconvexity that $r(x, y)$ can be chosen so that $\text{Re } r(x, y) > 0$ for all $x, y \in D$, so that the logarithm can be defined as the principal branch.

(c) $K_{\text{Szegő}}(x, y)$ is smooth on $\bar{\Omega} \times \Omega \setminus \mathcal{U}$, for any neighbourhood $\mathcal{U}$ of the boundary diagonal $\{(x, x) : x \in \partial\Omega\}$.

Finally, there is a device for converting this description of the boundary behaviour into the description of the Taylor components from Ligocka's formula.

5.8. Resolution of singularities

Recall that the power series $\sum_{k=0}^{\infty} k^j z^k$ converges on the unit disc $\mathbf{D}$, and its sum equals

$$\sum_{k=0}^{\infty} k^j z^k = \frac{j!}{(1-z)^{j+1}} + \sum_{k=1}^j \frac{a_{jk}}{(1-z)^k},$$

with some constants a_{jk} , if $j = 0, 1, 2, \dots$; and

$$\sum_{k=0}^{\infty} k^j z^k = \frac{(-1)^j}{j!} (1-z)^j \log(1-z) + F_j(z),$$

with some $F_j \in C^{-j}(\bar{\mathbf{D}})$, if $j = -1, -2, -3, \dots$. Also, by the familiar Cauchy estimates, if a holomorphic function $f(z) = \sum_k f_k z^k$ on the disc belongs to $C^j(\bar{\mathbf{D}})$, then its Taylor coefficients satisfy

$$f_k = O(k^{-j}) \quad \text{as } k \rightarrow +\infty.$$

Now suppose that $f(z) = \sum_k f_k z^k$ is a holomorphic function on $\mathbf{D}$ which satisfies

$$f(z) = \frac{a(z)}{(1-z)^{n+1}} + b(z) \log(1-z)$$

for some $a, b \in C^\infty(\mathbf{C})$. Taking the Taylor expansions of a, b around $z = 1$, this implies that there exist $\alpha_1, \dots, \alpha_{n+1}$ and $\beta_0, \beta_1, \beta_2, \dots$, with $\alpha_{n+1} = a(1)$, such that, for any $M = 0, 1, 2, \dots$,

$$f(z) = \sum_{j=1}^{n+1} \frac{\alpha_j}{(1-z)^j} + \sum_{j=0}^M \beta_j (1-z)^j \log(1-z) + F_M(z),$$

with $F_M \in C^M(\overline{\mathbf{D}})$. Combining this with the observations in the preceding paragraph, it transpires that

$$f_k \approx a_n k^n + a_{n-1} k^{n-1} + \cdots + a_0 + \frac{a_{-1}}{k} + \cdots, \quad a_n = \frac{a(1)}{n!}, \quad (34)$$

for some constants $a_n, a_{n-1}, \dots$, as $k \rightarrow \infty$.

5.9. Sketch of proof of Theorem B

As already mentioned in §5.3, the hypotheses of the theorem guarantee that the Hartogs domain

$$\tilde{\Omega} = \{(z, t) \in \Omega \times \mathbf{C} : |t|^2 < e^{-\Phi(z)}\}$$

is smoothly bounded, strictly pseudoconvex, and with a defining function

$$\tilde{r}(z, t) := e^{-\Phi(z)} - |t|^2.$$

Consider the Hardy space $H^2(X)$ on the boundary $X = \partial\tilde{\Omega}$. By Ligocka's formula from §5.6, we have

$$H^2(X) = \bigoplus_{k=n+1}^{\infty} L_{\text{hol}}^2(\Omega, e^{-k\Phi} \det[\partial\bar{\partial}\Phi]) \quad (35)$$

(where $n = \dim \Omega$, so $n+1 = \dim \tilde{\Omega}$), and

$$K_{\text{Szegö}}((x, s), (y, t)) = \frac{1}{2\pi} \sum_{k=0}^{\infty} K_{k+n+1}(x, y) (s\bar{t})^k,$$

where, for brevity, we are denoting the reproducing kernel of $L_{\text{hol}}^2(\Omega, e^{-k\Phi} \det[\partial\bar{\partial}\Phi])$ by $K_k(x, y)$.

