

GEOMETRIC ARVESON-DOUGLAS CONJECTURE

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ABSTRACT. We prove the Arveson-Douglas essential normality conjecture for graded Hilbert submodules that consist of functions vanishing on a given homogeneous subvariety of the ball, smooth away from the origin. Our main tool is the theory of generalized Toeplitz operators of Boutet de Monvel and Guillemin.

1. INTRODUCTION

Let $\mathbb{B}^d$ be the unit ball in $\mathbb{C}^d$, $d \geq 1$. The Drury-Arveson space H_d^2 consists of all holomorphic functions $f(z) = \sum_{\nu} f_{\nu} z^{\nu}$ on $\mathbb{B}^d$ such that

$$\|f\|_{DA}^2 := \sum_{\nu} |f_{\nu}|^2 \frac{\nu!}{|\nu|!} < \infty,$$

equipped with the corresponding norm and inner product. The operators

$$M_{z_j} : f(z) \mapsto z_j f(z)$$

of multiplication by the coordinate functions are bounded on H_d^2 , and commute with each other. This endows H_d^2 with the structure of a module over the polynomial ring $\mathbb{C}[z_1, \dots, z_d]$, a polynomial p corresponding to the operator $M_p = p(M_{z_1}, \dots, M_{z_d})$ of multiplication by p on H_d^2 . If $\mathcal{M} \subset H_d^2$ is a (closed) subspace invariant under all M_{z_j} , $j = 1, \dots, d$, we can therefore consider the restrictions $M_{z_j}|_{\mathcal{M}}$, which are commuting bounded linear operators on $\mathcal{M}$, as well as the compressions

$$S_j := P_{\mathcal{M}^{\perp}} M_{z_j}|_{\mathcal{M}^{\perp}}, \quad j = 1, \dots, d,$$

of the M_{z_j} to the orthogonal complement $\mathcal{M}^{\perp} = H_d^2 \ominus \mathcal{M}$, which are commuting bounded linear operators on $\mathcal{M}^{\perp}$.

The following conjecture was originally made by Arveson [1] with d in the place of $\dim Z(p)$, and refined to the current form by Douglas [8].¹

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¹In both cases, it was also formulated for the more general case of modules $\mathcal{M}$ in $H_d^2 \otimes \mathbb{C}^N$ generated by $\mathbb{C}^N$ -valued homogeneous polynomials, with some finite $N \geq 1$.

Arveson-Douglas Conjecture. *Assume $\mathcal{M}$ is generated, as a module, by finitely many homogeneous polynomials $p_1, \dots, p_m \in \mathbb{C}[z_1, \dots, z_d]$. Then the commutators $[S_j, S_k^*]$, $j, k = 1, \dots, d$, belong to the Schatten class $\mathcal{S}^q$ for all $q > \dim Z(p)$, where $\dim Z(p)$ is the complex dimension of the zero-set $Z(p) \equiv Z(p_1, \dots, p_m)$ of the polynomials $p_1, \dots, p_m$.*

The Arveson conjecture, and in some cases also its refined version due to Douglas, have so far been proved in various special settings: by Arveson himself [1] when $p_1, \dots, p_m$ are monomials; by Guo and Wang [17] for $m = 1$ or $d \leq 3$; by Douglas and Wang [9] when $m = 1$ and $\mathcal{M}$ is a submodule of the Bergman space $L^2_{\text{hol}}(\mathbb{B}^d)$ on $\mathbb{B}^d$ (instead of H^2_d) generated by an arbitrary, not necessarily homogeneous polynomial p ; by Fang and Xia [14] for submodules of the same type in certain weighted (Sobolev-)Bergman spaces on $\mathbb{B}^d$, which included $L^2_{\text{hol}}(\mathbb{B}^d)$ as well as the Hardy space $H^2(\partial\mathbb{B}^d)$ on $\mathbb{B}^d$, but not H^2_d (unless $d = 1$); by Kennedy and Shalit [20] when $p_1, \dots, p_m$ are homogeneous polynomials such that the linear spans of $Z(p_1), \dots, Z(p_m)$ in $\mathbb{C}^d$ have mutually trivial intersections; etc. See the recent survey paper by Shalit [23] for some more details and further information, as well as the original paper by Douglas [8] for more on the motivation and applications to K -homology and index theory.

There is also a reformulation of (a weaker version of) the Arveson-Douglas conjecture in terms of varieties. Namely, denote by $I(p)$ the ideal in $\mathbb{C}[z_1, \dots, z_d]$ generated by $p_1, \dots, p_m$; then $\mathcal{M}$ is the closure of $I(p)$ in H^2_d , and $I(p)$ is a *homogeneous* (or *graded*) ideal, meaning that

$$I(p) = \sum_{k \geq 0}^{\oplus} (I(p) \cap (\text{homogeneous polynomials of degree } k)).$$

Denoting for any ideal J in $\mathbb{C}[z_1, \dots, z_d]$ by

$$Z(J) := \{z \in \mathbb{C}^d : q(z) = 0 \ \forall q \in J\}$$

the zero set of J , we then have $Z(p) = Z(I(p))$, which is a homogeneous variety in $\mathbb{C}^d$, i.e. $z \in Z(p)$, $t \in \mathbb{C}$ implies $tz \in Z(p)$. Conversely, for any subset $X \subset \mathbb{C}^d$,

$$I(X) := \{q \in \mathbb{C}[z_1, \dots, z_d] : q(z) = 0 \ \forall z \in X\}$$

is an ideal in $\mathbb{C}[z_1, \dots, z_d]$, which is homogeneous if X is. The correspondences $J \mapsto Z(J)$, $X \mapsto I(X)$ are not one-to-one: one always has $I(Z(J)) \supset J$, with equality if and only if J is a radical ideal, i.e. $J = \sqrt{J}$ where $\sqrt{J} := \{q \in \mathbb{C}[z_1, \dots, z_d] : q^n \in J \text{ for some } n = 1, 2, \dots\}$; also, $Z(J_1) = Z(J_2)$ if and only if $\sqrt{J_1} = \sqrt{J_2}$ (this is Hilbert's Nullstellensatz). Specializing to modules generated by radical ideals, we thus get the following “geometric version” of the Arveson-Douglas conjecture [20].

Geometric Arveson-Douglas conjecture. *Let V be a homogeneous variety in $\mathbb{C}^d$ and $\mathcal{M} = \{f \in H^2_d : f(z) = 0 \text{ for all } z \in V \cap \mathbb{B}^d\}$. Then $[S_j, S_k^*] \in \mathcal{S}^q$ for all $q > \dim_{\mathbb{C}} V$.*

As already mentioned in passing, one can consider the above conjectures not only for H^2_d , but also for other spaces of holomorphic functions on $\mathbb{B}^d$ on which

the multiplication operators M_{z_j} , $j = 1, \dots, d$, act boundedly. These include the (weighted Bergman) spaces

$$A_\alpha^2(\mathbb{B}^d) \equiv A_\alpha^2 := L_{\text{hol}}^2(\mathbb{B}^d, d\mu_\alpha)$$

of holomorphic functions on $\mathbb{B}^d$ square-integrable with respect to the probability measure

$$d\mu_\alpha(z) := \frac{\Gamma(\alpha + d + 1)}{\Gamma(\alpha + 1)\pi^d} (1 - |z|^2)^\alpha dz, \quad \alpha > -1,$$

where dz denotes the Lebesgue volume on $\mathbb{C}^d$ and the restriction on α ensures that these spaces are nontrivial (and contain all polynomials). In terms of the Taylor coefficients $f(z) = \sum_\nu f_\nu z^\nu$, the norm in A_α^2 is given by

$$(1) \quad \|f\|_\alpha^2 = \sum_\nu |f_\nu|^2 \frac{\nu! \Gamma(d + \alpha + 1)}{\Gamma(|\nu| + d + \alpha + 1)}.$$

The right-hand side makes actually sense and is positive-definite for all $\alpha > -d - 1$, and we can thus extend the definition of A_α^2 also to α in this range; in particular, this will give, in addition to the weighted Bergman spaces for $\alpha > -1$ (including the ordinary — i.e. unweighted — Bergman space $L_{\text{hol}}^2(\mathbb{B}^d)$ for $\alpha = 0$), also the Hardy space

$$A_{-1}^2 = H^2(\partial\mathbb{B}^d, d\sigma)$$

with respect to the normalized surface measure $d\sigma$ on $\partial\mathbb{B}^d$ for $\alpha = -1$, as well as the Drury-Arveson space

$$A_{-d}^2 = H_d^2$$

for $\alpha = -d$. Furthermore, passing from (1) to the equivalent norm

$$(2) \quad \|f\|_{\alpha\circ}^2 := \sum_\nu \frac{|f_\nu|^2}{(|\nu| + 1)^{d+\alpha}} \frac{\nu!}{|\nu|!},$$

one can even define the corresponding spaces $A_{\alpha\circ}^2$ for any real α , with $A_{\alpha\circ}^2 = A_\alpha^2$ (as sets, with equivalent norms) for $\alpha > -d - 1$ (hence, in particular, $A_{-d,\circ}^2 = H_d^2$ for $\alpha = -d$, $A_{-1,\circ}^2 = H^2(\partial\mathbb{B}^d)$ for $\alpha = -1$, and $A_{\alpha\circ}^2 = A_\alpha^2$ for $\alpha > -1$). Actually, $A_{\alpha\circ}^2$ are precisely the subspaces of holomorphic functions

$$A_{\alpha\circ}^2 = W_{\text{hol}}^{-\alpha/2}(\mathbb{B}^d) := \{f \in W^{-\alpha/2}(\mathbb{B}^d) : f \text{ is holomorphic on } \mathbb{B}^d\}$$

in the Sobolev spaces $W^{-\alpha/2}(\mathbb{B}^d)$ on $\mathbb{B}^d$ of order $-\frac{\alpha}{2}$, for any real α . The coordinate multiplications M_{z_j} , $j = 1, \dots, d$, are continuous on $A_{\alpha\circ}^2$ for any $\alpha \in \mathbb{R}$, and one can consider the Arveson-Douglas conjecture in this setting.

Our main result is the proof of the geometric variant of the Arveson-Douglas conjecture — that is, proof of the Arveson-Douglas conjecture for subspaces $\mathcal{M}$ generated by a radical homogeneous ideal — in all these settings for smooth submanifolds.

Main Theorem. *Let V be a homogeneous variety in $\mathbb{C}^d$ such that $V \setminus \{0\}$ is a complex submanifold of $\mathbb{C}^d \setminus \{0\}$ of dimension n , $\alpha \in \mathbb{R}$, and $\mathcal{M}$ the subspace in $A_{\alpha,0}^2$, or in A_α^2 if $\alpha > -d-1$, of functions vanishing on $V \cap \mathbb{B}^d$. Then $[S_j, S_k^*] \in \mathcal{S}^q$, $j, k = 1, \dots, d$, for all $q > n$.*

Our method of proof relies on two ingredients: the results of Beatrous about restrictions of functions in $A_{\alpha,0}^2$ to submanifolds [7], and the theory of Boutet de Monvel and Guillemin of Toeplitz operators on the Hardy space with pseudodifferential symbols (so-called “generalized Toeplitz operators”) [4] [3]. It actually turns out that the Boutet de Monvel and Guillemin theory can also be used to replace the results of Beatrous from [7], at least those that we need here. The required prerequisites about the generalized Toeplitz operators of Boutet de Monvel and Guillemin are reviewed in Section 2, and those about restrictions to submanifolds in Section 3. With these tools it is possible to prove a variant of our main theorem with V a (not necessarily homogeneous) complex submanifold of $\mathbb{B}^d$ intersecting $\partial\mathbb{B}^d$ transversally; we do this in Section 4. The proof of Main Theorem, which builds on the same ideas but with some additional technicalities, is given in Section 5.

