SOME ASPECTS OF BERGMAN KERNELS

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Abstract. We discuss some recent advances, open problems, and new variations in Berezin quantization on Kähler manifolds and their relationships to the asymptotic behaviour of the Bergman kernel, pseudolocal estimates on nonsmooth or unbounded pseudoconvex domains, and the Lu Qi-Keng conjecture.

Let Ω be a domain in $\mathbf{C}^n$ and $\rho > 0$ a weight function on Ω. The weighted Bergman space $L^2_{\text{hol}}(\Omega, \rho)$ is the subspace of all holomorphic functions in $L^2(\Omega, \rho)$. Under mild assumptions on ρ (for instance, if ρ is continuous), this space has a reproducing kernel, the weighted Bergman kernel $K_\rho(x, y)$. We shall assume throughout that $K_\rho(x, x) > 0 \ \forall x$ (this is the case, for instance, whenever $\rho \in L^1(\Omega)$). Then the integral operator

$$\begin{aligned} B_\rho f(x) :&= \frac{1}{K_\rho(x,x)} \int_\Omega f(y) |K_\rho(x,y)|^2 \rho(y)\, dy \\ &= \langle f k_x, k_x \rangle_{L^2(\Omega,\rho)}, \qquad \text{where } k_x := K_\rho(\,\cdot\,, x)/\|K_\rho(\,\cdot\,, x)\|, \end{aligned}$$

is well defined for (at least) any $f \in L^\infty(\Omega)$, and is called the Berezin transform.

Example 1. For $\Omega = \mathbf{C}^n$ and $\rho(x) = (\alpha/\pi)^n e^{-\alpha|x|^2}$ (where $\alpha > 0$ is a parameter), we have $K(x, y) = e^{\alpha\langle x,y\rangle}$, and $B_\rho f(x) = (\frac{\alpha}{\pi})^n \int f(y) e^{-\alpha|x-y|^2}\, dy$; that is, B_ρ is the heat solution operator $Bf = e^{\Delta/4\alpha} f$.

Example 2. If $\Omega = \mathbf{D}$, the unit disc, and $\rho(x) = \frac{\alpha+1}{\pi}(1 - |x|^2)^\alpha$ (where $\alpha > -1$ is a parameter), then $K(x, y) = (1 - x\overline{y})^{-\alpha-2}$, and

$$\begin{aligned} Bf(x) &= \frac{\alpha+1}{\pi} \int_{\mathbf{D}} f(y) \left[\frac{(1-|x|^2)(1-|y|^2)}{|1-\overline{y}x|^2} \right]^{\alpha+2} \frac{dy}{(1-|y|^2)^2} \\ &= \int_{\mathbf{D}} f(\phi(y))\, \rho(y)\, dy \qquad \forall \phi \in \operatorname{Aut}(\mathbf{D}) \text{ such that } \phi(0) = x. \end{aligned}$$

The Bergman kernels and the Berezin transform occur in several areas of mathematics, for instance, in quantization on Kähler manifolds, in the Lu Qi-Keng conjecture, in invariant mean value theorems, analytic operator models, properties of Toeplitz and Hankel operators, decomposition of tensor products of representations of Lie groups, etc. In this talk, we want to discuss the first two of these in some detail, and mention some interesting problems and results relevant to them.

1. Quantization on Kähler manifolds. The original idea of quantization on $\mathbf{R}^{2n}$, as envisaged by Weyl, Dirac, von Neumann, and others, consisted in assigning to observables (= smooth real-valued functions f of $(p, q) \in \mathbf{R}^n \times \mathbf{R}^n$) self-adjoint operators $\mathtt{Op}(f)$ on a (separable) Hilbert space in such a way that the following axioms were satisfied:

(a) $f \mapsto \mathtt{Op}(f)$ is linear;
(b) (the von Neumann rule) for any polynomial $\phi : \mathbf{R} \to \mathbf{R}$,

$$\mathtt{Op}(\phi \circ f) = \phi(\mathtt{Op}(f))$$

(in particular: $\mathtt{Op}(\mathbf{1}) = I$);

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(c) $[\mathtt{Op}(f), \mathtt{Op}(g)] = -\frac{ih}{2\pi}\mathtt{Op}(\{f,g\})$, where $\{f,g\}$ is the Poisson bracket of f and g,

$$\{f,g\} := \sum_{j=1}^{n}\Big(\frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q_j} - \frac{\partial f}{\partial q_j}\frac{\partial g}{\partial p_j}\Big);$$

(d) (the Schrödinger representation) the operators corresponding to the coordinate functions p_j, q_j ($j = 1, \dots, n$) are unitarily equivalent to the operators

$$\mathtt{Op}(q_j) : f(q) \mapsto q_j f(q), \qquad \mathtt{Op}(p_j) : f(q) \mapsto -\frac{ih}{2\pi}\frac{\partial f(q)}{\partial q_j}$$

on $L^2(\mathbf{R}^n)$.