Fefferman's theorem for the Szegö kernel tells us that

$$K_{\text{Szegö}} = \frac{a}{\tilde{r}^{n+1}} + b \log \tilde{r},$$

for some (almost-sesquiholomorphic) functions $a, b \in C^\infty(\mathbf{C}^{n+1} \times \mathbf{C}^{n+1})$. Hence, in particular,

$$\begin{aligned} \frac{1}{2\pi} \sum_{k=0}^{\infty} K_{k+n+1}(x, x) s^k &= \tilde{K}_{\text{Szegö}}((x, s), (x, 1)) \\ &= \frac{a(x, s)}{(e^{-\Phi(x)} - s)^{n+1}} + b(x, s) \log(e^{-\Phi(x)} - s) \\ &= \frac{a(x, s)e^{(n+1)\Phi(x)}}{(1 - se^{\Phi(x)})^{n+1}} + b(x, s) \log(1 - se^{\Phi(x)}) - b(x, s)\Phi(x) \\ &\quad \underbrace{\hspace{1.5cm}}_{=: z} \\ &= \frac{A(x, z)}{(1 - z)^{n+1}} + B(x, z) \log(1 - z), \end{aligned}$$

where $A(x, z) = a(x, ze^{-\Phi(x)})e^{(n+1)\Phi(x)} - b(x, ze^{-\Phi(x)})\Phi(x)(1-z)^{n+1}$ and $B(x, z) = b(x, ze^{-\Phi(x)})$. So for each $x \in \Omega$,

$$\sum_{k=0}^{\infty} e^{-k\Phi(x)} K_{k+n+1}(x, x) z^k = \frac{A(x, z)}{(1-z)^{n+1}} + B(x, z) \log(1-z)$$

with functions $A, B \in C^\infty(\overline{\Omega} \times \overline{\mathbf{D}})$. Employing the resolution of singularities from §5.8 implies

$$K_k(x, x) \approx \frac{k^n}{\pi^n} e^{k\Phi(x)} \sum_{j=0}^{\infty} \frac{b_j(x)}{k^j}$$

as $k \rightarrow +\infty$, proving the first part of Theorem B. (The formula for b_0 follows from (31), (33) and (34).)

With a bit of technicalities which we omit, the last result can be extended also to $x \neq y$:

$$K_k(x, y) \approx \frac{k^n}{\pi^n} e^{k\Phi(x, y)} \sum_{j=0}^{\infty} \frac{b_j(x, y)}{k^j} \quad (36)$$

for (x, y) near the diagonal, where $\Phi(x, y)$, $b_j(x, y)$ are almost-sesquiholomorphic extensions of $\Phi(x) = \Phi(x, x)$ and $b_j(x) = b_j(x, x)$. (The technicalities involve an improved version of the resolution of singularities from §5.8, where $f(z)$, holomorphic in $z \in \mathbf{D}$, is replaced by $f(x, z)$, depending smoothly on x and holomorphic in z in the disc $|z| < r(x)$, where the radius $r(x)$ also depends smoothly on x ; see Lemma 7 in [20].)

The second part of Theorem B (concerning the asymptotics of the Berezin transform) is then proved by first showing that in the integral defining B_α ,

$$B_\alpha f(x) = \int_{\Omega} f(y) \frac{|K_\alpha(x, y)|^2}{K_\alpha(x, x)} e^{-\alpha\Phi(y)} \det[\partial\bar{\partial}\Phi(y)] dy$$

the main contribution, as $\alpha \rightarrow +\infty$, comes from a small neighbourhood of x . In that neighbourhood, one then replaces $K_\alpha(x, y)$ by the asymptotic expansion (36). This reduces the problem to finding the asymptotics as $\alpha \rightarrow +\infty$ of integrals of the form

$$\int_{\text{neighbourhood of } x} F(y) e^{\alpha(\Phi(x, y) + \Phi(y, x) - \Phi(x) - \Phi(y))} dy,$$

where F is an expression involving f , $\det[\partial\bar{\partial}\Phi]$, and the coefficient functions b_j from (36). Finally, this kind of integrals is handled by the standard stationary-phase (Laplace, WJKB) method, yielding the result in the theorem.

