APPROXIMATE SCHREIER DECORATIONS AND APPROXIMATE KÖNIG'S LINE COLORING THEOREM

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ABSTRACT. Following recent result of L. M. Tóth [arXiv:1906.03137] we show that every 2Δ-regular Borel graph $\mathcal{G}$ with a (not necessarily invariant) Borel probability measure admits approximate Schreier decoration. In fact, we show that both ingredients from the analogous statements for finite graphs have approximate counterparts in the measurable setting, i.e., approximate König's line coloring Theorem for Borel graphs without odd cycles and approximate balanced orientation for even degree Borel graphs.

It is a standard fact from finite combinatorics that every 2Δ-regular finite graph is a Schreier graph of the free group F_Δ on Δ generators. This means that every such graph admits an orientation and a Δ-labeling of the edges such that for every $\alpha \in \Delta$ and every vertex there is exactly one out-edge with label α and exactly one in-edge with label α. Such an orientation and labeling is called a *Schreier decoration*. Note that every Schreier decoration corresponds to an action of the free group F_Δ on the vertex set of the graph. We refer the reader to the introduction in [11] for more information about Schreier decorations.

The analogous statement for infinite graphs without any restriction on definability follows from the axiom of choice. In the measurable setting, i.e., when the vertex set is endowed with a standard probability (Borel) structure and we require the orientation and labeling to be measurable, the full analogue of the statement fails. This follows from the example of Laczkovich [9] who constructed an acyclic 2-regular bipartite graph on the unit interval that is not induced by an action of $\mathbb{Z}$ on any set of a full measure. However, Tóth recently proved [11] that if the measure is invariant one can always find a measurable Schreier decoration on a different graph that has the same local statistics. This can be stated in a compact form as follows: every 2Δ-regular unimodular random rooted graph has an invariant random Schreier decoration, see [11, Theorem 1]. An equivalent formulation in a language that is closer to the one in this paper is as follows, see [11, Corollary 4]: Every 2Δ-regular graphing $(\mathcal{G}, \mu)$ is a local isomorphic copy of some graphing $(\mathcal{G}', \mu')$ that is induced by a Borel action of F_Δ that preserves μ'.

The key steps in the proof of [11, Theorem 1] are

(I) a consequence of [11, Theorem 3]: for every Δ-regular bipartite graphing $(\mathcal{G}, \mu)$ and for every $\epsilon > 0$ there is a Borel map $c : E \to \Delta$ that is a proper edge coloring on a set of μ-measure at least $1 - \epsilon$,

The author was supported by Leverhulme Research Project Grant RPG-2018-424.

(II) [11, Theorem 2]: every 2Δ-regular unimodular random rooted graph admits a graphing representation $(\mathcal{G},\mu)$ that admits measurable balanced orientation, where an orientation is *balanced* if the in-degree is equal to out-degree at any vertex.

Note that the same example [9] shows that none of (I) and (II) admits a full measurable analogue.

The main purpose of this paper is to provide short and straightforward proofs of approximate versions of (I) and (II) that hold for any bounded degree Borel graph with any Borel probability measure. This implies immediately that every 2Δ-regular graphing admits an approximate Schreir decoration. As a consequence of the ultrapower technique one can get a strengthening of [11, Corollary 4], and consequently [11, Theorem 1], see Section 5.

Recall that a *Borel graph* is a triple $\mathcal{G}=(V,\mathcal{B},E)$ where $(V,\mathcal{B})$ is a standard Borel space, (V,E) is a graph and $E\subseteq V\times V$ is a Borel symmetric set in the product Borel structure. We denote as $\Delta(\mathcal{G})$ the maximum degree of $\mathcal{G}$ and we say that $\mathcal{G}$ is of *bounded degree* if $\Delta(\mathcal{G})\in\mathbb{N}$. An *orientation* S of $\mathcal{G}$ is a Borel set $S\subseteq E$ such that for every $x,y\in V$ that form an edge in E exactly one of (x,y) or (y,x) is in S. We define $\operatorname{out}_S(v)=\{(v,x)\in S\}$, $\operatorname{in}_S(v)=\{(x,v)\in S\}$ and $\mathbf{Corr}(S)=\{v\in V:|\operatorname{in}_S(v)|=|\operatorname{out}_S(v)|\}$.