More generally, by a quantization of a symplectic manifold (Ω, ω), one means a similar assignment $f \mapsto \mathtt{Op}(f)$, where now $f \in C^\infty(\Omega)$, which satisfies the same axioms (a)–(c), except that the Poisson brackets are taken with respect to ω, as well as (d) if $(\Omega, \omega) = \mathbf{R}^n$ with the standard symplectic form, and

(e) the assignment is functorial in the sense that for any diffeomorphism ϕ of one symplectic manifold $(\Omega^{(1)}, \omega^{(1)})$ onto another, $(\Omega^{(2)}, \omega^{(2)})$, there should be a unitary operator U_ϕ such that

$$\mathtt{Op}^{(1)}(f \circ \phi) = U_\phi^* \,\mathtt{Op}^{(2)}(f) U_\phi, \qquad \forall f.$$

Unfortunately, it turns out that the above axioms are inconsistent, even for $\mathbf{R}^{2n}$: the operator $\mathtt{Op}(p_1^2 q_1^2)$ can be computed in two different ways with two different results. There are two standard solutions to this disappointing situation. Both start by discarding the von Neumann rule (b), except for $\phi = \mathbf{1}$, i.e. $\mathtt{Op}(\mathbf{1}) = I$. The first solution is then to keep all other axioms, but restrict the space of quantizable observables (the domain of the map $f \mapsto \mathtt{Op}(f)$). For instance, on $\mathbf{R}^{2n}$ the recipe

$$\mathtt{Op}(f) : \psi \longmapsto -\frac{ih}{2\pi}\Big(\sum_j \frac{\partial f}{\partial p_j}\frac{\partial \psi}{\partial q_j}\Big) + \Big(f - \sum_j p_j \frac{\partial f}{\partial p_j}\Big)\psi,$$

where $\psi = \psi(q) \in L^2(\mathbf{R}^n)$, works if f is restricted to be at most linear in the variables p_j. For general symplectic manifolds, similar "restricting to functions depending on only half of the variables" requires the use of polarizations, and eventually leads to the so-called GEOMETRIC QUANTIZATION of Kostant [Ks] and Souriau [So].

The second solution is to relax (c) to hold only asymptotically as the Planck constant h tends to zero:

$$[\mathtt{Op}(f), \mathtt{Op}(g)] = -\frac{ih}{2\pi}\mathtt{Op}(\{f,g\}) + O(h^2). \tag{1}$$

The simplest example of such correspondence on $\mathbf{R}^{2n}$ is the Weyl calculus from the theory of pseudodifferential operators, given by

$$\mathtt{Op}(f) = \iint \hat{f}(\xi, \eta)\, e^{2\pi i(\xi\cdot \mathtt{Op}(p) + \eta \cdot \mathtt{Op}(q))}\, d\xi\, d\eta =: W(f),$$

where $\hat{f}$ is the Fourier transform of f and $\xi \cdot \mathtt{Op}(p) := \sum_j \xi_j \,\mathtt{Op}(p_j)$ with $\mathtt{Op}(p_j)$ given by the Schrödinger representation, and similarly for $\eta \cdot \mathtt{Op}(q)$. It can be shown that for sufficiently smooth f and g, $W(f)W(g) = W(f \sharp g)$, where the "twisted product" $f \sharp g$ has an asymptotic expansion

$$f \sharp g = \sum_{j=0}^{\infty} h^j C_j(f,g) \qquad \text{as } h \to 0,$$

where

$$\begin{gathered} C_0(f,g) = fg \quad \text{and} \quad C_1(f,g) - C_1(g,f) = -\frac{i}{2\pi}\{f,g\}, \\ C_j(f, \mathbf{1}) = C_j(\mathbf{1}, f) = 0 \qquad \forall j \geq 1. \end{gathered} \tag{2}$$

Hence $f \sharp g - g \sharp f = -\frac{ih}{2\pi}\{f,g\} + O(h^2)$, and we see that (1) holds for $\mathtt{Op}(f) = W_f$.

In fact, for the physical applications it is often not really necessary to have the operators $\mathtt{Op}(f)$, but suffices to have the noncommutative product like $\sharp$. This is the basic idea of the DEFORMATION QUANTIZATION, and the corresponding product is called a star-product. The formal definition runs as follows. Let $C^\infty(\Omega)[[h]]$ be the ring of all formal power series in h with coefficients in $C^\infty(\Omega)$. A star-product is an associative $\mathbf{C}[[h]]$-bilinear mapping $*$ such that

$$f * g = \sum_{j=0}^\infty h^j C_j(f,g), \qquad \forall f,g \in C^\infty(\Omega),$$

where the bilinear operators C_j satisfy (2).

Deformation quantization was introduced in [BF], and subsequently the existence and classification of star-products was established by various authors [DL] [Fe] [OM] [Kn]. However, there are some drawbacks. First of all, the star-product is just a formal power series — no convergence is guaranteed for a given value of h and functions $f,g \in C^\infty(\Omega)$. Second, in physical applications one is interested in the (suitably defined) spectra of the observables, and it turns out that isomorphic star products may lead to different spectra for the same observable. Hence, it is of interest to have some "natural" or "distinguished" star-products (e.g. defined canonically in terms of some geometric data of the manifold, etc.). For Kähler manifolds, it is a notable idea of Berezin that such star-products can be constructed with the aid of Bergman spaces and the Berezin transform: namely, the BEREZIN and the BEREZIN-TOEPLITZ star-products.