The first two terms in the asymptotic expansion for B_α can be evaluated explicitly, giving the desired outcomes $Q_0 = I$ and $Q_1 = \Delta$, and thus finishing completely the proof of Theorem B.

6. Berezin-Toeplitz quantization

For $f \in L^\infty(\Omega)$, let us denote, for brevity, the Toeplitz operator with symbol f on $L^2_{\text{hol}}(\Omega, e^{-m\Phi} \det[\partial\bar{\partial}\Phi])$ by $T_f^{(m)}$. The main result on the Berezin-Toeplitz quantization then reads as follows.

Theorem BT. *Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^n$, and $\Phi : \Omega \rightarrow \mathbf{R}$ a smooth strictly-PSH function such that $e^{-\Phi} =: r$ is a defining function for Ω . Then there exist bilinear differential operators C_j ($j = 0, 1, 2, \dots$) such that for any $f, g \in C^\infty(\bar{\Omega})$ and any $M = 0, 1, 2, \dots$,*

$$\left\| T_f^{(m)} T_g^{(m)} - \sum_{j=0}^M m^{-j} T_{C_j(f,g)}^{(m)} \right\| = O(m^{-M-1}) \quad \text{as } m \rightarrow \infty.$$

Furthermore,

$$C_0(f, g) = fg, \quad C_1(f, g) - C_1(g, f) = \frac{i}{2\pi} \{f, g\}.$$

Consequently, $f * g := \sum_{j=0}^\infty h^j C_j(f, g)$ defines a star-product on Ω .

Observe that the theorem establishes the expansion for the product of two Toeplitz operators (17) in the strongest possible sense, namely, in the operator norm.

As already mentioned, the proof of Theorem BT involves a sophisticated machinery, due to Boutet de Monvel and Guillemin, of Fourier integral operators of Hermite type — more specifically, of Toeplitz operators with pseudodifferential symbols. It is not our intention to introduce all the necessary notions and technicalities here; we will, however, try to highlight at least the main ideas.

Consider again the Hartogs domain $\tilde{\Omega}$ from Section 5,

$$\tilde{\Omega} = \{(z, t) \in \Omega \times \mathbf{C} : |t|^2 < e^{-\Phi(z)}\}.$$

Again, the hypotheses of Theorem BT guarantee that $\tilde{\Omega}$ is smoothly bounded, strictly pseudoconvex, and admits

$$\tilde{r}(z, t) := e^{-\Phi(z)} - |t|^2$$

as a defining function.

As before, consider the Szegő kernel on the compact manifold $X = \partial\tilde{\Omega}$ with respect to the measure

$$d\sigma := \frac{J[\tilde{r}]}{\|\partial\tilde{r}\|} dS.$$

We have already seen that (Ligocka's formula)

$$K_{\text{Szegő}}(x, s; y, t) = \frac{1}{2\pi} \sum_{k=0}^{\infty} K_{k+n+1}(x, y) (s\bar{t})^k,$$

$$H^2(X) = \bigoplus_{k=n+1}^{\infty} L^2_{\text{hol}}(\Omega, e^{-k\Phi} \det[\partial\bar{\partial}\Phi]). \quad (37)$$

The space $H^2(X)$ also admits its own “Hardy-space” Toeplitz operators: namely, if F is a function in, say, $C^\infty(X)$, one defines the Toeplitz operator T_F on $H^2(X)$ with symbol F as

$$T_F \psi := P_{\text{Szegö}}(F\psi), \quad \psi \in H^2(X),$$

where $P_{\text{Szegö}} : L^2(X, d\sigma) \rightarrow H^2(X)$ is the orthogonal projection (the *Szegö projection*).

Now if f is a smooth function on $\bar{\Omega}$, we can lift it to a function $F \in C^\infty(\bar{\Omega})$ by composing with the projection on the first variable, i.e.

$$F(x, t) := f(x).$$

An easy verification then reveals that under the orthogonal decomposition (37), the Toeplitz operators $T_f^{(m)}$ on $L^2_{\text{hol}}(\Omega, e^{-m\Phi} \det[\partial\bar{\partial}\Phi])$ and the Toeplitz operator T_F on $H^2(X)$ are related by

$$T_F = \bigoplus_{m=n+1}^{\infty} T_f^{(m)}.$$

The main ingredient in the whole proof is that, following the ideas of Boutet de Monvel and Guillemin, we can define Toeplitz operators T_Q on $H^2(X)$ by the same recipe not only for functions, but also for *pseudodifferential operators* (Ψ DO for short) Q on X as symbols. That is, for a Ψ DO Q on X , we define

$$T_Q \psi := P_{\text{Szegö}} Q \psi.$$

For Q the operator of multiplication by a function $F \in C^\infty(X)$, this recovers the Toeplitz operators T_F above as a particular case. Toeplitz operators on $H^2(X)$ with Ψ DO symbols are often called *generalized Toeplitz operators*.