2. GENERALIZED TOEPLITZ OPERATORS

Let Ω be a bounded strictly pseudoconvex domain with smooth (i.e. C^∞) boundary in W , where W is either $\mathbb{C}^n$ or, more generally, a complex manifold of dimension n ; an example is W a complex submanifold of dimension n in $\mathbb{C}^d$, $d > n$, and $\Omega = W \cap \mathbb{B}^d$. (One could even allow W to be a complex analytic variety of dimension n with singularities in Ω but not on $\partial\Omega$, an example being a homogeneous complex cone of dimension n in $\mathbb{C}^d$, $d > n$, again with $\Omega = W \cap \mathbb{B}^d$; see §2i in [3].) We fix a positively-signed “defining function” ρ for Ω , i.e. a function smooth on the closure $\bar{\Omega}$ of Ω such that $\rho > 0$ on Ω and $\rho = 0$, $\nabla\rho \neq 0$ on $\partial\Omega$; in the example above, we can take $\rho(z) = 1 - |z|^2$.

Let $L^2(\partial\Omega)$ be the Lebesgue space on the boundary $\partial\Omega$ with respect to the surface measure (i.e. the $(2n-1)$ -dimensional Hausdorff measure) $d\lambda$ on W ; we will denote the inner product and norm in $L^2(\partial\Omega)$ by $\langle \cdot, \cdot \rangle_{\partial\Omega}$ and $\| \cdot \|_{\partial\Omega}$, respectively, and similarly by $\langle \cdot, \cdot \rangle_\Omega$ the inner product in $L^2(\Omega, dz)$.² Let $\bar{\partial}$ denote the usual Cauchy-Riemann operator on W , and $\bar{\partial}^*$ its (formal) adjoint with respect to some fixed smooth Hermitian metric on W ; the harmonic functions on W are then those annihilated by the Laplacian $\Delta := -\bar{\partial}^* \bar{\partial}$. The Hardy space $H^2(\partial\Omega)$ is the subspace in $L^2(\partial\Omega)$ of functions whose Poisson extension into Ω is not only harmonic but holomorphic; or, equivalently, the closure in $L^2(\partial\Omega)$ of $C_{\text{hol}}^\infty(\partial\Omega)$, the space of boundary values of all the functions in $C^\infty(\bar{\Omega})$ that are holomorphic on Ω . We will also denote by $W^s(\partial\Omega)$, $s \in \mathbb{R}$, the Sobolev spaces on $\partial\Omega$, and by $W_{\text{hol}}^s(\partial\Omega)$ the closure of $C_{\text{hol}}^\infty(\partial\Omega)$ in $W^s(\partial\Omega)$. The Poisson extension operator

$$\mathbf{K} : C^\infty(\partial\Omega) \rightarrow C^\infty(\bar{\Omega}), \quad \Delta \mathbf{K}u = 0 \text{ on } \Omega, \quad \mathbf{K}u|_{\partial\Omega} = u,$$

then extends to a bounded operator from $W^s(\partial\Omega)$ onto $W_{\text{harm}}^{s+1/2}(\Omega)$, the subspace of harmonic functions in the Sobolev space $W^{s+1/2}(\Omega)$ on Ω , and from $W_{\text{hol}}^s(\partial\Omega)$

²In the case of W a manifold, we use the surface measure and volume with respect to some chosen Riemannian metric on W .

onto $W_{\text{hol}}^{s+1/2}(\Omega)$, the subspace of holomorphic functions in $W^{s+1/2}(\Omega)$. The operator of taking the boundary values (or “trace”)

$$\gamma : C^\infty(\overline{\Omega}) \rightarrow C^\infty(\partial\Omega), \quad \gamma f := f|_{\partial\Omega},$$

which acts from $W^s(\Omega)$ onto $W^{s-1/2}(\partial\Omega)$ for $s > \frac{1}{2}$ (this is the Sobolev trace theorem), similarly extends to a bounded map from $W_{\text{harm}}^s(\Omega)$ onto $W^{s-1/2}(\partial\Omega)$ and from $W_{\text{hol}}^s(\Omega)$ onto $W_{\text{hol}}^{s-1/2}(\partial\Omega)$, for any $s \in \mathbb{R}$, which is the right inverse to $\mathbf{K}$. On harmonic and holomorphic functions, γ and $\mathbf{K}$ are thus mutual inverses, establishing isomorphisms $W_{\text{harm}}^s(\Omega) \leftrightarrow W^{s-1/2}(\partial\Omega)$ and $W_{\text{hol}}^s(\Omega) \leftrightarrow W_{\text{hol}}^{s-1/2}(\partial\Omega)$, for any real s . See e.g. Lions and Magenes [21], Chapter 2, Section 7.3 for the proofs and further details.³

As usual, by a classical pseudodifferential operator (or Ψ DO for short) on $\partial\Omega$ of order m we will mean a pseudodifferential operator whose total symbol in any local coordinate system has an asymptotic expansion

$$p(x, \xi) \sim \sum_{j=0}^{\infty} p_{m-j}(x, \xi)$$

where p_{m-j} is C^∞ in x, ξ and positive homogeneous of degree $m-j$ in ξ for $|\xi| > 1$. Here m can be any real number, and the symbol “ $\sim$ ” means that the difference between p and $\sum_{j=0}^{k-1} p_{m-j}$ should belong to the Hörmander class S^{m-k} , for each $k = 0, 1, 2, \dots$; see [19]. The space of all such operators will be denoted Ψ^m . An operator in Ψ^m maps $W^s(\partial\Omega)$ into $W^{s-m}(\partial\Omega)$ for any $s \in \mathbb{R}$. Unless explicitly stated otherwise, all Ψ DOs in this paper will be classical.

If $A \in \Psi^m$, $m < 0$, is elliptic, i.e. its principal symbol $\sigma(A)(x, \xi) = a_m(x, \xi)$ does not vanish for $\xi \neq 0$, and is positive selfadjoint as an operator on $L^2(\partial\Omega)$ (i.e. $\langle Au, u \rangle > 0$ for all $u \in L^2(\partial\Omega)$, $u \neq 0$), then A is compact and its spectrum consists of isolated eigenvalues $\lambda_1 > \lambda_2 > \dots > 0$ of finite multiplicity, so one can define the power A^z for any $z \in \mathbb{C}$ by the spectral theorem. Similarly for positive (i.e. $\langle Au, u \rangle > 0$ for all $u \in \text{dom } A$, $u \neq 0$) selfadjoint elliptic $A \in \Psi^m$ with $m > 0$, one defines A^z as $(A^{-1})^{-z}$. It is then a classical result of Seeley that in both cases, A^z is a Ψ DO of order mz , with principal symbol $\sigma(A)^z$. In particular, if we define the space $\mathcal{H}_A$ as the completion of $C^\infty(\partial\Omega)$ with respect to the norm

$$\|u\|_A^2 := \langle Au, u \rangle_{\partial\Omega} = \|A^{1/2}u\|_{\partial\Omega}^2,$$

then $\mathcal{H}_A = W^{m/2}(\partial\Omega)$ as sets, with equivalent norms. All this remains in force also for operators of order $m = 0$; note that the positivity of A then implies, in particular, that A is injective and, hence, with bounded inverse on $L^2(\partial\Omega)$, so one can again define A^z for any $z \in \mathbb{C}$ by the spectral theorem for (bounded) selfadjoint operators.

For $P \in \Psi^m$, the generalized Toeplitz operator $T_P : W_{\text{hol}}^m(\partial\Omega) \rightarrow H^2(\partial\Omega)$ is defined as

$$T_P = \Pi P,$$

³Or page 29, fifth paragraph, in [2]; as mentioned explicitly on p. 13 there, this applies also to the case of manifolds Ω , not only to domains in $\mathbb{R}^n$.

where $\Pi : L^2(\partial\Omega) \rightarrow H^2(\partial\Omega)$ is the orthogonal projection (the Szegő projection). Alternatively, one may view T_P as the operator

$$T_P = \Pi P \Pi$$

on all of $W^m(\partial\Omega)$. Then T_P maps continuously $W^s(\partial\Omega)$ into $W_{\text{hol}}^{s-m}(\partial\Omega)$, for each $s \in \mathbb{R}$. The microlocal structure of generalized Toeplitz operators was described by Boutet de Monvel and Guillemin [3] [4], who proved in particular the following facts. Let Σ denote the half-line bundle

$$(3) \quad \Sigma := \{(x, \xi) \in T^*(\partial\Omega) : \xi = t\eta_x, t > 0\},$$

where η is the restriction to $\partial\Omega$ of the 1-form $\text{Im}(-\partial\rho) = (\bar{\partial}\rho - \partial\rho)/(2i)$; the strict pseudoconvexity of Ω implies that Σ is a symplectic submanifold of the cotangent bundle $T^*(\partial\Omega)$.

- (P1) For any T_P , $P \in \Psi^m$, there in fact exists $Q \in \Psi^m$ such that $T_P = T_Q$ and Q commutes with Π .
(Hence, $T_P = T_Q$ is just the restriction of Q to the Hardy space. It follows, in particular, that generalized Toeplitz operators T_P form an algebra.)
- (P2) It can happen that $T_P = T_Q$ for two different Ψ DOs P and Q . However, one can define unambiguously the order of T_Q as $\min\{\text{ord}(P) : T_P = T_Q\}$, and the symbol of T_Q as $\sigma(T_Q) := \sigma(Q)|_\Sigma$ if $\text{ord}(Q) = \text{ord}(T_Q)$.
- (P3) The order and the symbol obey the usual laws: $\text{ord}(T_Q T_P) = \text{ord}(T_Q) + \text{ord}(T_P)$ and $\sigma(T_Q T_P) = \sigma(T_Q)\sigma(T_P)$.
- (P4) If $\text{ord}(T_P) = m$, then T_P maps $W_{\text{hol}}^s(\partial\Omega)$ continuously into $W_{\text{hol}}^{s-m}(\partial\Omega)$, for any $s \in \mathbb{R}$. In particular, if $\text{ord}(T_P) = 0$ then T_P is a bounded operator on $L^2(\partial\Omega)$; if $\text{ord}(T_P) < 0$, then it is even compact.
- (P5) If $P \in \Psi^m$ and $\sigma(T_P) = 0$, then there exists $Q \in \Psi^{m-1}$ with $T_Q = T_P$.
- (P6) We will say that a generalized Toeplitz operator T_P is elliptic if $\sigma(T_P)$ does not vanish. Then T_P has a parametrix, i.e. there exists an elliptic generalized Toeplitz operator T_Q of order $-\text{ord}(T_P)$, with $\sigma(T_Q) = \sigma(T_P)^{-1}$, such that $T_P T_Q - I$ and $T_Q T_P - I$ are smoothing operators (i.e. have Schwartz kernel in $C^\infty(\partial\Omega \times \partial\Omega)$).

Note that from (P3) and (P5) we obtain, in particular, that

$$(4) \quad \text{ord}[T_P, T_Q] \leq \text{ord}(T_P) + \text{ord}(T_Q) - 1.$$

For an elliptic generalized Toeplitz operator T_P of order $m > 0$ or $m < 0$ which is positive selfadjoint as an operator on $H^2(\partial\Omega)$, it again follows from (P1), (P6) and the result of Seeley recalled above that the complex powers T_P^z , $z \in \mathbb{C}$, defined by the spectral theorem, are elliptic generalized Toeplitz operators of order mz , with symbol $\sigma(T_P)^z$ (see Proposition 16 in [10] for the details); and, likewise, the space $\mathcal{H}_{T_P}$ defined as the completion of $C_{\text{hol}}^\infty(\partial\Omega)$ with respect to the norm

$$(5) \quad \|u\|_{T_P}^2 := \langle T_P u, u \rangle_{\partial\Omega} = \|T_P^{1/2} u\|_{\partial\Omega}^2$$

coincides with $W_{\text{hol}}^{m/2}(\partial\Omega)$, with equivalent norms. The corresponding space

$$\mathbf{K}\mathcal{H}_{T_P} := \{\mathbf{K}u : u \in \mathcal{H}_{T_P}\}$$

of holomorphic functions on Ω thus coincides with

$$(6) \quad \mathbf{K}\mathcal{H}_{T_P} = \mathbf{K}W_{\text{hol}}^{m/2}(\partial\Omega) = W_{\text{hol}}^{(m+1)/2}(\Omega),$$

with equivalent norms.

We conclude this section with a simple criterion for Schatten class membership of generalized Toeplitz operators.