For simplicity, we consider only domains in $\mathbf{C}^n$ (not manifolds). The idea is to find a family $\rho = \rho_h$ of weights on Ω, depending on the Planck constant h, so that the corresponding Berezin transforms $B_{\rho_h} =: B_h$ have an asymptotic expansion

$$B_h = Q_0 + Q_1 h + Q_2 h^2 + \dots \qquad \text{as } h \to 0, \tag{3}$$

where Q_j are some differential operators such that

$$Q_0 = I \quad \text{(the identity)}, \qquad Q_1 = \tilde{\Delta} \quad \text{(the Laplace-Beltrami operator)}.$$

Then if $c_{j\alpha\beta}$ are the coefficients of Q_j, in the sense that

$$Q_j f = \sum_{\alpha,\beta \text{ multiindices}} c_{j\alpha\beta}\, \partial^\alpha \overline{\partial}^\beta f,$$

then setting

$$f * g := \sum_{j=0}^\infty h^j C_j(f,g), \qquad \text{with } C_j(f,g) := \sum_{\alpha,\beta} c_{j\alpha\beta}\, (\overline{\partial}^\beta f)(\partial^\alpha g) \tag{4}$$

gives a star-product. Further, this star-product has some nice additional properties: namely, it is differential (i.e. C_j are differential operators), it has the property of separation of variables (i.e. $f*g = fg$ if either f or $\overline{g}$ is holomorphic), and it is not only formal, but for a lot of functions f,g in fact converges. We may call (4) the Berezin star-product.

To describe the other quantization, recall that the Toeplitz operator $T_\phi^{(\rho)}$ corresponding to a function (called its "symbol") $\phi \in L^\infty(\Omega)$ is defined on $L^2_{\text{hol}}(\Omega,\rho)$ by

$$T_\phi^{(\rho)} f := P_\rho(\phi f),$$

where $P_\rho : L^2 \to L^2_{\text{hol}}(\Omega,\rho)$ is the orthogonal projection. (In other words, $T_\phi^{(\rho)}$ is an integral operator on $L^2_{\text{hol}}(\Omega,\rho)$ with kernel $\phi(x)K_\rho(y,x)$.) Now, again, the idea is to find a family of weights ρ_h such that the corresponding Toeplitz operators $T^{(\rho_h)} =: T^{(h)}$ satisfy

$$T_f^{(h)} T_g^{(h)} = \sum_{j=0}^\infty h^j T^{(h)}_{C_j(f,g)} \quad \text{as } h \to 0, \qquad \forall f,g \in C^\infty(\Omega), \tag{5}$$

where C_j satisfy (2). Then C_j define a differential star-product with separation of variables, called the Berezin-Toeplitz star-product.

Here (5) is to be understood in the sense of operators norms, i.e.

$$\left\| T_f^{(h)} T_g^{(h)} - \sum_{j=0}^{m} h^j T_{C_j(f,g)}^{(h)} \right\| = O(h^{-m-1}), \qquad \forall m = 1, 2, \dots.$$

Of course, the first problem in both cases is how to choose the weights ρ_h. Here one can take guidance from the following group-invariance consideration. Assume there is a group G acting on Ω by biholomorphic transformations preserving the symplectic form ω. We then want the product $*$ to be G-invariant, i.e. to satisfy $(f \circ \phi) * (g \circ \phi) = (f * g) \circ \phi$, $\forall \phi \in G$. Working through the definitions of e.g. the Berezin quantization reveals that this happens if and only if the ratio $\rho(\phi(x))/\rho(x)$ is a squared modulus of a holomorphic function. Writing $\rho(x)\,dx = w(x)\omega^n(x)$, where ω^n is the (G-invariant) symplectic volume element on Ω, the last condition translates into $w(\phi(x)) = w(x)|f_\phi(x)|^2$, for some holomorphic function f_ϕ. Hence, the form $\partial\overline{\partial}\log w$ is G-invariant. But the simplest examples of G-invariant forms — and if G is sufficiently "ample", the only ones — are clearly the constant multiples of ω. Thus if ω is not only symplectic but Kähler (i.e. lies in the range of $\partial\overline{\partial}$), we are lead to taking

$$\rho_h(x)\,dx = e^{c\Phi(x)} \cdot \omega^n(x),$$

where Φ is a (real-valued) Kähler potential for ω, and $c = c(h)$ depends only on h. Observe that the potential Φ is then always strictly plurisubharmonic, i.e. the matrix $[\partial\overline{\partial}\Phi(x)]$ of mixed second derivatives is positive definite, for any $x \in \Omega$. (Indeed, the Riemannian metric associated to ω is given by $ds^2 = \sum_{j,k} g_{j\overline{k}}\,dz_j d\overline{z}_k$, where $g_{j\overline{k}} = \partial^2\Phi/\partial z_j \partial\overline{z}_k$, and $\omega^n(x) = \det[\partial\overline{\partial}\Phi(x)]\,dx$.)

Thus our recipes for the Berezin and the Berezin-Toeplitz quantizations run as follows: let Ω be a domain in $\mathbf{C}^N$, Φ a strictly PSH (=plurisubharmonic) function on Ω, $K_c(x,y)$ the Bergman kernel of $L^2_{\mathrm{hol}}(\Omega, e^{c\Phi}\omega^n)$, B_c and $T^{(c)}$ the associated Berezin transform and Toeplitz operators, respectively, and see if $c = c(h)$ can be chosen so that the asymptotic expansions (3) or (5) hold.

It turns out that this is indeed often true, and the right choice is $c(h) = -1/h$. (Thus $c \to -\infty$ as $h \searrow 0$.)