One proceeds to define the order $\text{ord}(T_Q)$ and the symbol $\sigma(T_Q)$ of T_Q as the order of Q and the restriction of the principal symbol $\sigma(Q)$ of Q to the symplectic submanifold

$$\Sigma := \{(x, \xi) : \xi = t(\bar{\partial}\tilde{r} - \partial\tilde{r})_x, t > 0\}$$

of the cotangent bundle of X , respectively. It can be shown that these two definitions are unambiguous: although it may happen that $T_Q = T_{Q'}$ for two different Ψ DOs Q, Q' (which is peculiar for Ψ DO symbols — it is never the case that $T_F = T_{F'}$ for $F \neq F'$), in that case either Q, Q' have the same order and their symbols coincide on Σ , or one of them — say, Q — has greater order than the other and its symbol vanishes on Σ to order $\text{ord}(Q) - \text{ord}(Q')$. Also, the order and the symbol of T_Q obey the usual rules one would expect, as well as some additional ones:

(P1) the generalized Toeplitz operators form an algebra under composition

(i.e. $\forall Q_1, Q_2 \exists Q_3 : T_{Q_1} T_{Q_2} = T_{Q_3}$);

(P2) $\text{ord}(T_{Q_1} T_{Q_2}) = \text{ord}(T_{Q_1}) + \text{ord}(T_{Q_2})$; $\sigma(T_{Q_1} T_{Q_2}) = \sigma(T_{Q_1}) \sigma(T_{Q_2})$;

(P3) $\sigma([T_{Q_1}, T_{Q_2}]) = \{\sigma(T_{Q_1}), \sigma(T_{Q_2})\}_\Sigma$;

(P4) if $\text{ord}(T_Q) = 0$, then T_Q is a bounded operator on H^2 ;

- (P5) if $\text{ord}(T_{Q_1}) = \text{ord}(T_{Q_2}) = k$ and $\sigma(T_{Q_1}) = \sigma(T_{Q_2})$, then $\text{ord}(T_{Q_1} - T_{Q_2}) \leq k - 1$;
(P6) for $F \in C^\infty(X)$ and $(x, \xi) \in \Sigma$, $\sigma(T_F)(x, \xi) = F(x)$.

Returning to the proof of Theorem BT, let $\mathcal{T}$ be the subalgebra of all generalized Toeplitz operators on $H^2(X)$ which commute with the rotations

$$U_\theta : f(z, w) \mapsto f(z, e^{i\theta} w), \quad (z, w) \in X, \quad \theta \in \mathbf{R},$$

in the fiber variable. Clearly, the operators T_F with $F(x, t) = f(x)$ for some function $f \in C^\infty(\overline{\Omega})$ (i.e. with F constant along fibers) belong to $\mathcal{T}$.

Denote by $D : H^2(X) \rightarrow H^2(X)$ the infinitesimal generator of the semigroup U_θ . Then D acts as multiplication by im on the m -th summand in (37), for each m :

$$D = \bigoplus_m imI;$$

and also

$$D = T_{\partial/\partial\theta}$$

is a generalized Toeplitz operator of order 1.

Using (P1)–(P6) it can be shown that if $T \in \mathcal{T}$ is of order 0, then

$$T = T_F + D^{-1}R$$

for some (uniquely determined) $F \in C^\infty(X)$ which is constant along the fibers (hence, descends to a function on Ω), and $R \in \mathcal{T}$ of order 0. Repeated application of this formula shows that, for each $k \geq 0$,

$$T = \sum_{j=0}^k D^{-j} T_{F_j} + D^{-k-1} R_k,$$

with $F_j(x, t) = f_j(x)$ for some $f_j \in C^\infty(\overline{\Omega})$ and $R_k \in \mathcal{T}$ of order 0. Invoking the fact that zero order operators are bounded, it follows that

$$D^{k+1} \left(T - \sum_{j=0}^k D^{-j} T_{F_j} \right) = R_k$$

is a bounded operator on H^2 .