Let $k\in\mathbb{N}$. A partial Borel map $c;E\to k$ is called a *partial edge coloring* if $\operatorname{dom}(c)\subseteq E$ is a Borel set and $c(x,y)=c(y,x)$ whenever $(x,y)\in\operatorname{dom}(c)$. We say that c is a *proper* partial edge coloring if $c(e)\neq c(f)$ whenever $e\neq f\in\operatorname{dom}(c)$ and there is $v\in e\cap f$. For any partial edge coloring c we define $v\in\mathbf{Corr}(c)$ if $N(v)=\{e\in E:v\in e\}\subseteq\operatorname{dom}(c)$ and $c\restriction N(v)$ is a proper partial edge coloring. When $\operatorname{dom}(c)=E$, then we say that c is an *edge coloring*.

A pair (S,c), where S is an orientation and $c;E\to k$ is a partial edge coloring, is called a *partial Schreier decoration*. Define $v\in\mathbf{Corr}(S,c)$ if $v\in\mathbf{Corr}(S)$ and c is injective when restricted to both $\operatorname{in}_S(v)$ and $\operatorname{out}_S(v)$. If $\mathcal{G}$ is 2Δ-regular, then we say that (S,c) is a *Schreier decoration* if $c:E\to\Delta$ and $\mathbf{Corr}(S,c)=V$.

With this notation it is easy to see that S is a *balanced orientation* if $\mathbf{Corr}(S)=V$ and $c:E\to k$ is a proper edge coloring if $\mathbf{Corr}(c)=V$.

Definition 0.1. *Let $\mathcal{G}$ be a Borel graph of bounded degree. The* approximate chromatic index *of $\mathcal{G}$, in symbols $\chi'_{App}(\mathcal{G})$, is defined as the minimal $k\in\mathbb{N}$ such that for every Borel probability measure μ and every $\epsilon>0$ there is an edge coloring $c:E\to k$ such that*

$$\mu(\mathbf{Corr}(c))>1-\epsilon.$$

We say that $\mathcal{G}$ admits approximate balanced orientation *if for every for every Borel probability measure μ and every $\epsilon>0$ there is an orientation S of $\mathcal{G}$ such that*

$$\mu(\mathbf{Corr}(S))>1-\epsilon.$$

If $\mathcal{G}$ is 2Δ-regular, then we say that $\mathcal{G}$ admits approximate Schreier decoration *if for every Borel probability measure μ and every $\epsilon>0$ there is a partial Schreier decoration (S,c) of $\mathcal{G}$ where $c;E\to\Delta$ such that*

$$\mu(\mathbf{Corr}(S,c))>1-\epsilon.$$

It follows from [5, Theorem 1.8] that $\chi'_{App}(\mathcal{G}) \leq \Delta(\mathcal{G}) + 1$ for any bounded degree Borel graph $\mathcal{G}$, i.e., this is the corresponding approximate version of Vizing's Theorem. Since we use this result in the proof of the approximate version of König line coloring Theorem (Theorem 0.2 (I)) we would like to stress that its proof is significantly more easier than the main result of [5, Theorem 1.6]. In particular, the combinatorial idea reflects the proof of the Vizing's Theorem for finite graphs in the same way as the proof of Theorem 0.2 (I) reflects the proof of König's line coloring Theorem for finite bipartite graphs.

The result (I), a consequence of [11, Theorem 3], mentioned above is equivalent to saying that if we restrict our attention only to invariant probability measures, then the approximate chromatic index (restricted to invariant measures) of a bipartite Δ-regular Borel graph is Δ.

Theorem 0.2. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a bounded degree Borel graph.*

(I) *Suppose that $\mathcal{G}$ is bipartite. Then $\chi'_{App}(\mathcal{G}) = \Delta(\mathcal{G})$.*
(II) *Every vertex of $\mathcal{G}$ has even degree if and only if $\mathcal{G}$ admits approximate balanced orientation.*
(III) *Suppose that $\mathcal{G}$ is 2Δ-regular where $\Delta \in \mathbb{N}$. Then $\mathcal{G}$ admits approximate Schreier decoration.*

Note that $\mathcal{G}$ being bipartite is the same as saying that $\mathcal{G}$ does not contain odd cycles. This is strictly weaker then $\mathcal{G}$ being Borel bipartite Borel graph.