To see how this comes about, note that the asymptotic behaviour of both the Berezin transform B_c and the Toeplitz operators $T^{(c)}$ clearly depends on the corresponding asymptotics of Bergman kernels K_c. To unravel the latter, we will study the more general kernels

(6) $$K_m(x,y) := \text{ the reproducing kernel of } L^2_{\mathrm{hol}}(\Omega,\, e^{-m\Phi-\Psi})$$

as $m \to \infty$, where Ψ is another (auxiliary) PSH function on Ω. Since $\omega^n = \det[\partial\overline{\partial}\Phi]\,dx$, the required result will then follow upon choosing Ψ so that

(7) $$\det[\partial\overline{\partial}\Phi] = e^{m_0\Phi-\Psi}$$

for some $m_0 \in \mathbf{Z}$.

Our first result is then as follows. Recall that an almost-analytic extension of a function $F(x)$ on Ω is a smooth function $f(x,y)$ on $\Omega \times \Omega$ such that $f(x,x) = F(x)$ $\forall x$ and both $\partial f/\partial\overline{x}$ and $\partial f/\partial y$ vanish to an infinite order on the diagonal $x = y$. It is known that every $F \in C^\infty(\Omega)$ has such an extension.

Theorem 1. (kernel asymptotics) *Let Ω be a pseudoconvex domain in $\mathbf{C}^N$, Φ, Ψ two real-valued PSH functions on Ω, and let $x_0 \in \Omega$ be a point in a neighbourhood of which Φ and Ψ are C^∞ and Φ is strictly PSH. Then there is a smaller neighborhood $\mathcal{U}$ of x_0 such that for the Bergman kernels* (6), *the asymptotic expansion*

$$K_m(x,y) = \frac{m^N}{\pi^N} e^{m\Phi(x,y)+\Psi(x,y)} \sum_{j=0}^{\infty} \frac{b_j(x,y)}{m^j}$$

holds uniformly for all $x, y \in \mathcal{U}$ as $m \to \infty$, in the sense that for each $k > 0$,

$$\sup_{x,y\in\mathcal{U}} \left| e^{-m(\Phi(x)+\Phi(y))/2} \Big[(\text{LHS}) - \Big(\text{RHS with } \sum_{j=0}^{N+k-1} \Big) \Big] \right| = O(m^{-k}) \qquad \text{as } m \to \infty.$$

Here $\Phi(x,y),\Psi(x,y)$ are almost-analytic extensions of $\Phi(x)$ and $\Psi(x)$, respectively.

Further, the coefficients $b_j(x,y)$ are almost-analytic extensions of functions $b_j(x)$ on $\mathcal{U}$ whose jets at x_0 depend only on the jets of Φ and Ψ at x_0. In particular, $b_0=\det[\partial\overline{\partial}\Phi]$.

In the situation of the last theorem, fix an integer $M\geq 0$ and consider the domain

$$\widetilde{\Omega}=\{(z_1,z_2,z_3)\in\Omega\times\mathbf{C}^M\times\mathbf{C}:\quad e^{\Psi(z_1)}|z_2|^2+e^{\Phi(z_1)}|z_3|^2<1\}. \tag{8}$$

Theorem 2. (Berezin quantization) *Assume that the hypotheses of Theorem 1 are fulfilled, and in addition $\widetilde{\Omega}$ is smoothly bounded and of finite type. Then for any $f\in L^\infty(\Omega)$ which is C^∞ in a neighborhood of x_0, the Berezin transforms have an asymptotic expansion*

$$B_m f(x)=\sum_{j=0}^\infty Q_j f(x)\cdot m^{-j}\qquad \text{as } m\to\infty,$$

uniformly for all x in a neighbourhood of x_0.

Here Q_j are linear differential operators whose coefficients involve only the derivatives of Φ and Ψ, and $Q_0=I$ and $Q_1=\tilde{\Delta}$.

Finally, if Ψ is chosen according to (7), then we have also more explicit information about the operators Q_j:

Theorem 3. *If the hypotheses of Theorem 2 are fulfilled, and in addition (7) holds, then the operators Q_m are finite sums of differential operators of the form*

$$f\mapsto \sum_{i_1,\dots,i_k,j_1,\dots,j_l} C^{i_1\dots i_k\overline{j}_1\dots\overline{j}_l} f_{/\overline{j}_1\dots\overline{j}_l i_1\dots i_k}$$

with $k,l\leq m$, where the slash stands for the covariant differentiation (with respect to the metric induced by ω) and $C^{i_1\dots i_k\overline{j}_1\dots\overline{j}_l}$ are tensor fields on Ω, symmetric in $i_1,\dots,i_k$ and in $\overline{j}_1,\dots,\overline{j}_l$, that are contractions of tensor products of the contravariant metric tensor $g^{\overline{j}k}$, the curvature tensor $R_{i\overline{j}k\overline{l}}$, and the latter's covariant derivatives.

In particular, in addition to $Q_0=I$ and $Q_1=\tilde{\Delta}$, we have

$$Q_2=\frac{1}{2}\tilde{\Delta}^2+\frac{1}{2}\sum_{j,k}\mathrm{Ric}^{\overline{j}k}\,\frac{\partial^2}{\partial\overline{x}_j\partial x_k},$$

where $\mathrm{Ric}^{\overline{j}k}$ are the contravariant components of the Ricci tensor.