In view of the decomposition $T_F = \bigoplus_m T_f^{(m)}$, this means that

$$\left\| T \Big|_{L^2(\Omega, e^{-m\Phi} \det[\partial\bar{\partial}\Phi])} - \sum_{j=0}^k m^{-j} T_{f_j}^{(m)} \right\| = O(m^{-k-1})$$

as $m \rightarrow +\infty$. Taking for T the product $T_F T_G$, with $F(x, t) = f(x)$, $G(x, t) = g(x)$ for some $f, g \in C^\infty(\overline{\Omega})$, and setting $C_j(f, g) := f_j$, we obtain the desired asymptotic expansion for $T_f^{(m)} T_g^{(m)}$.

Finally, the assertions concerning C_0 and C_1 follow from the above properties (P2) and (P3) of the symbol by a routine calculation.

7. Concluding remarks

This paper is by no means intended as an exhaustive survey of quantization methods, or even of the Berezin and the Berezin-Toeplitz quantizations; its main goal was to serve as a first introduction into the subject for a new-comer interested in the area. From the many surveys and overviews of various quantization techniques, the reader is referred e.g. to [1] for a somewhat more in-depth account of many (but not all) things discussed here, as well as for abundant references to other literature. Two good surveys of traditional deformation quantization (i.e. on the level of formal power series) are Gutt [29] and Sternheimer [42]; a very nice recent overview focused on the Berezin-Toeplitz quantization discussed here is Schlichenmaier [40]. Some more technical aspects of several points left out here can be found in the author's article [21]. An excellent reading about the material discussed in Section 1 are several books by Folland, in particular [27].

It should, finally, be mentioned that the subject of Berezin and Berezin-Toeplitz quantization is still far from being understood completely, and there are many things waiting still to be resolved in a satisfactory way. For instance, in both Theorem B and Theorem BT the semiclassical limit $\alpha = \frac{1}{h} \rightarrow +\infty$ is taken only for α ranging through the integers; this is of course natural if Ω is a compact manifold (as was the original context in [9]), but is only an artifact of the methods of proof for Ω a domain in $\mathbf{C}^n$. Removing this restriction, i.e. extending the asymptotics of the reproducing kernels K_α , the Berezin transforms B_α , and the Toeplitz operators $T_f^{(\alpha)}$ also to non-integer $\alpha \rightarrow +\infty$ would be most desirable.

Another highly active area concerns the generalizations of Fefferman's theorem on the Szegő kernel from §5.7 (and the analogous theorem of his for the Bergman kernel, which was not mentioned here) to domains which are only weakly (i.e. not necessarily strictly) pseudoconvex; at the moment, there are only some partial results for special types of domains (see e.g. [30]). Having a result of that kind would make it possible to extend Theorems B and BT to more general domains. Similarly, having a result of that kind for domains which are not necessarily smoothly bounded — more specifically, for Hartogs domains $\tilde{\Omega}$ whose the radius-function $e^{-\phi}$ has a logarithmic singularity at the boundary of Ω — would make it possible to quantize metrics whose Kähler potential behaves like that at the boundary; the latter includes, for instance, the important Cheng-Yau metric on Ω (the Kähler-Einstein metric; see [5] for more information on this). Carrying out the Berezin-Toeplitz quantization in the last case by the method described in Section 6 would also require an extension of the Boutet de Monvel and Guillemin theory of generalized Toeplitz operators to noncompact manifolds, which is another open problem at present.

Closely related ideas concern also the boundary behaviour of weighted Bergman kernels with respect to weights having some kind of singularity at the boundary (e.g. involving the logarithm of the defining function); some results of the present author in that direction can be found in [22]. Interestingly, the same technique

as in that paper can also be used to establish that the weighted Bergman kernels $K_\alpha(x, y)$ appearing in the previous sections can be continued to meromorphic functions of α in the entire complex plane [24]; this is somewhat reminiscent of the resonances occurring in scattering theory, and is related to zeta functions of elliptic operators. A subject of a completely different flavour is the extension of the Theorems B and BT above also to the setting of harmonic, rather than holomorphic, functions; although this seems not to have any direct relevance for quantization, the results are equally interesting, and, apparently, much more intriguing, than in the holomorphic case (see e.g. [23]).