Proposition 1. *A generalized Toeplitz operator T_Q of order $-q$ on $\partial\Omega$, $q > 0$, belongs to $\mathcal{S}^p$ for all $p > n/q$, $n = \dim_{\mathbb{C}} \Omega$.*

Proof. Choose a positive selfadjoint elliptic generalized Toeplitz operator of order -1 on $\partial\Omega$ with positive symbol, for instance, T_{Λ} where $\Lambda = \mathbf{K}^* \mathbf{K}$, cf. the beginning of the next section. Then $T_{\Lambda}^{-q} T_Q$ is a bounded operator; since $\mathcal{S}^p$ is an ideal, it therefore suffices to show that $T_{\Lambda}^q \in \mathcal{S}^p$ for p as indicated.

To prove the latter, we proceed as in Theorem 3 in [13]: namely, let $0 < \lambda_1 \leq \lambda_2 \leq \dots$ be the eigenvalues of T_{Λ}^{-1} , counting multiplicities, and denote

$$N(\lambda) = \text{card}\{j : \lambda_j < \lambda\}$$

the corresponding counting function. By Theorem 13.1 in [4],

$$N(\lambda) = c\lambda^n + O(\lambda^{n-1}) \quad \text{as } \lambda \rightarrow +\infty$$

with some positive constant c ; implying that

$$\frac{1}{\lambda} = \left(\frac{N(\lambda)}{c} \right)^{-1/n} [1 + O(N(\lambda)^{-1/n})].$$

Consequently,

$$\begin{aligned} \|T_{\Lambda}^q\|_{\mathcal{S}^p}^p &= \sum_{j=1}^{\infty} \lambda_j^{-pq} = \int_{[\lambda_1, \infty)} \lambda^{-pq} dN(\lambda) \\ &= \int_1^{\infty} \left(\frac{c}{N} \right)^{pq/n} [1 + O(N^{-1/n})] dN, \end{aligned}$$

which is finite for $\frac{pq}{n} > 1$, i.e. for $p > \frac{n}{q}$. $\square$

From the last proof one can in fact show that T_{Λ}^q and, hence, T_Q belongs to the ideal $\mathcal{S}^{n/q, \infty}$ of operators T whose singular numbers satisfy $s_j(T) = O(j^{-q/n})$ as $j \rightarrow \infty$, and which is properly contained in $\mathcal{S}^p$ for all $p > \frac{n}{q}$.

3. SOBOLEV-BERGMAN SPACES AND RESTRICTIONS

The Poisson operator $\mathbf{K}$ is in particular bounded from $L^2(\partial\Omega)$ into $L^2(\Omega)$, and we denote by $\mathbf{K}^* : L^2(\Omega) \rightarrow L^2(\partial\Omega)$ its adjoint. Operators of the form

$$\Lambda_w := \mathbf{K}^* w \mathbf{K},$$

where w is a function on Ω , are governed by a calculus developed by Boutet de Monvel [2]. Namely, for w of the form

$$w = \rho^{\alpha} g, \quad \alpha > -1, \quad g \in C^{\infty}(\overline{\Omega}),$$

Λ_w is an operator in $\Psi^{-\alpha-1}$, with principal symbol

$$\sigma(\Lambda_w)(x, \xi) = \frac{\Gamma(\alpha+1)}{2|\xi|^{\alpha+1}} g(x) \|\eta_x\|^{\alpha}.$$

In particular, we obtain that $\Lambda := \Lambda_1 = \mathbf{K}^* \mathbf{K}$ is an elliptic operator in Ψ^{-1} , and more generally, $\Lambda_{\rho^\alpha} = \mathbf{K}^* \rho^\alpha \mathbf{K}$ is an elliptic operator in $\Psi^{-\alpha-1}$, for any $\alpha > -1$. From the simple computation

$$(7) \quad \begin{aligned} \int_{\Omega} |\mathbf{K}u|^2 w \, dz &= \langle w \mathbf{K}u, \mathbf{K}u \rangle_{\Omega} \\ &= \langle \Lambda_w u, u \rangle_{\partial\Omega} \\ &= \langle T_{\Lambda_w} u, u \rangle_{\partial\Omega}, \end{aligned}$$

valid for any $u \in C_{\text{hol}}^\infty(\partial\Omega)$, and (6) we thus see that the space

$$A_{\alpha, \rho}^2(\Omega) := L_{\text{hol}}^2(\Omega, \rho^\alpha \, dz), \quad \alpha > -1,$$

coincides with $W_{\text{hol}}^{-\alpha/2}(\Omega)$, with equivalent norms, independently of the choice of the defining function ρ . This suggests extending the definition of the spaces $A_{\alpha\rho}^2$ in this manner to all real α : namely, let us introduce the notation

$$A_{\alpha*}^2 := W_{\text{hol}}^{-\alpha/2}(\Omega), \quad \alpha \in \mathbb{R}.$$

It was shown by Beatrous [7] for smoothly bounded strictly pseudoconvex domains Ω in a Stein manifold W that there exist many equivalent norms on $A_{\alpha*}^2$, beside the Sobolev norm inherited from $W^{-\alpha/2}(\Omega)$. Namely, if m is a nonnegative integer and $m > -\frac{\alpha+1}{2}$, then $f \in A_{\alpha*}^2$ if and only if $\partial^\nu f$ belongs to $A_{\alpha+2m, \rho}^2$ for all multiindices ν with $|\nu| \leq m$, and the norm in $A_{\alpha*}^2$ is equivalent to

$$(8) \quad \|f\|_{\alpha\#m\rho} := \left(\sum_{|\nu| \leq m} \|\partial^\nu f\|_{\alpha+2m, \rho}^2 \right)^{1/2}.$$

Furthermore, in fact one need not consider all the derivatives in (8), but only “radial” ones: namely, if $\mathcal{D}$ is the holomorphic vector field on Ω given by

$$\mathcal{D} := \sum_{j=1}^n (\bar{\partial}_j \rho) \partial_j,$$

then a holomorphic function f belongs to $A_{\alpha*}^2$ if and only if $\mathcal{D}^j f \in L_{\alpha+2m, \rho}^2$ for all $0 \leq j \leq m$, and

$$(9) \quad \|f\|_{\alpha b m \rho} := \left(\sum_{j=0}^m \|\mathcal{D}^j f\|_{\alpha+2m, \rho}^2 \right)^{1/2}$$

is an equivalent norm in $A_{\alpha*}^2$. Here $L_{\alpha\rho}^2 := L^2(\Omega, \rho^\alpha \, dz)$ for $\alpha > -1$, and we use $\|\cdot\|_{\alpha\rho}$ to denote the norm in $L_{\alpha\rho}^2$. Again, both in (8) and in (9), ρ can be an arbitrary defining function, and different choices of ρ lead to equivalent norms. We remark that a proof of all the above facts can be given based on (6) and the machinery of generalized Toeplitz operators reviewed in the preceding section (which is completely different from the methods used in [7]): namely, one checks that the norms in (8) and (9) are special cases of the norm (5), with

$$(10) \quad P = P_{\alpha\#m\rho} := \sum_{|\nu| \leq m} \mathbf{K}^* \bar{\partial}^\nu \rho^{\alpha+2m} \partial^\nu \mathbf{K}$$

and

$$(11) \quad P = P_{\alpha \flat m, \rho} := \sum_{j=0}^m \mathbf{K}^* \mathcal{D}^{*j} \rho^{\alpha+2m} \mathcal{D}^j \mathbf{K},$$

respectively, and that P is a positive selfadjoint elliptic Ψ DO on $\partial\Omega$ of order $-\alpha-1$; see Sections 5–7 in [10] for the details.

Finally, [7] also gives a result concerning restrictions of functions in $A_{\alpha*}^2$ to complex submanifolds which intersect $\partial\Omega$ transversally. (In fact [7] treats even the case of L^p -Sobolev spaces of holomorphic functions for any $p > 0$, not only $p = 2$.) Namely, if V is such a submanifold in a neighbourhood of $\overline{\Omega}$, then Corollary 1.7 in [7] asserts that the restriction map

$$(12) \quad R_V : f \longmapsto f|_V$$

actually sends each $A_{\alpha*}^2(\Omega)$ continuously onto $A_{\alpha+k,*}^2(\Omega \cap V)$:

$$(13) \quad R_V A_{\alpha*}^2(\Omega) = A_{\alpha+k,*}^2(\Omega \cap V) \quad \text{continuously,} \quad \forall \alpha \in \mathbb{R},$$

where

$$k = n - \dim_{\mathbb{C}} V$$

is the codimension of V in the n -dimensional Stein manifold W .

We will need a somewhat more precise information on the nature of the restriction operator R_V and its relationships to the inner products like (2), (8), (9) on $A_{\alpha*}^2(\Omega)$ and $A_{\alpha+k,*}^2(\Omega \cap V)$. To that end, we now review some properties of the Szegő projection $\Pi : L^2(\partial\Omega) \rightarrow H^2(\partial\Omega)$ due to Boutet de Monvel and Sjöstrand [5].

Recall that a *Fourier integral distribution* is an integral of the form

$$(14) \quad u(x) = I(a, \phi)(x) := \int e^{i\phi(x, \theta)} a(x, \theta) d\theta.$$

Here a is a (classical) symbol in the Hörmander class $S^m(U \times \mathbb{R}^N)$, $U \subset \mathbb{R}^n$, $m \in \mathbb{R}$, and $\phi \in C^\infty(U \times \dot{\mathbb{R}}^N)$, $\dot{\mathbb{R}}^N := \mathbb{R}^N \setminus \{0\}$, is a *nondegenerate phase function*, meaning that ϕ is real-valued, $\phi(x, \lambda\theta) = \lambda\phi(x, \theta)$ for $\lambda > 0$, $d_{(x, \theta)}\phi \neq 0$, and $d_{(x, \theta)}\frac{\partial\phi}{\partial\theta_j}$, $j = 1, \dots, N$, are linearly independent on the set where $d_\theta\phi = 0$. The integral (14) converges absolutely when $m < -N$, and can be defined as a distribution on U for any real m . The image Λ_ϕ of the set $\{(x, \theta) : d_\theta\phi(x, \theta) = 0\}$ under the map $(x, \theta) \mapsto (x, d_x\phi(x, \theta))$ is then a conical Lagrangian submanifold of $T^\bullet U := T^*U \setminus \{0\}$, the cotangent bundle of U with zero section removed. The set of all distributions of the form (14) turns out to depend not on ϕ but only on Λ_ϕ , modulo smooth functions: namely, if $\psi \in C^\infty(U \times \dot{\mathbb{R}}^M)$ is another phase function such that $\Lambda_\phi = \Lambda_\psi$ in a neighbourhood of $(x_0, \xi_0) \in T^\bullet U$, and $a \in S^m(U \times \mathbb{R}^N)$ is supported in a small conical neighbourhood of (x_0, ξ_0) , then there exists $b \in S^{m'}(U \times \mathbb{R}^M)$, where $m' = m + (M - N)/2$, supported in a small conical neighbourhood of (x_0, ξ_0) , such that $I(a, \phi) - I(b, \psi) \in C^\infty(U)$. Moreover, if a is elliptic, then so is b . Given a conical Lagrangian submanifold Λ of $T^\bullet U$, one can therefore unambiguously define the space of associated Fourier integral distributions

$$(15) \quad I^m(U, \Lambda) := \{u = I(a, \phi) + v \text{ locally, } \Lambda_\phi = \Lambda, a \in S^{m - \frac{N}{2} + \frac{n}{4}}, v \in C^\infty\},$$

and its subset I_{ell}^m with a elliptic. (The reason for the shift by $\frac{n}{4}$ will become apparent in a moment.) The whole construction carries over in a straightforward manner from subsets $U \subset \mathbb{R}^n$ also to real manifolds of dimension n .