The first two theorems were first established by Berezin for bounded symmetric domains with the invariant (Kähler) metric [Be]. Complete proofs of Theorems 1–3 can be found in [E4], [E8] and [E5]; in the last reference, a formula for the operator Q_3 is also given. However, we can at least indicate here the main ingredients.

Sketch of the proofs. The hypotheses of Theorem 1 imply that $\widetilde{\Omega}$ is pseudoconvex, with $\boldsymbol{x}:=(x_0,0,e^{-\Phi(x_0)/2})$ a smooth strictly-pseudoconvex boundary point.

1) By a formula of Ligocka, the (unweighted) Bergman kernel of $\widetilde{\Omega}$ satisfies

$$\widetilde{K}((x_1,0,x_3),(y_1,0,y_3))=\sum_{m=0}^\infty\frac{(m+M+1)!}{\pi^{M+1}\,m!}\,K_{m+1}(x_1,y_1)\,(x_3\overline{y}_3)^m.$$

2) Consider first the case when $\widetilde{\Omega}$ is <u>bounded</u>. Then by Fefferman's asymptotic expansion, there exist functions $a,b\in C^\infty(\widetilde{\Omega}\cup\{\boldsymbol{x}\})$ with almost-analytic extensions $a(x,y),b(x,y)$ such that

$$\widetilde{K}=\frac{a}{r^{N+M+2}}+b\log r,$$

where $r(x,y)$ is an almost-analytic extension of $r(x)=1-e^{\Psi(x_1)}|x_2|^2-e^{\Phi(x_1)}|x_3|^2$.

3) Next, recall that

$$\sum_{j=1}^{\infty} j^m t^j = \begin{cases} m!\,(1-t)^{-m-1} + O((1-t)^{-m}) & \text{if } m \geq 0, \\ \frac{(-1)^m}{m!}\,(1-t)^m \log(1-t) + C^m(\overline{\mathbf{D}}) & \text{if } m < 0. \end{cases}$$

4) Combining the last three items, the desired asymptotic formula for $K_m(x,y)$ as $m \to \infty$ follows. (Some care is needed to get the uniformity on $\mathcal{U}$.) This settles Theorem 1 (for $\widetilde{\Omega}$ bounded).

5) If $\widetilde{\Omega}$ is in addition C^∞-bounded and of finite type, then by a result of Bell, $\widetilde{K}$ extends smoothly to $\overline{\widetilde{\Omega}} \times \overline{\widetilde{\Omega}}$ minus the boundary diagonal. Ligocka's formula then implies that $K_m(x,y)$ decays exponentially, as $m \to \infty$, for $x \neq y$, and it transpires that in the integral defining the Berezin transform

$$B_m f(y) = K_m(y,y)^{-1} \int_\Omega f(x) |K_m(x,y)|^2 \, e^{-m\Phi(x) - \Psi(x)} \, dx$$

the main contribution comes from a small neighbourhood $\mathcal{U}$ of y.

6) Using Theorem 1, the latter contribution reduces to the standard Laplace integral

$$\int_{\mathcal{U}} f(x) \, \exp\Big(m[\Phi(x,y) + \Phi(y,x) - \Phi(x) - \Phi(y)] \Big) \, \gamma(x) \, dx$$

(with some expression γ involving Φ, Ψ and the coefficients b_j), whose evaluation gives the asymptotics of $B_m f(y)$, proving Theorem 2.

7) Analyzing the standard formulas for Laplace integrals (=the method of stationary phase, or WJKB method) shows that the coefficients in the asymptotic expansion — the operators Q_m — must be expressions of certain special form, involving summations over derivatives of Φ, Ψ at y. Introducing normal coordinates and using the transformation properties of the quantities involved, it turns out that Q_m indeed come as sums of covariant derivatives as indicated. This proves Theorem 3.

8) Finally, to remove the boundedness assumption made in **2)** above, it is enough to extend to unbounded domains the asymptotic expansion of Fefferman, i.e. to show that for any pseudoconvex $\Omega \subset \mathbf{C}^d$, the unweighted Bergman kernel K satisfies $K = a/r^{d+1} + b \log r$, with $a, b \in C^\infty(\Omega \cup \{\text{a neighborhood of } \boldsymbol{x}\})$ and $-r$ a defining function near a smooth strictly-pseudoconvex boundary point $\boldsymbol{x}$. This, in turn, follows once we can extend to the unbounded case the traditional subelliptic estimates and pseudolocal estimates. The last, finally, can be established from the corresponding results for the bounded case by means of a sequence of smoothly bounded finite type domains exhausting Ω. (Some care is needed, however, since the $\overline{\partial}$-Neumann operator on Ω is no longer bounded nor everywhere defined; see [E7] for the details.) □

Remarks. (1) One can also extend to the unbounded/nonsmooth domains the above result of Bell, i.e. that $K \in C^\infty\big((\Omega \times \Omega) \cup \{\text{a neighbourhood of } (x,y)\}\big)$ for any two distinct smooth boundary points $x \neq y$ of finite type. As a corollary, it follows that if Ω is any pseudoconvex domain in $\mathbf{C}^N$ (possibly unbounded and non-smooth) which has at least one smooth strictly-pseudoconvex boundary point, then $L^2_{\text{hol}}(\Omega) \neq \{0\}$.