There is also a variety of problems, though again not directly related to quantization, concerning the range of the Berezin symbol map $T \mapsto \tilde{T}$ (see e.g. Coburn [15] and Bommier-Hato [8]), while notable applications of Toeplitz operators and the Berezin transform appear in operator theory and in time-frequency analysis; let us mention at least [16], [36], [2], [3] and [45].

References

- [1] S. Twareque Ali, M. Engliš: *Quantization methods: a guide for physicists and analysts*, Rev. Math. Phys. **17** (2005), 391–490.
- [2] S. Axler, D. Zheng: *Compact operators via the Berezin transform*, Indiana Univ. Math. J. **47** (1998), 387–400.
- [3] W. Bauer, L.A. Coburn, J. Isralowitz: *Heat flow, BMO, and the compactness of Toeplitz operators*, J. Funct. Anal. **259** (2010), 57–78.
- [4] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer: *Deformation theory and quantization*, Ann. Phys. **111** (1978), 61–110 (part I), 111–151 (part II).
- [5] M. Beals, C. Fefferman, and R. Grossman: *Strictly pseudoconvex domains in $\mathbf{C}^n$* , Bull. Amer. Math. Soc. **8** (1983), 125–326.
- [6] F.A. Berezin: *General concept of quantization*, Comm. Math. Phys. **40** (1975), 153–174.
- [7] R. Berman, B. Berndtsson, J. Sjöstrand: *A direct approach to Bergman kernel asymptotics for positive line bundles*, Ark. Mat. **46** (2008), 197–217.
- [8] H. Bommier-Hato: *Lipschitz estimates for the Berezin transform*, J. Funct. Spaces Appl. **8** (2010), 103–128.
- [9] M. Bordemann, E. Meinrenken, M. Schlichenmaier: *Toeplitz quantization of Kähler manifolds and $\mathfrak{gl}(N)$, $N \rightarrow \infty$ limits*, Comm. Math. Phys. **165** (1994), 281–296.
- [10] D. Borthwick, A. Lesniewski, H. Upmeyer: *Nonperturbative deformation quantization of Cartan domains*, J. Funct. Anal. **113** (1993), 153–176.
- [11] L. Boutet de Monvel, V. Guillemin: *The spectral theory of Toeplitz operators*, Ann. Math. Studies, vol. 99, Princeton University Press, Princeton 1981.
- [12] L. Boutet de Monvel, J. Sjöstrand: *Sur la singularité des noyaux de Bergman et de Szegő*, Astérisque **34–35** (1976), 123–164.
- [13] D. Catlin: *The Bergman kernel and a theorem of Tian*, Analysis and geometry in several complex variables (Katata 1997), pp. 1–23, Trends in Math., Birkhäuser, Boston 1999.