If X, Y are two compact real manifolds, and Λ is a conical Lagrangian submanifold of $T^\bullet X \times T^\bullet Y \subset T^\bullet(X \times Y)$, an operator from $C^\infty(X)$ into $\mathcal{D}'(Y)$ whose distributional (Schwartz) kernel belongs to $I^m(Y \times X, \Lambda)$ is called a Fourier integral operator (FIO for short) of order m . The set

$$C := \{(x, \xi, y, -\zeta) : ((x, y), (\xi, \zeta)) \in \Lambda\} \subset T^\bullet X \times T^\bullet Y$$

is called the “canonical relation” corresponding to Λ , and we denote the space of all FIOs (with classical symbols) from X into Y of order m and with canonical relation C by $I^m(X, Y, C)$, and by $I_{\text{ell}}^m(X, Y, C)$ its subset of elliptic elements. The L^2 adjoint (with respect to some smooth volume elements on X and Y) of an operator $A \in I^m(X, Y, C)$ belongs to $I^m(Y, X, C^t)$, where $C^t = \{(y, \zeta, x, \xi) : (x, \xi, y, \zeta) \in C\}$. If X, Y, Z are three compact real manifolds and $A_1 \in I^{m_1}(X, Y, C_1)$, $A_2 \in I^{m_2}(Y, Z, C_2)$, where C_1 and C_2 intersect nicely⁴, then

$$(16) \quad A_2 A_1 \in I^{m_1+m_2}(X, Z, C_1 \circ C_2),$$

and similarly for I^m replaced by I_{ell}^m . (This composition law — i.e. that $I^{m_1} \circ I^{m_2} \subset I^{m_1+m_2}$ — is the reason for the shift by $\frac{n}{4}$ in (15).)

Pseudodifferential operators are special case of FIOs corresponding to the phase function $\phi(x, y, \theta) = (x - y) \cdot \theta$; thus $C = \text{diag } T^\bullet X$ is the identity relation $\{(x, \xi, x, \xi) : (x, \xi) \in T^\bullet X\}$, and Ψ DOs can be composed with any FIO, yielding a FIO with the same canonical relation as the original FIO.

Finally, up to a number of technicalities which we will not go into here (see the references mentioned below for the details), the calculus of FIOs extends also to complex valued phase functions ϕ with $\text{Im } \phi \geq 0$. The technicalities stem from the fact that the set $\{(x, \theta) : d_\theta \phi(x, \theta) = 0\}$ is no longer a (real) manifold in $U \times \mathbb{R}^N$ in general, and needs to be replaced, roughly speaking, by the “real part” of its “almost analytic” complex extension; the same applies to the conical Lagrangian manifolds Λ_ϕ and canonical relations C . With these modifications, the whole formalism of Fourier integral distributions and FIOs just described remains in force also for complex-valued phase functions.

The reader is referred e.g. to Grigis and Sjöstrand [16], Hörmander [19] (Chapter 25), Melin and Sjöstrand [22] and Treves [25] (Chapters VIII and X) for full accounts of the theory of FIOs with real as well as complex valued phase functions.

The main result of [5] then says that for any smoothly-bounded strictly pseudoconvex domain Ω as in Section 2, the Szegő kernel $S(x, y)$ is a Fourier integral distribution in $I^0(\partial\Omega \times \partial\Omega, \text{diag } \Sigma_{\partial\Omega})$, and the Szegő projection $\Pi : L^2(\partial\Omega) \rightarrow H^2(\partial\Omega)$ is an elliptic FIO with complex valued phase function in $I_{\text{ell}}^0(\partial\Omega, \partial\Omega, \text{diag } \Sigma_{\partial\Omega})$, where

$$\begin{aligned} \text{diag } \Sigma_{\partial\Omega} &= \{(\Upsilon, \Upsilon) : \Upsilon \in \Sigma_{\partial\Omega}\} \\ &= \{(x, t\eta_x, x, t\eta_x) : x \in \partial\Omega, t > 0\} \subset T^\bullet(\partial\Omega) \times T^\bullet(\partial\Omega), \end{aligned}$$

⁴In detail: if Δ denotes the “diagonal” $\Delta = \{(a, b, c) : a \in T^\bullet X, b \in T^\bullet Y, c \in T^\bullet Z\}$, then (i) $C_1 \times C_2$ should intersect Δ transversally (i.e. the sum of the tangent spaces should be equal to the full tangent space of $T^\bullet X \times (T^\bullet Y)^2 \times T^\bullet Z$ at each point of intersection), and (ii) the natural projection $(C_1 \times C_2) \cap \Delta \rightarrow T^\bullet(X \times Z)$ should be injective and proper; its image is denoted $C_1 \circ C_2$.

with $\Sigma = \Sigma_{\partial\Omega}$ as in (3). More specifically, one has⁵

$$(17) \quad S(x, y) = \int_0^\infty e^{-\theta\rho(x, y)} a(x, y, \theta) d\theta,$$

where a is an elliptic symbol in $S^{n-1}(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}_+)$ and $\rho(x, y)$ is an “almost analytic” extension of the defining function ρ , namely $\rho(\cdot, \cdot) \in C^\infty(\overline{\Omega} \times \overline{\Omega})$ satisfies $\rho(x, x) = \rho(x)$, $\rho(y, x) = \overline{\rho(x, y)}$, while $\bar{\partial}_x \rho(x, y), \partial_y \rho(x, y)$ vanish to infinite order on the diagonal $x = y$, and $2\operatorname{Re} \rho(x, y) \geq \rho(x) + \rho(y) + c|x - y|^2$ for all $x, y \in \overline{\Omega}$ for some $c > 0$. It follows that an (elliptic) generalized Toeplitz operator T_P on $\partial\Omega$ of order m is an (elliptic) FIO in $I^m(\partial\Omega, \partial\Omega, \operatorname{diag} \Sigma_{\partial\Omega})$, and in fact generalized Toeplitz operators of order m on $\partial\Omega$ are precisely those operators $A \in I^m(\partial\Omega, \partial\Omega, \operatorname{diag} \Sigma_{\partial\Omega})$ for which $A = \Pi A \Pi$ ($= \Pi A = A \Pi$); see [6], p. 21, §v.⁶

After these preparations, we can state an observation which in some sense is our main result of this section. Let V be a complex submanifold in a neighbourhood of $\overline{\Omega}$ which intersects $\partial\Omega$ transversally, $k = n - \dim_{\mathbb{C}} V$ and let R_V be the restriction operator (12). We will denote the Szegő projections on $\partial\Omega$ and $\partial(\Omega \cap V) = \partial\Omega \cap V$ by Π and Π_V , respectively, and similarly by ρ and ρ_V the respective defining functions as well as their almost-analytic extensions (thanks to the transversality hypothesis, one can take $\rho_V = \rho|_{V \times V}$, and we will assume this from now on), by $\eta = \frac{1}{2i}(\bar{\partial}\rho - \partial\rho)|_{\partial\Omega}$ and $\eta^V = \frac{1}{2i}(\bar{\partial}\rho_V - \partial\rho_V)|_{\partial\Omega \cap V}$ the corresponding one-forms on the boundary, $\mathbf{K}$ and $\mathbf{K}_V$ the respective Poisson operators, etc. Finally, we denote by

$$R_{\partial V} := \gamma_V R_V \mathbf{K} : u \mapsto u|_{\partial\Omega \cap V}$$

the action of R_V on boundary values.

Proposition 2. $R_{\partial V} \Pi = \Pi_V R_{\partial V} \Pi$ is an elliptic FIO from $\partial\Omega$ to $\partial\Omega \cap V$ of order $k/2$, with canonical relation⁷

$$(18) \quad \Sigma_{\partial\Omega|V} := \{(x, t\eta_x, x, t\eta_x^V) : x \in \partial\Omega \cap V, t > 0\} \subset T^\bullet(\partial\Omega) \times T^\bullet(\partial\Omega \cap V)$$

(the “restriction” of $\operatorname{diag} \Sigma_{\partial\Omega}$ to $T^\bullet(\partial\Omega) \times T^\bullet(\partial\Omega \cap V)$).

Furthermore, if T is a generalized Toeplitz operator on $\partial\Omega$ of order $s \in \mathbb{R}$, then $(R_{\partial V} \Pi) T (R_{\partial V} \Pi)^*$ is a generalized Toeplitz operator on $\partial\Omega \cap V$ of order $s+k$, which is elliptic if T is.⁸

⁵For $\Omega = \mathbb{B}^d$ with $\rho(x, y) = 1 - \langle x, y \rangle$, one has simply $a(x, y, \theta) = \theta^{d-1}/\lambda(\partial\mathbb{B}^d)$.

⁶Page 253 in [3] gives a construction of an operator H from $L^2(\mathbb{R}^n)$ onto the Hardy space such that $H^*H = I$ while $HH^* = \Pi$; H is in fact a FIO of order 0. An operator T satisfying $T = \Pi T \Pi$ then equals $T = HQH^*$ where $Q = H^*TH$. Now the paragraph before (1.7) on the same page 253 of [3] outlines a proof that any such Q can be obtained as $Q = H^*PH$ for some Ψ DO P . It follows that $T = \Pi P \Pi = T_P$ is a generalized Toeplitz operator, as claimed.

⁷More precisely, (18) is the “real part” of the canonical relation.

⁸Throughout the paper, unless explicitly stated otherwise, the adjoint X^* of an operator X acting on a Hilbert space of functions on a domain Ω or its boundary $\partial\Omega$ (or between two such spaces) is always meant with respect to the L^2 products on Ω or $\partial\Omega$. This is in line with the standard convention in Ψ DO theory — if X is a Ψ DO of order m , then X^* is also of order m , and both X and X^* (sic!) map W^s into W^{s-m} for any real s . (Normally, the adjoint of $X : W^s \rightarrow W^{s-m}$ would be $X^* : W^{s-m} \rightarrow W^s$, the reason of course being that the latter adjoint is taken with respect to the W^s and W^{s-m} inner products and not with respect to the L^2 products. The only place where we will use the genuine, instead of L^2 , adjoints are the operators T^* in the proofs of Theorem 4 in Section 4 and of Main Theorem in Section 5.)

Strictly speaking, by the L^2 inner product above we also mean its extension to the duality pairing between W^s and W^{-s} , $s \in \mathbb{R}$, which coincides with the L^2 pairing when both arguments are smooth functions.

Note that the second part of the proposition would actually follow immediately by (16) if the canonical relations $\Sigma_{\partial\Omega|V}$, $\Sigma_{\partial\Omega|V}^t$ and $\text{diag } \Sigma_{\partial\Omega}$ intersected nicely. However this does not seem to be the case (unless $k = 0$); fortunately it is possible to give a direct proof.

Proof. Set temporarily, for brevity, $A := R_{\partial V}\Pi$. Since the restriction of a holomorphic function to a complex submanifold is again holomorphic, it is clear that $A = \Pi_V A$. Also by (17), the Schwartz kernel of A is simply the restriction

$$R_{\partial V, x}S(x, y) = \int_0^\infty e^{-\theta\rho(x, y)} a(x, y, \theta) d\theta, \quad x \in \partial\Omega \cap V, y \in \partial\Omega.$$

Comparing this with (14) shows that, just as (17), the right-hand side is a Fourier integral distribution, and, hence, A is a FIO, with phase function $i\theta\rho(x, y)$, $x \in \partial\Omega \cap V$, $y \in \partial\Omega$, and canonical relation given by (18). Since a is an elliptic symbol in S^{n-1} , the order of $R_{\partial V}\Pi$ is $n - 1 + \frac{\dim \mathbb{R}_+}{2} - \frac{\dim \partial\Omega + \dim \partial\Omega \cap V}{4} = n - 1 + \frac{1}{2} - \frac{(2n-1)+(2n-2k-1)}{4} = \frac{k}{2}$, proving the first part of the proposition.