Moreover, it will be obvious from the proof that in (II) we can get a somewhat stronger statement: there is a sequence $\{S_n\}_{n\in\mathbb{N}}$ of orientations such that $\mu(\mathbf{Corr}(S_n)) \to 1$ for every Borel probability measure μ and $\mu(\mathbf{Corr}(S_n)) \geq 1 - \frac{1}{n}$ for every $\mathcal{G}$-invariant measure μ. Similarly, one can modify the proof of (I) to get that there is a sequence of Borel colorings $\{c_n : E \to \Delta(\mathcal{G})\}_{n\in\mathbb{N}}$ such that $\mu(\mathbf{Corr}(c_n)) > 1 - \frac{1}{n}$ for every $\mathcal{G}$-invariant Borel probability measure μ. Consequently, in (III) there is a sequence (S_n, c_n) of Schreier decorations such that $\mu(\mathbf{Corr}(S_n, c_n)) > 1 - \frac{1}{n}$ for every $\mathcal{G}$-invariant measure μ. This follows from the same principle as the result of Elek and Lippner about a sequence of matchings without short augmenting paths, see [4].

1. Preliminaries

Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a bounded degree Borel graph. We write $[v]_\mathcal{G}$ for the $\mathcal{G}$-connected component of $v \in V$ and $[A]_\mathcal{G}$ for the $\mathcal{G}$-saturation of $A \subseteq V$. We denote as $F_\mathcal{G}$ the countable Borel equivalence relation on V that is generated by E.

A Borel probability measure μ on $(V, \mathcal{B})$ is called $\mathcal{G}$-quasi-invariant if for every $A \in \mathcal{B}$ we have $\mu(A) = 0$ if and only if $\mu([A]_\mathcal{G}) = 0$. We denote as $\rho_\mu : F_\mathcal{G} \to (0, +\infty)$ the corresponding cocycle, see [7, Proposition 8.3], i.e., a Borel map that satisfies

$$\mu(B) = \int_A \rho_\mu(\psi(v), v) \; d\mu$$

for every $A, B \in \mathcal{B}$ and a Borel bijection $\psi : A \to B$ such that $\psi(v) \in [v]_\mathcal{G}$ for every $v \in A$.

Claim 1.1. [5, Proposition 3.2] *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a bounded degree Borel graph and μ be a Borel probability measure on $(V, \mathcal{B})$. Then there is a $\mathcal{G}$-quasi-invariant Borel probability measure ν on $(V, \mathcal{B})$ that satisfies $\mu(A) \leq 2\nu(A)$ for every $A \in \mathcal{B}$.*

We denote as $\mathcal{E} = (E, \mathcal{C}, I_{\mathcal{G}})$ the corresponding line graph where we abuse the notation and write E for the set of unordered pairs $\{x, y\} \subseteq X$ such that $(x, y) \in E$ (and consequently $(y, x) \in E$), $\mathcal{C}$ is the σ-algebra inherited from $E \subseteq V^2$ and $(e, f) \in I_{\mathcal{G}}$ if $e \cap f \neq \emptyset$. It is easy to see that $\mathcal{E} = (E, \mathcal{C}, I_{\mathcal{G}})$ is a bounded degree Borel graph.

Claim 1.2. [5, Proposition 3.1] *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a bounded degree Borel graph and μ be a Borel probability measure on $(V, \mathcal{B})$. Then there is a Borel probability measure η on $(E, \mathcal{C})$ that satisfies*

$$\mu(\{v \in V : \exists e \in A\ v \in e\}) \leq \Delta(\mathcal{G})\eta(A)$$

for every $A \in \mathcal{C}$. Moreover, if μ is $\mathcal{G}$-invariant (quasi-invariant), then η is $\mathcal{E}$-invariant (quasi-invariant).

2. Approximate König's line coloring Theorem

Let $c; E \to (\Delta(\mathcal{G}) + 1)$ be a proper partial edge coloring. We put $m_c(v) = (\Delta(\mathcal{G}) + 1) \setminus \{c(e) : e \in N(v)\}$ for the set of colors missing at $v \in V$. Note that $m_c(v)$ is always non-empty. We fix a distinguished color $\mathbf{a} \in (\Delta(\mathcal{G}) + 1)$. If $\beta, \gamma \in (\Delta(\mathcal{G}) + 1)$ and $v \in V$, then we write $P^c_{\beta/\gamma}(v) = (f_0, f_1, \dots)$ for the unique (finite or infinite) alternating β/γ-path that starts in v, i.e., $c(f_0) = \beta$, $c(f_1) = \gamma$, etc.

Let $e \in E$ be such that $c(e) = \mathbf{a}$. Using [6, Theorem 18.10] we find a Borel function that assigns to e one of its endpoints $v(e)$ and colors $\beta(e), \gamma(e) \in (\Delta(\mathcal{G})+1) \setminus \mathbf{a}$ such that $\gamma(e) \in m_c(v(e))$ and $\beta(e) \in m_c(w)$ where $e = (v(e), w)$. We put $P^c(e) = e^\frown P^c_{\beta(e)/\gamma(e)}(v(e)) = (e, f_0, f_1, \dots)$.