(2) If Ω is bounded and Φ, Ψ are assumed to be not only C^∞ but C^ω near x_0, the assertion of Theorem 1 can be substantially sharpened, namely, the factor $e^{-m[\Phi(x)+\Phi(y)]/2}$ can be replaced by $e^{-m\Phi(x,y)}$:

$$\sup_{x,y \in \mathcal{U}} \Big| e^{-m\Phi(x,y)} K_m(x,y) - \frac{m^N}{\pi^N} e^{\Psi(x,y)} \sum_{j=0}^{N+k-1} \frac{b_j(x,y)}{m^j} \Big| = O(m^{-k}).$$

Also, the coefficients $b_j(x,y)$ are not merely almost-analytic, but holomorphic in $x, \overline{y}$ on $\mathcal{U} \times \mathcal{U}$. All this follows from the real-analytic extension of Fefferman's expansion due to Kashiwara and Kaneko. For $x \neq y$, the last estimate is better than the original one by an exponential factor.

(3) Theorem 1 fails if Φ is only assumed to be PSH but not strictly PSH at x_0. (So that $\widetilde{\Omega}$ is pseudoconvex but not strictly.) Indeed, computations for the "ellipsoids" $|x_1|^2 + |x_2|^{2l} < 1$ seem to suggest that in that case one gets an asymptotic expansion not in the negative powers of m, but instead in negative powers of $m^{2/p}$, where p is the type (assumed to be finite) of the corresponding boundary

point. Though growth estimates for the Bergman kernel near finite and infinite type points exist [Oh], [BY], [DH], [KL], they seem insufficient for extending Theorem 1 to this case.

(4) If, on the other hand, Φ is allowed to be $-\infty$ (so that $\widetilde{\Omega}$ has a cusp), then computations suggest that $K_m(x,x)$ can grow faster then predicted by Theorem 1 (for instance, as $c(x)\, m^Q e^{m\Phi(x)}$ with some $Q > N$). In physics, the cusps correspond to allowing Poisson (instead of only symplectic) manifolds.

(5) In a sense, the hypotheses of Theorem 2 are not completely satisfactory, for the following two reasons. The first is the relation (7): it can be shown that to be able to choose such *plurisubharmonic* Ψ, one must have

$$\exists C > 0 : \qquad C g_{j\overline{k}}(x) - \mathrm{Ric}_{j\overline{k}}(x) \qquad \text{is positive definite } \forall x \in \Omega.$$

This is an unpleasant restriction. Second, the condition of finite type, and even of smooth boundedness, of the domain $\widetilde{\Omega}$ in (8) is also rather unsatisfactory, since two standard metrics on Ω — the Bergman and the Cheng-Yau metric — do not satisfy it in general. Indeed, the Bergman metric corresponds to the choice $\Phi(z) = \log K(z,z)$, with $K(z,z)$ the unweighted Bergman kernel of Ω, so that $e^{-\Phi}$ is almost never C^∞ up to the boundary, even for Ω strictly pseudoconvex, owing to the presence of the logarithmic term in Fefferman's expansion. Similarly, for the Cheng-Yau metric on a strictly pseudoconvex domain, the potential Φ has a similar logarithmic singularity at the boundary, by a result of Lee and Melrose. Thus in both cases, $\widetilde{\Omega}$ fails to be C^∞ on the "equator" $\{(x,0,0) : x \in \partial\Omega\} \subset \partial\widetilde{\Omega}$.

What prevents us from extending Theorem 2 to such situations is the lack of estimates for the "tail" of the integral defining the Berezin transform,

$$\int_{\Omega \backslash \text{a neighborhood of } y} f(x) \, \frac{|K_\rho(x,y)|^2}{K_\rho(y,y)} \, \rho(x) \, dx.$$

What is thus needed is a direct proof of the following assertion:

Problem. *For Φ, Ψ plurisubharmonic on Ω, and any neighbourhood U of $y \in \Omega$ there exists $\delta > 0$ such that the Bergman kernels $K_m(x,y)$ of $L^2_{\mathrm{hol}}(\Omega, e^{-m\Phi-\Psi})$ satisfy*

$$\sup_{x \in \Omega \backslash U} \left[\frac{|K_m(x,y)|^2}{K_m(x,x) K_m(y,y)} \right]^{1/m} \leq 1 - \delta \qquad \forall m \gg 0.$$

(6) So far, we have assumed that the potential Φ of the Kähler form ω exists globally. This restriction is easily removed by passing from functions on Ω to sections of holomorphic line bundles. In this setting, and for Ω a compact Kähler manifold, analogs of Theorem 1 — the asymptotics of $K_m(x,y)$ — were obtained by Zelditch [Ze] (inspired by Tian [Ti]; only for $x = y$) and Catlin [Ca]. Analogues of Theorem 3 (computation of the first three terms of the asymptotic expansion) in that context was done by Lu [Lu]. (See also Donaldson [Do].) Their method is, however, different from ours: for Theorem 1, it follows the proof of Bordemann, Meinrenken and Schlichenmaier [BM] using the theory of Fourier integral operators (or rather Hermite operators [BG]); and for Theorem 3, it uses *peak sections*.

(7) The limits of $K_m(x,x)$ are of interest in the study of Pauli operators with magnetic fields [Er], [R1].