For the second part, note that from the formulas

$$Af(x) = \int_{y \in \partial\Omega} f(y)S(x, y) d\lambda(y), \quad A^*g(y) = \int_{x \in \partial\Omega \cap V} g(x)S(y, x) d\lambda_V(x),$$

we get

$$ATA^*f(x) = \int_{y \in \partial\Omega} S(x, y)T_y \int_{x_1 \in \partial\Omega \cap V} f(x_1)S(y, x_1) d\lambda_V(x_1) d\lambda(y),$$

where the subscript y in T_y refers to the variable T is being applied to. Thus the Schwartz kernel of ATA^* is

$$\mathcal{K}_{ATA^*}(x, x_1) := \int_{y \in \partial\Omega} S(x, y)T_y S(y, x_1) d\lambda(y) = (TS_{x_1})(x)$$

by the reproducing property of the Szegő kernel, where $S_y(x) := S(x, y)$. Now we may assume that $T = T_P$ for some Ψ DO P of the same order which commutes with Π ; and by the standard symbol calculus for Ψ DOs (see, for instance, Theorem 4.2 in Hörmander [18]) we have quite generally

$$(19) \quad T_x \int_0^\infty e^{-\theta\rho(x, y)} a(x, y, \theta) d\theta = \int_0^\infty e^{-\theta\rho(x, y)} b(x, y, \theta) d\theta$$

where $b \in S^{m+s}(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}_+)$ if $a \in S^m(\overline{\Omega} \times \overline{\Omega} \times \mathbb{R}_+)$, and with b elliptic if a is elliptic. (See the proof of Theorem 5 in [11] for the details.) Restricting to $x, y \in \overline{\Omega} \cap V$ we thus see that ATA^* is a FIO, elliptic if T is elliptic, of order $s + k$ and with the same canonical relation as Π_V , i.e. $\text{diag } \Sigma_{\partial\Omega \cap V}$. Since $A = \Pi_V A$ and, hence, $ATA^* = \Pi_V ATA^* \Pi_V$, it therefore follows (see the end of the paragraph after (17) above) that ATA^* is a generalized Toeplitz operator on $\partial\Omega \cap V$, proving the second part of the proposition. $\square$

Corollary 3. *Under the same hypotheses as for the preceding proposition, $A = R_{\partial V}\Pi$ is bounded from $W^{-s/2}(\partial\Omega)$ into $W_{\text{hol}}^{-(s+k)/2}(\partial\Omega \cap V)$, for any $s \in \mathbb{R}$, and its range has finite codimension.*

In particular, $R_{\partial V} : W_{\text{hol}}^{-s/2}(\partial\Omega) \rightarrow W_{\text{hol}}^{-(s+k)/2}(\partial\Omega \cap V)$ and $R_V : A_{\alpha*}^2(\Omega) \rightarrow A_{\alpha+k,*}^2(\Omega \cap V)$ are bounded for any $s, \alpha \in \mathbb{R}$, with ranges of finite codimension.

Proof. Choose an invertible elliptic generalized Toeplitz operator T_P of order $\frac{s}{2}$ on $\partial\Omega$ (for instance, $\Lambda^{-s/2}$ with $\Lambda = \mathbf{K}^*\mathbf{K}$ as before), and similarly an invertible elliptic generalized Toeplitz operator T_Q of order $-\frac{k+s}{2}$ on $\partial\Omega \cap V$. Then by the last proposition and the properties of generalized Toeplitz operators, $T_Q A T_P T_P^* A^* T_Q^*$ is a generalized Toeplitz operator of order 0, hence, a bounded operator on $H^2(\partial\Omega \cap V)$. Since an operator X between Hilbert spaces is bounded if and only if XX^* is, $T_Q A T_P$ must be bounded from $H^2(\partial\Omega)$ into $H^2(\partial\Omega \cap V)$. By the mapping properties of generalized Toeplitz operators again, this means that A is bounded from $W_{\text{hol}}^{-s/2}(\partial\Omega)$ into $W_{\text{hol}}^{-(s+k)/2}(\partial\Omega \cap V)$, proving the first claim. Furthermore, since $T_Q A T_P T_P^* A^* T_Q^*$ is elliptic, by the property (P6) of generalized Toeplitz operators it has a parametrix and, hence, is a Fredholm operator on $H^2(\partial\Omega \cap V)$; thus it has (closed) range of finite codimension. Since again an operator X has range of finite codimension if and only if XX^* does, while T_P and T_Q are isomorphisms of $H^2(\partial\Omega)$ onto $W_{\text{hol}}^{-s/2}(\partial\Omega)$ and of $W_{\text{hol}}^{-(s+k)/2}(\partial\Omega \cap V)$ onto $H^2(\partial\Omega \cap V)$, respectively, the second claim about A follows. The second half of the corollary is immediate from the first. $\square$

From the results of Beatrous, we know that for Ω in $\mathbb{C}^n$ or in a Stein manifold, the restriction operator $R_V : A_{\alpha*}^2(\Omega) \rightarrow A_{\alpha+k,*}^2(\Omega \cap V)$ is actually onto. On the other hand, consider the situation when W is the tautological line bundle over the complex projective space $\mathbb{C}P^1$, i.e. $W = \{(\mathbb{C}z, cz) : c \in \mathbb{C}, z \in \mathbb{C}^2, z \neq 0\}$, let Ω be the unit disc bundle $\{(\mathbb{C}z, cz) \in W : \|cz\| < 1\}$, and take $V = \{(\mathbb{C}z, cz) \in W : c \in \mathbb{C}, z_1^2 - z_2^2 = 0\}$. Then $\Omega \cap V$ consists of the two fibers of Ω over the points $(1 : 1)$ and $(1 : -1)$ of $\mathbb{C}P^1$, i.e. two disjoint discs. The function equal to 0 on one disc and to 1 on the other one is holomorphic in $\Omega \cap V$, but cannot be the restriction to V of a holomorphic function f on Ω : any such f must be constant on the zero section of Ω (which is a compact complex submanifold of Ω), hence assumes one and the same value in the centers of the two discs that form $\Omega \cap V$. Thus finite codimension of the range of R_V is indeed the best one can get (in this example, the codimension is 1).

The statement of the last corollary should be contrasted with the situation for full Sobolev spaces W^s (instead of W_{hol}^s): there the restriction map R_V maps $W^s(\Omega)$ into $W^{s-(k/2)}(\Omega \cap V)$ (and $R_{\partial V}$ maps $W^s(\partial\Omega)$ into $W^{s-(k/2)}(\partial\Omega \cap V)$) only for $s > k/2$, by the Sobolev trace theorem; whereas for the subspaces W_{hol}^s of holomorphic functions this holds for all real s . This is completely parallel to what happens for the “boundary-value” operator γ from Section 2, which maps $W^s(\Omega) \rightarrow W^{s-1/2}(\partial\Omega)$ only for $s > 1/2$, but $W_{\text{hol}}^s(\Omega) \rightarrow W_{\text{hol}}^{s-1/2}(\partial\Omega)$ and $W_{\text{harm}}^s(\Omega) \rightarrow W^{s-1/2}(\partial\Omega)$ for any real s .

Finally, note that when R_V is onto, then $R_V R_V^*$ must be invertible (by Banach’s inverse mapping theorem), and $R_V^*(R_V R_V^*)^{-1}$, being a right inverse to R_V , is then a bounded extension operator from $A_{\alpha+k,*}^2(\Omega \cap V)$ into $A_{\alpha*}^2(\Omega)$, for any $\alpha \in \mathbb{R}$.

4. THE CASE OF A SMOOTH SUBMANIFOLD

Throughout the rest of this paper, the space $A_{\alpha\circ}^2$ on $\mathbb{B}^d$, $\alpha \in \mathbb{R}$, will always be understood to be equipped with a norm of the form

$$(20) \quad \|f\|^2 = \langle T_Y \gamma f, \gamma f \rangle_{\partial \mathbb{B}}$$

for some positive selfadjoint elliptic generalized Toeplitz operator T_Y on $\partial \mathbb{B}^d$ of order $-\alpha-1$. In particular, this includes the weighted Bergman spaces $L_{\text{hol}}^2(\mathbb{B}^d, \rho^\alpha dz)$ for $\alpha > -1$, with $Y = \Lambda_{\rho^\alpha}$ (cf. (7)), the Hardy space on $\partial \mathbb{B}^d$ (with Y the identity operator), or any of the Sobolev norms (8) or (9), for any $\alpha \in \mathbb{R}$ (with Y given by (10) and (11), respectively). It also includes the original norms $\|f\|_{\alpha\circ}$ from (2): namely, by computing the inner products $\langle z^\nu, z^\mu \rangle$, where μ, ν are two multiindices, in the Hardy space $H^2(\partial \mathbb{B}^d)$ and in the unweighted Bergman space $L_{\text{hol}}^2(\mathbb{B}^d)$ and comparing the results, one sees that T_Λ , $\Lambda = \mathbf{K}^* \mathbf{K}$, is the operator on $H^2(\partial \mathbb{B}^d)$ diagonalized by the monomial basis $\{z^\nu\}_\nu$ with eigenvalues

$$T_\Lambda z^\nu = \frac{1}{2(|\nu| + d)} z^\nu.$$

Thus if $F \in C^\infty(\mathbb{R})$ is a function satisfying

$$F(2x + 2d) = \frac{\Gamma(x + d)}{\Gamma(x + 1)(x + 1)^{d+\alpha}} \quad \text{for } x \geq 0, \quad F(x) = 0 \quad \text{for } x \leq -1,$$

then the operator $B := F(T_\Lambda^{-1})$ (defined by the functional calculus for selfadjoint operators) satisfies

$$\langle B z^\nu, z^\mu \rangle_{\partial \mathbb{B}} = \langle z^\nu, z^\mu \rangle_{\alpha\circ}.$$

Thus the norm (2) is of the form (20) with B in the place of T_Y . Now by the property (P1) of generalized Toeplitz operators, there exists an elliptic Ψ DO Q of order -1 , commuting with Π , such that $T_Q = T_\Lambda$. By elementary properties of the functional calculus and since $\Pi Q = Q \Pi$, this implies $B = F(T_Q^{-1}) = T_{F(Q^{-1})}$. Finally, the function F is easily seen to belong to the Hörmander class $S^{-\alpha-1}(\mathbb{R})$, so by the classical result of Strichartz [24], $F(Q^{-1})$ is also an elliptic Ψ DO, of order $-\alpha-1$. So taking $Y = F(Q^{-1})$ shows that the norm (2) is of the form (20), as claimed. (Note that for $\alpha = -d$ this includes, in particular, also the Drury-Arveson norm $\|f\|_{DA}$ that interests us most of all.) A slight modification of this construction (with $F(2x + 2d) = \Gamma(x + d)/\Gamma(x + d + \alpha + 1)$ for $x \geq 0$) likewise shows that (20) includes also the norms in $A_\alpha^2(\mathbb{B}^d)$ for all $\alpha > -d - 1$.

The proof of the result below uses the same ideas as that of our main theorem in the next section, but is somewhat simpler.

Theorem 4. *Let V be a complex submanifold of $\mathbb{C}^d$ of dimension n that intersects $\partial \mathbb{B}^d$ transversally, $\alpha \in \mathbb{R}$, and $\mathcal{M}$ the subspace in $A_{\alpha\circ}^2$ of functions vanishing on $V \cap \mathbb{B}^d$. Then $[S_j, S_k^*] \in \mathcal{S}^q$, $j, k = 1, \dots, d$, for all $q > n$.*

Proof. Let $R_V : f \mapsto f|_V$ be the restriction map (12) for $\Omega = \mathbb{B}^d$. Then $\mathcal{M} = \text{Ker } R_V$, and by the result of Beatrous in [7], we know that R_V maps $A_{\alpha\circ}^2$ boundedly onto $A_{\alpha+k,*}^2(\mathbb{B}^d \cap V)$, $k = d - n$. The restriction of R_V to $\mathcal{M}^\perp = (\text{Ker } R_V)^\perp$ is thus an injective map of $\mathcal{M}^\perp$ onto $A_{\alpha+k,*}^2(\mathbb{B}^d \cap V)$. Keeping the notations from the end

of Section 3, let T_X be a positive selfadjoint elliptic generalized Toeplitz operator of order $-\alpha-k-1$ on $\partial(\mathbb{B}^d \cap V) = \partial\mathbb{B}^d \cap V$ so that $\langle T_X \gamma_V f, \gamma_V g \rangle_{\partial\mathbb{B}^d \cap V}$ is an equivalent inner product in $A_{\alpha+k,*}^2(\mathbb{B}^d \cap V)$ (for instance, T_X can be one of the operators (10) or (11) corresponding to the inner products (8) and (9), respectively). Similarly, as discussed at the beginning of this section, let T_Y be a positive selfadjoint elliptic generalized Toeplitz operator of order $-\alpha-1$ on $\partial\mathbb{B}^d$ such that $\langle T_Y \gamma f, \gamma g \rangle_{\partial\mathbb{B}}$ is the inner product in $A_{\alpha o}^2$. The composed map

$$T := T_X^{1/2} \gamma_V R_V = T_X^{1/2} R_{\partial V} \gamma : A_{\alpha o}^2 \rightarrow H^2(\partial\mathbb{B}^d \cap V)$$

then satisfies $\text{Ker } T = \mathcal{M}$, maps boundedly $A_{\alpha o}^2$ onto $H^2(\partial\mathbb{B}^d \cap V)$ and is an isomorphism of $\mathcal{M}^\perp$ onto $H^2(\partial\mathbb{B}^d \cap V)$. Let $T^* : H^2(\partial\mathbb{B}^d \cap V) \rightarrow A_{\alpha o}^2$ be its adjoint. By abstract operator theory, TT^* is then invertible and for all $f, g \in A_{\alpha o}^2$,

$$(21) \quad \langle P_{\mathcal{M}^\perp} f, g \rangle_{A_{\alpha o}^2} = \langle (TT^*)^{-1} T f, T g \rangle_{\partial\mathbb{B}^d \cap V}.$$