- [14] L.A. Coburn: *Deformation estimates for the Berezin-Toeplitz quantization*, Comm. Math. Phys. **149** (1992), 415–424; *Berezin-Toeplitz quantization*, Algebraic methods in operator theory, pp. 101–108, Birkhäuser, Boston, 1994.
- [15] L.A. Coburn: *A Lipschitz estimate for Berezin’s operator calculus*, Proc. Amer. Math. Soc. **133** (2005), 127–131.
- [16] L. Coburn: *Symbol calculus for Gabor-Daubechies windowed Fourier localization operators*, preprint, 2005.
- [17] M. DeWilde, P.B.A. Lecomte: *Existence of star products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys. **7** (1983), 487–496.
- [18] M. Engliš: *A Forelli-Rudin construction and asymptotics of weighted Bergman kernels*, J. Funct. Anal. **177** (2000), 257–281.
- [19] M. Engliš: *The asymptotics of a Laplace integral on a Kähler manifold*, J. reine angew. Math. **528** (2000), 1–39.
- [20] M. Engliš: *Weighted Bergman kernels and quantization*, Comm. Math. Phys. **227** (2002), 211–241.
- [21] M. Engliš: *Berezin and Berezin-Toeplitz quantizations for general function spaces*, Rev. Mat. Complut. **19** (2006), 385–430.
- [22] M. Engliš: *Weighted Bergman kernels for logarithmic weights*, Pure Appl. Math. Quarterly (Kohn special issue) **6** (2010), 781–813.
- [23] M. Engliš: *Berezin transform on the harmonic Fock space*, J. Math. Anal. Appl. **367** (2010), 75–97.
- [24] M. Engliš: *Analytic continuation of weighted Bergman kernels*, J. Math. Pures Appl. **94** (2010), 622–650.
- [25] B.V. Fedosov: *A simple geometric construction of deformation quantization*, J. Diff. Geo. **40** (1994), 213–238.
- [26] C. Fefferman: *The Bergman kernel and biholomorphic mappings of pseudoconvex domains*, Inv. Math. **26** (1974), 1–65.
- [27] G.B. Folland, *Harmonic analysis in phase space*, Annals of Mathematics Studies, vol. 122, Princeton University Press, Princeton, 1989.
- [28] F. Forelli, W. Rudin: *Projections on spaces of holomorphic functions in balls*, Indiana Univ. Math. J. **24** (1974), 593–602.
- [29] S. Gutt: *Variations on deformation quantization*, Conference Moshe Flato (Dijon, 1999), Vol. I, pp. 217–254, Math. Phys. Stud. 21, Kluwer, Dordrecht, 2000.
- [30] J. Kamimoto: *Newton polyhedra and the Bergman kernel*, Math. Z. **246** (2004), 405–440.
- [31] A.V. Karabegov: *Deformation quantization with separation of variables on a Kähler manifold*, Comm. Math. Phys. **180** (1996), 745–755.
- [32] S. Klimek, A. Lesniewski: *Quantum Riemann surfaces, I: The unit disc*, Comm. Math. Phys. **146** (1992), 103–122; *II: The discrete series*, Lett. Math. Phys. **24** (1992), 125–139; *III: The exceptional cases*, Lett. Math. Phys. **32** (1994), 45–61.
- [33] M. Kontsevich: *Deformation quantization of Poisson manifolds*, preprint (1997), arXiv:q-alg/9709040.

- [34] B. Kostant: *Quantization and unitary representations*, Lecture Notes in Math., vol. 170, Springer, Berlin, 1970.
- [35] E. Ligocka: *On the Forelli-Rudin construction and weighted Bergman projections*, Studia Math. **94** (1989), 257–272.
- [36] M.-L. Lo: *The Bargmann transform and windowed Fourier localization*, Integral Eqs. Oper. Theory **57** (2007), 397–412.
- [37] X. Ma, G. Marinescu, *Holomorphic Morse inequalities and Bergman kernels*, Progress in Mathematics, vol. 254, Birkhäuser Verlag, Basel, 2007.
- [38] H. Omori, Y. Maeda, A. Yoshioka: *Weyl manifolds and deformation quantization*, Adv. Math. **85** (1991), 224–255.
- [39] W.-D. Ruan: *Canonical coordinates and Bergman metrics*, Comm. Anal. Geom. **6** (1998), 589–631.
- [40] M. Schlichenmaier: *Berezin-Toeplitz quantization for compact Kähler manifolds. A review of results*, Adv. Math. Phys. (2010), Art. ID 927280, 38 pp.
- [41] J.-M. Souriau, *Structure des systèmes dynamiques*, Dunod, Paris, 1969.
- [42] D. Sternheimer: *Deformation quantization: twenty years after*, Particles, Fields and Gravitation (Łódź, 1998), pp. 107–145, AIP Conf. Proc. vol. 453, Amer. Inst. Phys., Woodbury, 1998.
- [43] G. Tian: *On a set of polarized Kähler metrics on algebraic manifolds*, J. Diff. Geom. **32** (1990), 99–130.
- [44] S. Zelditch: *Szegő kernels and a theorem of Tian*, Int. Math. Res. Not. **6** (1998), 317–331.
- [45] K. Zhu, *Operator theory in function spaces*, 2nd edition, Amer. Math. Soc., Providence, 2007.

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