(8) While the boundary behaviour of $\widetilde{K}$ at the points of $\widetilde{\Omega}$ lying above Ω gives the asymptotics of $K_m(x,y)$ as $m \to \infty$, the behaviour at points on the "equator" $\{(x,0,0) : x \in \partial\Omega\}$ gives the asymptotics of $K_m(x,y)$ for m fixed and $x, y \to \partial\Omega$. The latter has been studied, even for weights rapidly decaying at the boundary (so that $\widetilde{\Omega}$ is nonsmooth on the equator), like $\exp\{-\exp[\alpha/(1-|z|^2)^\beta]\}$ on $\mathbf{D}$ or $\exp[-\exp(\alpha|z|^\beta)]$ on $\mathbf{C}^n$, in [HR] and [Kr]. There is also a strange result of Raikov [R2] for weights $e^{-\Phi}$ on $\mathbf{C}$ such that $\Delta\Phi$ is almost periodic.

(9) Conversely, our Theorem 3 yields formulas for the first coefficients of Fefferman's asymptotic expansion for the Hartogs domain $\widetilde{\Omega}$. For domains in $\mathbf{C}^2$ much stronger results of this type have been obtained by Hirachi, Komatsu and Nakazawa [HK].

We now proceed to the Berezin-Toeplitz quantization. Let us again consider the Bergman spaces $L^2_{\mathrm{hol}}(\Omega, e^{-m\Phi} \det[\partial\overline{\partial}\Phi])$, and denote by $T^{(m)}_f$ the corresponding Toeplitz operators with symbol $f \in L^\infty(\Omega)$.

Theorem 4. (Berezin-Toeplitz quantization) *Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbf{C}^N$, and $\Phi : \Omega \to \mathbf{R}$ a smooth strictly PSH function such that $-e^{-\Phi}$ is a defining function for Ω. Then:*

(i) *for any $f \in C^\infty(\overline{\Omega})$, $\|T_f^{(m)}\| \to \|f\|_\infty$ as $m \to \infty$;*

(ii) *there exist bilinear differential operators C_j ($j = 0, 1, 2, \dots$) such that for any $f, g \in C^\infty(\overline{\Omega})$ and any integer M,*

$$\left\| T_f^{(m)} T_g^{(m)} - \sum_{j=0}^{M} m^{-j} T^{(m)}_{C_j(f,g)} \right\| = O(m^{-M-1}) \qquad \text{as } m \to \infty.$$

Further, C_0 and C_1 satisfy (2), and, hence, $f * g := \sum_{j=0}^\infty h^j C_j(f,g)$ defines a star-product on Ω.

This was first proved for Ω a compact Kähler manifold (and functions replaced by sections of line bundles) in [BM]. Their proof, however, immediately extends also to the case when only a certain circle bundle (corresponding to the boundary of the domain $\widetilde{\Omega}$ below) is compact.

Sketch of proof. Consider this time the domain $\widetilde{\Omega}$ without the auxiliary function Ψ:

$$\widetilde{\Omega} = \{(z,t) \in \Omega \times \mathbf{C} : \quad |t|^2 < e^{-\Phi(z)}\}.$$

The hypotheses imply that $\widetilde{\Omega}$ is smoothly bounded, strictly pseudoconvex, with $r(z,t) := |t|^2 - e^{-\Phi(z)}$ a defining function. Consider this time the Szegö kernel on the compact manifold $X := \partial\widetilde{\Omega}$ with respect to the measure $\dfrac{J[r]}{\|\partial r\|}\, dS$, where dS stands for the surface measure on X and $J[r] = -\det \begin{bmatrix} r & \partial r \\ \bar\partial r & \partial\bar\partial r \end{bmatrix}$ for the Monge-Ampére determinant of r. It can be shown that (in analogy with Ligocka's formula)

$$K_{\text{Szegö}}((x,t),(y,s)) = \frac{1}{2\pi} \sum_{m=0}^\infty K_m(x,y)(s\bar t)^m,$$

that the Hardy space $H^2(X)$ is the direct sum of the Bergman spaces

$$H^2(X) = \bigoplus_{m=N+1}^\infty L^2_{\text{hol}}(\Omega, e^{-m\Phi} \det[\partial\bar\partial\Phi]), \tag{9}$$

and that

$$\bigoplus_{m=N+1}^\infty T_f^{(m)} = T_F, \qquad \text{with } F(x,t) := f(x),$$

where T_F denotes the Toeplitz operator on $H^2(X)$ with symbol F. Let further $D : H^2(X) \to H^2(X)$ be the operator which acts as multiplication by m on the m-th summand in (9), for each m. Then by the ideas of Boutet de Monvel-Guillemin, the Toeplitz operators T_F can be defined also for F a pseudodifferential operator on X, and from an analogue of the symbol calculus for pseudodifferential operators one can deduce the expansion

$$T_F T_G = \sum_{j=0}^\infty D^{-j} T_{H_j}$$

with some functions $H_j(z,t)$ on X depending only on z, i.e. descending to some functions $h_j(z) = H_j(z,t)$ on Ω. Setting $C_j(f,g) := h_j$, one eventually gets the result. See [E8] for the details. □

Remarks. Again, the last theorem does not apply to Φ the potential of the Bergman or of the Cheng-Yau metric. For this, we would need an extension of the Boutet de Monvel–Guillemin machinery to noncompact manifolds X. In the special case of $X = \{(z,t) \in \mathbf{C}^N \times \mathbf{C} : \quad |t|^2 = e^{-|z|^2}\}$, this was done by Borthwick [Bw]. In general, the main problem is that various smoothing error terms (which occur everywhere in the theory of Fourier integral operators) need no longer be bounded. (On compact manifolds, they are automatically even compact.)