(Indeed, the restriction τ of T to $\mathcal{M}^\perp = (\text{Ker } T)^\perp$ is injective and onto, hence invertible; as $TT^* = \tau\tau^*$, the invertibility of TT^* follows. As for (21), both sides vanish if f or g belongs to $\mathcal{M} = \text{Ker } T$; while for $f, g \in \mathcal{M}^\perp$, the right-hand side coincides with $\langle (\tau\tau^*)^{-1} \tau f, \tau g \rangle = \langle \tau^{*-1} \tau^{-1} \tau f, \tau g \rangle = \langle f, g \rangle$.) We claim that

$$(22) \quad TT^* = T_X^{1/2} T_Q T_X^{1/2} \quad \text{for an elliptic generalized Toeplitz operator } T_Q \text{ on } \partial\mathbb{B}^d \cap V \text{ of order } \alpha + k + 1.$$

To see this, note that for any $u \in H^2(\partial\mathbb{B}^d \cap V)$ and $f \in A_{\alpha o}^2$,

$$\begin{aligned} \langle T^* u, f \rangle_{A_{\alpha o}^2} &= \langle u, T f \rangle_{\partial\mathbb{B}^d \cap V} = \langle u, T_X^{1/2} R_{\partial V} \Pi \gamma f \rangle_{\partial\mathbb{B}^d \cap V} \\ &= \langle (R_{\partial V} \Pi)^* T_X^{1/2} u, \gamma f \rangle_{\partial\mathbb{B}}, \end{aligned}$$

while

$$\langle T^* u, f \rangle_{A_{\alpha o}^2} = \langle T_Y \gamma T^* u, \gamma f \rangle_{\partial\mathbb{B}}$$

by the definition of the inner product in $A_{\alpha o}^2$. Thus

$$T^* = \mathbf{K} T_Y^{-1} (R_{\partial V} \Pi)^* T_X^{1/2}$$

and

$$TT^* = T_X^{1/2} R_{\partial V} \Pi T_Y^{-1} (R_{\partial V} \Pi)^* T_X^{1/2}.$$

Since

$$T_Q := (R_{\partial V} \Pi) T_Y^{-1} (R_{\partial V} \Pi)^*$$

is an elliptic generalized Toeplitz operator of order $\alpha + k + 1$ by Proposition 2, (22) follows.

For the compressions S_j of M_{z_j} to $\mathcal{M}^\perp$, $j = 1, \dots, d$, we thus obtain by (21) and (22) for any $f, g \in \mathcal{M}^\perp$,

$$\begin{aligned} \langle S_j f, g \rangle_{\mathcal{M}^\perp} &= \langle P_{\mathcal{M}^\perp} M_{z_j} f, g \rangle_{A_{\alpha o}^2} \\ &= \langle (T_X^{1/2} T_Q T_X^{1/2})^{-1} T_X^{1/2} \gamma_V R_V M_{z_j} f, T g \rangle_{\partial\mathbb{B}^d \cap V} \\ &= \langle T_X^{-1/2} T_Q^{-1} \gamma_V R_V M_{z_j} f, T g \rangle_{\partial\mathbb{B}^d \cap V} \\ &= \langle T_X^{-1/2} T_Q^{-1} z_j \gamma_V R_V f, T g \rangle_{\partial\mathbb{B}^d \cap V} \\ &= \langle (T_X^{-1/2} T_Q^{-1} z_j T_X^{-1/2}) T f, T g \rangle_{\partial\mathbb{B}^d \cap V}, \end{aligned}$$

where, abusing notation, we have used z_j to denote also the operator of multiplication by (the boundary value of) the restriction $z_j|_{\partial\mathbb{B}^d \cap V}$ on $\partial\mathbb{B}^d \cap V$. Thus under the isomorphism $T : \mathcal{M}^\perp \rightarrow H^2(\partial\mathbb{B}^d \cap V)$,

$$S_j = T^* \mathcal{T}_j T$$

where

$$\mathcal{T}_j := T_X^{-1/2} T_Q^{-1} T_{z_j} T_X^{-1/2}$$

is a generalized Toeplitz operator on $\partial\mathbb{B}^d \cap V$ of order

$$\frac{\alpha + k + 1}{2} - (\alpha + k + 1) + 0 + \frac{\alpha + k + 1}{2} = 0.$$

Hence, for all $j, l = 1, \dots, d$, by an elementary computation

$$\begin{aligned} [S_j, S_l^*] &= [T^* \mathcal{T}_j T, T^* \mathcal{T}_l^* T] \\ &= T^* (\mathcal{T}_j [TT^*, \mathcal{T}_l^*] + [\mathcal{T}_j, \mathcal{T}_l^*] TT^* + \mathcal{T}_l^* [\mathcal{T}_j, TT^*]) T. \end{aligned}$$

As $\mathcal{T}_j$ (and, hence, $\mathcal{T}_l^*$) are generalized Toeplitz operators of order 0, and so is TT^* (by (22)), by (4) the last three commutators are generalized Toeplitz operators of order -1 (or less). Since $\mathcal{S}^p$ is an ideal and T is bounded, taking $q = 1$ and $\Omega = \mathbb{B}^d \cap V$ in Proposition 1 thus yields

$$[S_j, S_l^*] \in \mathcal{S}^p \quad \forall p > n,$$

proving the theorem. $\square$

5. PROOF OF MAIN THEOREM

Let now V be as in Main Theorem, i.e. a homogeneous variety in $\mathbb{C}^d$ such that $V \setminus \{0\}$ is a complex submanifold of $\mathbb{C}^d \setminus \{0\}$ of dimension n . Our idea is, loosely speaking, to proceed as in the preceding proof after removing (“blowing up”) the singularity of V at the origin.

For $z \in \mathbb{C}^d \setminus \{0\}$, denote by $\mathbb{C}z = \{cz : c \in \mathbb{C}\}$ the one-dimensional complex subspace through z ; the set of all such subspaces is the complex projective space $\mathbb{C}P^{d-1}$. The hypotheses on V mean precisely that $\mathcal{V} := \{\mathbb{C}z : 0 \neq z \in V\}$ is a complex submanifold of $\mathbb{C}P^{d-1}$ (of dimension $n-1$). Consider the tautological line bundle $\mathcal{L}$ over $\mathbb{C}P^{d-1}$, i.e. the fiber over a point $\mathbb{C}z \in \mathbb{C}P^{d-1}$ is the very same complex line $\mathbb{C}z$; in other words, $\mathcal{L}$ consists of all points $(\mathbb{C}z, cz) \in \mathbb{C}P^{d-1} \times \mathbb{C}^d$ of the form $(\mathbb{C}z, cz)$, $z \in \mathbb{C}^d \setminus \{0\}$, $c \in \mathbb{C}$. (The only role of c is to allow the second coordinate to be also 0.) Let $\mathcal{L}_V := \{(\mathbb{C}z, cz) : z \in V \setminus \{0\}, c \in \mathbb{C}\}$ be the part of $\mathcal{L}$ lying over $\mathcal{V}$. Finally let $\mathcal{B} := \{(\mathbb{C}z, cz) \in \mathcal{L} : \|cz\| < 1\}$ and $\Omega := \mathcal{B} \cap \mathcal{L}_V$ be the unit disc bundles of $\mathcal{L}$ and $\mathcal{L}_V$, respectively. Then $\mathcal{L}$ and $\mathcal{L}_V$ are complex manifolds of dimensions d and n , respectively, the subsets $\mathcal{B} \subset \mathcal{L}$ and $\Omega \subset \mathcal{L}_V$ are strictly-pseudoconvex domains with smooth boundary, and Ω contains no singular points — the origin has been blown up into the zero section of $\mathcal{L}_V$. The domains $\mathcal{B}$ and Ω will play the same roles as $\mathbb{B}^d$ and $\mathbb{B}^d \cap V$ did in the proof of Theorem 4.

The map π sending $(\mathbb{C}z, cz)$ into cz sends Ω back into $\mathbb{B}^d \cap V$ (and $\mathcal{B}$ into $\mathbb{B}^d$), and is bijective except for the zero section which is taken into the origin. This

map translates holomorphic functions on Ω into holomorphic functions on $\mathbb{B}^d \cap V$ (any such function must be constant on the zero section, as the latter, being a closed submanifold of $\mathbb{C}P^{d-1}$, is a compact complex manifold, so the translated function on $\mathbb{B}^d \cap V$ is single-valued at the origin). Similarly, $\mathcal{B}$ is mapped to $\mathbb{B}^d$, bijectively except for the zero section being mapped into the origin, and holomorphic functions on $\mathcal{B}$ correspond precisely to holomorphic functions on $\mathbb{B}^d$.

One defines harmonic functions on $\mathcal{B}$ as those annihilated by $\bar{\partial}^* \bar{\partial}$, where $\bar{\partial}^*$ is the formal adjoint of $\bar{\partial}$ with respect to some Hermitian metric on $\mathcal{L}$, for instance the Cartesian product of the Fubini-Study metric on $\mathbb{C}P^{d-1}$ and the Euclidean metric in the fibers; similarly for Ω . It should be noted that under the map π from the preceding paragraph, these functions do *not* correspond to harmonic functions on $\mathbb{B}^d$ (or $\mathbb{B}^d \cap V$).⁹ In particular, such pushforwards to $\mathbb{B}^d$ of harmonic functions on $\mathcal{B}$ can be multi-valued at the origin.

With these prerequisites, we now have the Poisson operator, Szegő projection, defining function, etc., on $\mathcal{B}$ (denoted by $\mathbf{K}$, Π , ρ and so on; for the two-variable defining function one can take the pullback $\rho((\mathbb{C}z, cz), (\mathbb{C}w, qw)) = 1 - \langle cz, qw \rangle$ under π of the two-variable defining function $\rho(x, y) = 1 - \langle x, y \rangle$ for $\mathbb{B}^d$), as well as the analogous objects on $\Omega = \mathcal{B} \cap \mathcal{L}_V$ (denoted $\mathbf{K}_V$, Π_V , $\rho_V = \rho|_{\mathcal{L}_V \times \mathcal{L}_V}$, and so on), together with the restriction operator R_V from $\mathcal{B}$ to Ω . (The role of the “ambient” manifold W from Section 2 is played by $\mathcal{L}$ for $\mathcal{B}$ and by $\mathcal{L}_V$ for Ω .) Note that from the above biholomorphism π of $\mathcal{B} \setminus \{\text{zero section}\}$ onto $\mathbb{B}^d \setminus \{0\}$ sending $\Omega \setminus \{\text{zero section}\}$ onto $\mathbb{B}^d \cap V \setminus \{0\}$, it follows that Ω (or, more precisely, $\mathcal{L}_V$) intersects $\partial\mathcal{B}$ transversally (since V intersects $\partial\mathbb{B}^d$ transversally, thanks to the homogeneity of V).