Another interesting application of the asymptotics of $K_m(x,y)$ is to the Lu Qi-Keng conjecture.

2. Kernel asymptotics and the Lu Qi-Keng conjecture. Let us keep the scenario from the preceding section, i.e. let Ω a domain in $\mathbf{C}^N$, Φ, Ψ two functions on Ω, which however we now allow to be only continuous (not necessarily PSH), and let $K_m(x,y)$ the Bergman kernel of $L^2_{\mathrm{hol}}(\Omega, e^{-m\Phi-\Psi})$. It follows from Theorem 1 that if Ω is pseudoconvex, Φ, Ψ are PSH, and Φ is strictly-PSH at x, then

$$K_m(x,x)^{1/m} \rightarrow e^{\Phi(x)} \qquad \text{as } m \to \infty.$$

What can be said about this limit in general?

Theorem 5. [E1] *Let Φ, Ψ be any continuous functions on Ω. Assume that there exists $\alpha > 0$ such that $e^{-\alpha\Phi-\Psi} \in L^1(\Omega)$. Let*

$$\Phi^*(x) := \sup\{\operatorname{Re} g(x) :\ g \text{ holomorphic on } \Omega,\ \operatorname{Re} g \le \Phi\},$$
$$\Phi^\#(x) := \sup\{\psi(x) :\ \psi \text{ PSH on } \Omega,\ \psi \le \Phi\}.$$

(Thus $\Phi^* \le \Phi^\# \le \Phi$, and $\Phi^\# = \Phi$ iff Φ is PSH.) Then

$$\exp(\Phi^*(x)) \le \liminf_{m\to\infty} K_m(x,x)^{1/m} \le \limsup_{m\to\infty} K_m(x,x)^{1/m} \le \exp(\Phi^\#(x)).$$

Since linear functions are holomorphic, the following corollary is immediate.

Corollary 6. *If Φ is convex, then $\lim_{m\to\infty} K_m(x,x)^{1/m} = e^{\Phi(x)}$.*

It turns out, perhaps a little surprisingly, that as soon as we can handle the limits $\lim_{m\to\infty} K_m(x,x)^{1/m}$, then we can also deal with the limits of $K_m(x,y)^{1/m}$ for $x \ne y$. Obviously, in order to be able to define the root, we need that $K(x,y)$ be nonvanishing. It turns out that this is the only obstacle!

Theorem 7. *Assume that $K_m(x,x)^{1/m} \to e^{\Phi(x)}$ $\forall x$, and that there exists an open connected set $U \subset \Omega$ and a sequence $m_1, m_2, \dots \to \infty$ such that*

$$K_m(x,y) \ne 0 \qquad \text{for all } x,y \in U \text{ and } m = m_1, m_2, \dots .$$

Then $\Phi(x)$ extends to a function $\Phi(x,y)$ on $U \times U$, holomorphic in $x, \overline{y}$, such that $\Phi(x,x) = \Phi(x)$ and

$$K_{m_j}(x,y)^{1/m_j} \to e^{\Phi(x,y)} \qquad \text{as } j \to \infty, \quad \forall x,y \in U,$$

where the branches of the roots are chosen to be positive on the diagonal $x = y$.

Corollary 8. *If Φ is convex on Ω but not real-analytic at some point x_0, then $K_m(x,y)$ has a zero near (x_0,x_0) for all m sufficiently large.*

The Lu Qi-Keng conjecture (1958) asserted that for any simply connected domain $\Omega \in \mathbf{C}^n$, the Bergman kernel (with respect to the Lebesgue measure) is zero-free. While this is true for planar domains, counterexamples were given by Skwarczynski and Boas for higher dimensions. However, until quite recently a possibility remained open for the conjecture to hold at least for convex domains. Two counterexamples to the latter were given in [PY] and [BS]. The last Corollary can be used to generate a whole infinite family of such counterexamples, and to show that in some sense these "counterexamples" are even generic. See [E6].

Let us conclude by mentioning some open problems related to the last two theorems:

1) In all known examples the limit as $m \to \infty$ of $K_m(x,x)^{1/m}$ always exists and equals $\exp(\Phi^\#(x))$. Is this true in general?
   It can be shown that for Ω pseudoconvex, $\limsup_{m\to\infty} K_m(x,x)^{1/m} = \exp(\Phi^\#(x))$.
2) More generally, if $K^\#_m$ is the analogous kernel for Φ replaced by $\Phi^\#$, is it true that $K^\#_m(x,x)$ and $K_m(x,x)$ have the same asymptotics as $m \to \infty$?
3) The sequence of absolute values $|K(x,y)|^{1/m}$ is always locally uniformly bounded on all of $\Omega \times \Omega$ (!) as $m \to \infty$. Almost nothing seems to be known about its limiting behaviour.
   For smoothly bounded $\Omega \subset \mathbf{C}$ with Φ the potential of the Poincaré metric and $\Psi = \mathbf{1}$, the author has shown in [E2] that the limit $\lim_{m\to\infty} |K(x,y)|^{1/m}$ exists for all (x,y) not in the cut locus of Ω (i.e. such that there is a unique shortest geodesic connecting x to y), while for (x,y) in the cut locus the sequence can exhibit an oscillatory behaviour as $m \to \infty$. Understanding the limiting behaviour in general therefore seems to be quite intriguing.

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