The biholomorphism π also identifies the spaces $A_{\alpha*}^2(\mathbb{B}^d)$ with $A_{\alpha*}^2(\mathcal{B})$, for any $\alpha \in \mathbb{R}$. Namely, using π to transport the Lebesgue measure on $\mathbb{C}^d \setminus \{0\}$ to $\mathcal{L} \setminus \{\text{zero section}\}$ (the resulting volume element on $\mathcal{L}$ actually coincides with the one induced by the Hermitian metric mentioned in the penultimate paragraph, provided the Fubini-Study metric has been normalized so that $\mathbb{C}P^{d-1}$ has volume one), the fact that π is a biholomorphism (except for the zero section being sent to the origin, but these are both of measure zero) implies that $f \mapsto f \circ \pi$ is a unitary map from $L^2(\mathbb{B}^d)$ onto $L^2(\mathcal{B})$ taking $L_{\text{hol}}^2(\mathbb{B}^d)$ unitarily onto $L_{\text{hol}}^2(\mathcal{B})$. Similarly, since we are taking for the defining function ρ on $\mathcal{B}$ the pullback under π of the standard defining function $\rho_{\mathbb{B}}(z) = 1 - |z|^2$ on $\mathbb{B}^d$, the last map acts unitarily from $L^2(\mathbb{B}^d, \rho_{\mathbb{B}}^\alpha)$ onto $L^2(\mathcal{B}, \rho^\alpha)$ for any $\alpha \in \mathbb{R}$, taking $A_{\alpha}^2(\mathbb{B}^d)$ unitarily onto $A_{\alpha}^2(\mathcal{B})$ for any $\alpha > -1$. By the same argument with $\partial\mathbb{B}^d$ and $\partial\mathcal{B}$ in the places of $\mathbb{B}^d$ and $\mathcal{B}$, respectively, the Hardy space $H^2(\partial\mathbb{B}^d)$ is mapped by π unitarily onto the Hardy space on $\partial\mathcal{B}$, and the observation in the preceding sentence means that (cf. (7)) the generalized Toeplitz operator $T_{\Lambda_{\rho_{\mathbb{B}}^\alpha}^{\mathbb{B}}}$, $\Lambda_w^{\mathbb{B}} = \mathbf{K}_{\mathbb{B}}^* w \mathbf{K}_{\mathbb{B}}$, on $\partial\mathbb{B}^d$ is mapped by composition with π to the generalized Toeplitz operator $T_{\Lambda_{\rho^\alpha}}$, $\Lambda_w = \mathbf{K}^* w \mathbf{K}$, on $\partial\mathcal{B}$ (even though $\Lambda_{\rho_{\mathbb{B}}^\alpha}^{\mathbb{B}}$ is *not* in general mapped into Λ_{ρ^α} , because, as was already remarked above, π does not preserve harmonic functions). Similarly, repeating the arguments from the beginning of Section 4, one can check that the various norms on $A_{\alpha*}^2(\mathbb{B}^d)$, $\alpha \in \mathbb{R}$, discussed there correspond under the composition with the biholomorphism π to norms of the form (20) with T_Y an appropriate generalized Toeplitz operator on $\partial\mathbb{B}^d$. Quite generally, one can arrive at this conclusion

⁹And there is little reason why they should, because in dimension greater than 1 biholomorphic maps do not preserve harmonicity in general (though they preserve holomorphy).

also using the characterization of generalized Toeplitz operators as FIOs with the canonical relation $\text{diag } \Sigma$ that commute with Π : indeed, as π is a biholomorphism, the composition with it is a FIO whose canonical relation takes $\Sigma_{\partial\mathbb{B}}$ isomorphically onto $\Sigma_{\partial\mathcal{B}}$, and, as we have seen above, intertwines the Szegő projections on $\partial\mathbb{B}^d$ and $\partial\mathcal{B}$; it follows that the conjugation $T \mapsto \pi^{-1}T\pi$ is a symbol-preserving map from $I^m(\partial\mathbb{B}^d, \partial\mathbb{B}^d, \text{diag } \Sigma_{\partial\mathbb{B}})$ onto $I^m(\partial\mathcal{B}, \partial\mathcal{B}, \text{diag } \Sigma_{\partial\mathcal{B}})$ that sends operators commuting with $\Pi_{\mathbb{B}}$ into those commuting with Π ; by the above criterion, it is thus an isomorphism from generalized Toeplitz operators on $\partial\mathbb{B}^d$ onto those on $\partial\mathcal{B}$.

From now on, we will therefore simply identify the spaces $A_{\alpha\circ}^2(\mathbb{B}^d)$, $A_{\alpha*}^2(\mathbb{B}^d)$, and so on, discussed in the beginning of Section 4, with the corresponding pullbacks $A_{\alpha*}^2(\mathcal{B})$ via π on $\mathcal{B}$, knowing that the inner products in the latter are again of the form (20) with $\partial\mathbb{B}^d$ replaced by $\partial\mathcal{B}$ and T_Y a suitable generalized Toeplitz operator on $\partial\mathcal{B}$; again, this includes in particular also the pullback $A_{-d,*}^2(\mathcal{B})$ under π of the Drury-Arveson space A_{-d}^2 on $\mathbb{B}^d$.

Proof of Main Theorem. As before, the restriction of R_V to $\mathcal{M}^\perp = (\text{Ker } R_V)^\perp \subset A_{\alpha\circ}^2(\mathbb{B}^d) \equiv A_{\alpha*}^2(\mathcal{B})$ is a continuous injective map of $\mathcal{M}^\perp$ into $A_{\alpha+k,*}^2(\Omega)$, which is now no longer onto in general but by Corollary 3 its range is (closed and) of finite codimension. Let again T_X be a positive selfadjoint elliptic generalized Toeplitz operator of order $-\alpha - k - 1$ on $\partial\Omega$ so that $\langle T_X \gamma_V f, \gamma_V g \rangle_{\partial\Omega}$ is the inner product in $A_{\alpha+k,*}^2(\Omega)$, and T_Y a positive selfadjoint elliptic generalized Toeplitz operator of order $-\alpha - 1$ on $\partial\mathcal{B}$ such that $\langle T_Y \gamma f, \gamma g \rangle_{\partial\mathcal{B}}$ is the inner product in $A_{\alpha*}^2(\mathcal{B}) \equiv A_{\alpha\circ}^2(\mathbb{B}^d)$. The composed map

$$T = T_X^{1/2} \gamma_V R_V = T_X^{1/2} R_{\partial V} \gamma : A_{\alpha\circ}^2 \rightarrow H^2(\partial\Omega)$$

then satisfies $\text{Ker } T = \mathcal{M}$, maps boundedly $A_{\alpha\circ}^2$ into $H^2(\partial\Omega)$, and induces an isomorphism from $\mathcal{M}^\perp$ onto a (closed) subspace $\mathcal{N}$ in $H^2(\partial\Omega)$ of finite codimension. Let $T^* : H^2(\partial\Omega) \rightarrow A_{\alpha\circ}^2$ be its adjoint. By abstract operator theory, the operator $G := (TT^*)|_{\mathcal{N}} \oplus I_{\mathcal{N}^\perp}$ on $\mathcal{N} \oplus \mathcal{N}^\perp = H^2(\partial\Omega)$ is then invertible and for all $f, g \in A_{\alpha\circ}^2$

$$(23) \quad \langle P_{\mathcal{M}^\perp} f, g \rangle_{A_{\alpha\circ}^2} = \langle G^{-1} T f, T g \rangle_{\partial\Omega}.$$

(Indeed, the restriction τ of T to $\mathcal{M}^\perp = (\text{Ker } T)^\perp$ is injective and maps onto $\text{Ran } T = \mathcal{N}$, hence is invertible as an operator from $\mathcal{M}^\perp$ onto $\mathcal{N}$; as $TT^* = \tau\tau^*$ and $\mathcal{N} = \text{Ran } T = \text{Ran } TT^* = (\text{Ker } TT^*)^\perp$, the invertibility of G follows. As for (23), both sides vanish if f or g belongs to $\mathcal{M} = \text{Ker } T$; while for $f, g \in \mathcal{M}^\perp$, the right-hand side reduces to $\langle (\tau\tau^*)^{-1} \tau f, \tau g \rangle = \langle f, g \rangle$.)

Arguing as in the proof of Theorem 4, we see from Proposition 2 that TT^* is an elliptic generalized Toeplitz operator on $\partial\Omega$ of order 0. Let T_H be its parametrix (guaranteed by the property (P6) of generalized Toeplitz operators); thus T_H is of order 0 (hence bounded) and $TT^*T_H - I$ is a generalized Toeplitz operator of order $-\infty$. By Proposition 1, $TT^*T_H - I \in \mathcal{S}^p$ for any $p > 0$; for brevity, let us temporarily denote an operator (not necessarily the same one at each occurrence) belonging to $\bigcap_{p>0} \mathcal{S}^p$ by $\mathcal{C}$. Since $G - TT^* = \mathcal{C}$ (as $I_{\mathcal{N}^\perp}$ has finite rank), we thus have $GT_H = (TT^* + \mathcal{C})T_H = TT^*T_H + \mathcal{C} = I + \mathcal{C}$, whence $T_H = G^{-1}(I + \mathcal{C}) = G^{-1} + \mathcal{C}$. Noting again that

$$TM_{z_j} = T_X^{1/2} \gamma_V R_V M_{z_j} = T_X^{1/2} \gamma_V M_{z_j} R_V = T_X^{1/2} T_{z_j} \gamma_V R_V = T_X^{1/2} T_{z_j} T_X^{-1/2} T,$$

we thus get from (23) for any $f, g \in \mathcal{M}^\perp$,

$$\begin{aligned}\langle S_j f, g \rangle_{\mathcal{M}^\perp} &= \langle P_{\mathcal{M}^\perp} M_{z_j} f, g \rangle_{A_{\alpha_0}^2} \\ &= \langle G^{-1} T M_{z_j} f, T g \rangle_{\partial\Omega} \\ &= \langle G^{-1} T_X^{1/2} T_{z_j} T_X^{-1/2} T f, T g \rangle_{\partial\Omega},\end{aligned}$$

that is,

$$S_j = \tau^* G^{-1} T_X^{1/2} T_{z_j} T_X^{-1/2} \tau = \tau^* (T_H + \mathcal{C}) T_X^{1/2} T_{z_j} T_X^{-1/2} \tau = \tau^* \mathcal{T}_j \tau + \mathcal{C}$$

(since $T_X^{1/2} T_{z_j} T_X^{-1/2}$, being a generalized Toeplitz operator of order 0, as well as τ are bounded), where

$$\mathcal{T}_j := T_H T_X^{1/2} T_{z_j} T_X^{-1/2}$$

is a generalized Toeplitz operator of order 0. For any $j, l = 1, \dots, d$, we thus obtain analogously as before

$$\begin{aligned}[S_j, S_l^*] &= [\tau^* \mathcal{T}_j \tau + \mathcal{C}, \tau^* \mathcal{T}_l^* \tau + \mathcal{C}] = [\tau^* \mathcal{T}_j \tau, \tau^* \mathcal{T}_l^* \tau] + \mathcal{C} \\ &= \tau^* (\mathcal{T}_j [\tau \tau^*, \mathcal{T}_l^*] + [\mathcal{T}_j, \mathcal{T}_l^*] \tau \tau^* + \mathcal{T}_l^* [\mathcal{T}_j, \tau \tau^*]) \tau + \mathcal{C}.\end{aligned}$$

Once again, the last three commutators are generalized Toeplitz operators of order -1 (or less) by (4), hence belong to $\mathcal{S}^p$ for all $p > n$ by Proposition 1, and since the Schatten class $\mathcal{S}^p$ forms an ideal in the algebra of all bounded operators, we get $[S_j, S_l^*] \in \mathcal{S}^p \forall p > n$, proving the main theorem. $\square$

Again, we have in fact proved that even $[S_j, S_l^*] \in \mathcal{S}^{n, \infty}$. Using the machinery of [13], it is not difficult to give e.g. a formula for the Dixmier trace of $[S_j, S_l^*]^n$.

We finally conclude by observing that our results can be extended also to the case when V is a disjoint union of smooth submanifolds (away from the origin) of possibly different dimensions.

Theorem 5. *Let $V_1, \dots, V_m$ be homogeneous varieties in $\mathbb{C}^d$ such that $V_j \setminus \{0\}$ is a complex submanifold of $\mathbb{C}^d \setminus \{0\}$ of dimension n_j , $j = 1, \dots, m$, and $V_j \cap V_k = \{0\}$ for $j \neq k$. Let $\alpha \in \mathbb{R}$ and let $\mathcal{M}$ be the subspace in $A_{\alpha_0}^2$ (or in A_α^2 if $\alpha > -d-1$) of functions that vanish on $V \cap \mathbb{B}^d := \bigcup_{j=1}^m V_j \cap \mathbb{B}^d$. Then $[S_j, S_k^*] \in \mathcal{S}^q$, $j, k = 1, \dots, d$, for all $q > \max(n_1, \dots, n_m)$.*

Proof. Let R_{V_j} be the restriction operator for $\mathcal{L}_{V_j} = \pi^{-1}(V_j)$ and T_{X_j} be the positive selfadjoint elliptic generalized Toeplitz operator of order $-\alpha - (d - n_j) - 1$ on $\partial\Omega_j$, $\Omega_j := \mathcal{L}_{V_j} \cap \mathcal{B}$, as in the preceding proof for V_j in the place of the V there, $j = 1, \dots, m$; and let also T_Y be as in the preceding proof. Denote by T the column block matrix with entries $T_j := T_{X_j}^{1/2} \gamma_{V_j} R_{V_j} = T_{X_j}^{1/2} R_{\partial V_j} \gamma$, $j = 1, \dots, m$; thus T acts continuously from $A_{\alpha_0}^2(\mathbb{B}^d) \equiv A_{\alpha_*}^2(\mathcal{B})$ into the Hilbert space direct sum $\mathcal{H} := \bigoplus_{j=1}^m H^2(\partial\Omega_j)$, and as before $\text{Ker } T = \mathcal{M}$. We have seen in the preceding proof that each $T_j T_j^*$ is a Fredholm operator on $H^2(\partial\Omega_j)$, by Proposition 2; on the other hand, from the proof of that proposition we also see that $T_j T_k^*$ for $j \neq k$ is an operator from $H^2(\partial\Omega_k)$ into $H^2(\partial\Omega_j)$ whose Schwartz kernel is in $C^\infty(\partial\Omega_j \times \partial\Omega_k)$: namely, the latter kernel is the restriction of $(T_{X_j}^{1/2} \otimes \tilde{T}_{X_k}^{1/2}) T_Y^{-1} S_x(y)$ to $x \in \partial\Omega_j$ and $y \in \partial\Omega_k$

(where the tensor product notation means that $T_{X_j}^{1/2}$ acts on the x variable and $\tilde{T}_{X_k}^{1/2}$ on the y variable, and for any operator A one defines $\tilde{A}f := \overline{A^*f}$, with bar denoting complex conjugation), and $T_Y^{-1}S_x(y)$ has singularities only on the diagonal $x = y$ by (19) while $\partial\Omega_j \cap \partial\Omega_k = \emptyset$ by hypothesis (also $T_{X_j}^{1/2} \otimes \tilde{T}_{X_k}^{1/2}$ maps $C^\infty(\partial\Omega_j \times \partial\Omega_k)$ into itself, in view of the way generalized Toeplitz operators act on Sobolev spaces). Thus $T_j T_k^*$ is a smoothing operator for $j \neq k$, and hence belongs to all $\mathcal{S}^p$, $p > 0$. Denoting temporarily by D the $m \times m$ block matrix with $T_j T_j^*$, $j = 1, \dots, m$, on the main diagonal and zeroes elsewhere, we therefore have

$$(24) \quad TT^* = D + \mathcal{C}$$

where $\mathcal{C}$ has the same meaning as in the preceding proof. It follows that TT^* is again a Fredholm operator, and the restriction τ of T to $\mathcal{M}^\perp$ is an isomorphism of $\mathcal{M}^\perp$ onto the (closed) subspace $\mathcal{N} := \text{Ran } T$ in $\mathcal{H}$ of finite codimension; the operator $G := (TT^*)|_{\mathcal{N} \oplus I_{\mathcal{N}^\perp}}$ on $\mathcal{N} \oplus \mathcal{N}^\perp = \mathcal{H}$ is invertible and for all $f, g \in A_{\alpha_0}^2$,

$$\langle P_{\mathcal{M}^\perp} f, g \rangle_{A_{\alpha_0}^2} = \langle G^{-1} T f, T g \rangle_{\mathcal{H}}.$$

Let T_{H_j} be a parametrix for $T_j T_j^*$, $j = 1, \dots, m$ (this is an elliptic generalized Toeplitz operator of order 0 on $\partial\Omega_j$), and H the $m \times m$ block matrix with T_{H_j} on the main diagonal and zeroes elsewhere. By (24), $TT^*H - I \in \mathcal{C}$ and arguing as before, we get $H = G^{-1} + \mathcal{C}$ and

$$S_l = \tau^* \mathcal{T}_l \tau + \mathcal{C},$$

where $\mathcal{T}_l$ is the $m \times m$ block matrix with $T_{H_j} T_{X_j}^{1/2} T_{z_l} T_{X_j}^{-1/2}$, $j = 1, \dots, m$, on the main diagonal and zeroes everywhere else; thus $\mathcal{T}_l$ is a direct sum over j of generalized Toeplitz operators of order 0 on $H^2(\partial\Omega_j)$. For any $k, l = 1, \dots, d$ we thus again get

$$\begin{aligned} [S_j, S_l^*] &= [\tau^* \mathcal{T}_j \tau + \mathcal{C}, \tau^* \mathcal{T}_l^* \tau + \mathcal{C}] = [\tau^* \mathcal{T}_j \tau, \tau^* \mathcal{T}_l^* \tau] + \mathcal{C} \\ &= \tau^* (\mathcal{T}_j [\tau \tau^*, \mathcal{T}_l^*] + [\mathcal{T}_j, \mathcal{T}_l^*] \tau \tau^* + \mathcal{T}_l^* [\mathcal{T}_j, \tau \tau^*]) \tau + \mathcal{C} \\ &= \tau^* (\mathcal{T}_j [D, \mathcal{T}_l^*] + [\mathcal{T}_j, \mathcal{T}_l^*] D + \mathcal{T}_l^* [\mathcal{T}_j, D]) \tau + \mathcal{C}. \end{aligned}$$

The last three commutators are $m \times m$ block matrices with generalized Toeplitz operators of order -1 on the main diagonal and zeroes elsewhere, hence belong to $\bigoplus_{j=1}^m \mathcal{S}^{p_j}(H^2(\partial\Omega_j))$ for all $p_j > n_j$, $j = 1, \dots, m$, by Proposition 1, so to $\mathcal{S}^p(\mathcal{H})$ if $p > n := \max(n_1, \dots, n_m)$. Since the Schatten classes form an ideal and T is bounded, the theorem follows. (And again, in fact $[S_k, S_l^*] \in \mathcal{S}^{n, \infty}$.) $\square$

We remark that once we know that the block matrix operator TT^* is Fredholm, and, hence, T has finite-codimensional range, it follows by abstract operator theory that both T and T^* have closed ranges; but the range of the row block operator T^* is precisely the sum $\mathcal{M}_1^\perp + \dots + \mathcal{M}_m^\perp$, where $\mathcal{M}_j$ is the subspace in $A_{\alpha_0}^2$ of functions vanishing on V_j . By Proposition 3.4(2) in Kennedy and Shalit [20], the closedness of this sum together with our Main Theorem imply that $[S_k, S_l^*] \in \mathcal{S}^p$, $k, l = 1, \dots, d$, if $p > \max(n_1, \dots, n_m)$, thus yielding an alternative way to conclude the last proof.

An elementary argument using the decomposition of homogeneous varieties into their irreducible components shows that every homogeneous variety in $\mathbb{C}^d$ which is

smooth at each of its non-zero points is a finite union of varieties V_j as in Theorem 5. Therefore Theorem 5 generalizes our main theorem to the case of arbitrary homogeneous varieties V in $\mathbb{C}^d$ that are smooth outside the origin. That is, the refinement of the Arveson Conjecture as formulated by Douglas holds in this case. (Note that the dimension of the analytic set $V = \bigcup_{j=1}^m V_j$ at the origin is given by $\dim_0 V = \max(n_1, \dots, n_m)$, see Section 5.3 in [15].)

REFERENCES

- [1] W. Arveson: *p*-summable commutators in dimension d , J. Operator Theory **54** (2005), 101–117.
- [2] L. Boutet de Monvel: *Boundary problems for pseudo-differential operators*, Acta Math. **126** (1971), 11–51.
- [3] L. Boutet de Monvel: *On the index of Toeplitz operators in several complex variables*, Invent. Math. **50** (1979), 249–272.
- [4] L. Boutet de Monvel, V. Guillemin: *The spectral theory of Toeplitz operators*, Ann. Math. Studies, vol. 99, Princeton University Press, Princeton, 1981.
- [5] L. Boutet de Monvel, J. Sjöstrand: *Sur la singularité des noyaux de Bergman et de Szegő*, Astérisque **34–35** (1976), 123–164.
- [6] L. Boutet de Monvel: *Symplectic cones and Toeplitz operators*, Multidimensional Complex Analysis and Partial Differential Equations, pp. 15–24, Contemporary Math., vol. 205, Amer. Math. Soc., Providence, 1997.
- [7] F. Beatrous: *Estimates for derivatives of holomorphic functions in pseudoconvex domains*, Math. Z. **191** (1986), 91–116.
- [8] R. Douglas: *A new kind of index theorem*, Analysis, geometry and topology of elliptic operators, pp. 369–382, World Sci. Publ., Hackensack, 2006.
- [9] R.G. Douglas, K. Wang: *Essential normality of cyclic submodule generated by any polynomial*, preprint, 2011, arXiv:1101.0774.
- [10] M. Engliš: *Toeplitz operators and weighted Bergman kernels*, J. Funct. Anal. **255** (2008), 1419–1457.
- [11] M. Engliš: *Weighted Bergman kernels for logarithmic weights*, Pure Appl. Math. Quarterly (Kohn special issue) **6** (2010), 781–813.
- [12] M. Engliš: *Analytic continuation of weighted Bergman kernels*, J. Math. Pures Appl. **94** (2010), 622–650.
- [13] M. Engliš, G. Zhang: *Hankel operators and the Dixmier trace on strictly pseudoconvex domains*, Docum. Math. **15** (2010), 601–622.
- [14] Q. Fang, J. Xia: *Essential normality of polynomial-generated submodules: Hardy space and beyond*, preprint, 2011, available at <http://www.acsu.buffalo.edu/~jxia/Preprints/poly.pdf>.
- [15] H. Grauert, R. Remmert, *Coherent analytic sheaves*, Springer, Berlin, 1984.
- [16] A. Grigis, J. Sjöstrand, *Microlocal analysis for differential operators*, London Math. Soc. Lecture Notes, vol. 196, Cambridge Univ. Press, Cambridge, 1994.
- [17] K. Guo and K. Wang, *Essentially normal Hilbert modules and K-homology*, Math. Ann. **340** (2008), 907–934.
- [18] L. Hörmander: *Pseudo-differential operators*, Comm. Pure Appl. Math. **18** (1965), 501–517.
- [19] L. Hörmander, *The analysis of linear partial differential operators, vol. I–IV*, Grundlehren der mathematischen Wissenschaften, Springer-Verlag, 1985.
- [20] M. Kennedy, O.M. Shalit: *Essential normality and the decomposability of algebraic varieties*, New York J. Math. **18** (2012), 877–890.
- [21] J.-L. Lions, E. Magenes, *Problèmes aux limites non homogènes et applications*, vol. 1, Dunod, Paris, 1968.
- [22] A. Melin, J. Sjöstrand: *Fourier integral operators with complex valued phase functions*, Fourier integral operators and partial differential equations, Lecture Notes Math., vol. 459, pp. 120–223, Springer Verlag, Berlin-Heidelberg, 1975.
- [23] O.M. Shalit: *Operator theory and function theory in Drury-Arveson space and its quotients*, preprint, 2013, arxiv:1308.1081.
- [24] R. Strichartz: *A functional calculus for elliptic pseudo-differential operators*, Amer. J. Math. **94** (1972), 711–722.

- [25] F. Treves, *Introduction to pseudodifferential and Fourier integral operators*, Plenum, New York, 1980.

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