Let $d; E \to (\Delta(\mathcal{G}) + 1)$ be a proper partial edge coloring. We say that d *improves* c if $\operatorname{dom}(d) = \operatorname{dom}(c)$ and $d^{-1}(\mathbf{a}) \subseteq c^{-1}(\mathbf{a})$. A particular way how to find d that improves c is as follows. Let $e \in E$ be such that $c(e) = \mathbf{a}$ and suppose that the last vertex (if it exists) of $P^c(e)$ is not $w \in V$ where $e = (v(e), w)$. Write f for the last edge of $P^c(e)$ if it exists. Put $d(h) = c(h)$ for every $h \in \operatorname{dom}(c) \setminus P^c(e)$ and define $d(e) = \beta(e)$, $d(f_i) = c(f_{i+1})$ for every $f_i \in P^c(e) \setminus \{f\}$ and $d(f) = \{\beta(e), \gamma(e)\} \setminus \{c(f)\}$. One can easily verify that d is indeed a proper partial edge coloring, $\operatorname{dom}(d) = \operatorname{dom}(c)$ and $d^{-1}(\mathbf{a}) = c^{-1}(\mathbf{a}) \setminus \{e\}$.

Claim 2.1. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a bounded degree Borel graph that does not contain odd cycles. Let $c; E \to (\Delta(\mathcal{G}) + 1)$ be a proper partial edge coloring and $A \subseteq c^{-1}(\mathbf{a})$ be a Borel set such that $P^c(e)$ and $P^c(e')$ are vertex disjoint for every $e \neq e' \in A$. Then there is a proper partial edge coloring $d; E \to (\Delta(\mathcal{G})+1)$ that is improvement of c and $A \cap d^{-1}(\mathbf{a}) = \emptyset$.*

Proof. The fact that the $\mathcal{G}$ does not contain odd cycles implies that $w \in V$ is not the last vertex of $P^c(e)$ for any $e \in A$ where $(v(e), w) = e$. Running the augmenting procedure defined in the preceding paragraph for all $P^c(e)$ simultaneously gives a partial map $d; E \to (\Delta(\mathcal{G}) + 1)$. It follows from the fact that $\{P^c(e)\}_{e \in A}$ is a collection of pairwise vertex disjoint paths that d is well-defined proper partial edge coloring with $\operatorname{dom}(d) = \operatorname{dom}(c)$

and $d^{-1}(\mathbf{a}) \cap A = \emptyset$. Moreover, since $e \mapsto P^c(e)$ is a Borel assignment we see that $d; E \to (\Delta(\mathcal{G})+1)$ is a partial Borel map and the proof is finished. □

Proposition 2.2. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a bounded degree Borel graph that does not contain odd cycles, $c; E \to (\Delta(\mathcal{G})+1)$ be a proper partial edge coloring, η be a $\mathcal{E}$-quasi-invariant Borel probability measure on $(E, \mathcal{C})$, where $\mathcal{E} = (E, \mathcal{C}, I_{\mathcal{G}})$ is the line graph, and $\epsilon > 0$. Then there is a proper partial edge coloring $d; E \to (\Delta(\mathcal{G})+1)$ that improves c such that $\eta(d^{-1}(\mathbf{a})) < \epsilon$.*

Proof. First we show a sufficient condition for d to satisfy the conclusion of the statement. Write ρ for the cocycle of η.

Claim. *Let $L \in \mathbb{N}$ and $d; E \to (\Delta(\mathcal{G})+1)$ be a proper partial edge coloring such that*

$$\sum_{f \in P^d(e)} \rho(f, e) \geq 2\Delta(\mathcal{G})L$$

for every $e \in E$ such that $d(e) = \mathbf{a}$. Then $\eta(d^{-1}(\mathbf{a})) \leq \frac{1}{L}$.

Proof. Let $f \in E$. Then there are at most $2\Delta(\mathcal{G})$-many edges $e \in E$ such that $d(e) = \mathbf{a}$ and $f \in P^d(e)$. The cocycle relation gives

$$2\Delta(\mathcal{G})L\eta(d^{-1}(\mathbf{a})) \leq \int_{d^{-1}(\mathbf{a})} \sum_{f \in P^d_e} \rho(f, e)\ d\eta \leq 2\Delta(\mathcal{G})$$

and that finishes the proof. □

Let $L \in \mathbb{N}$ be such that $\frac{1}{L} < \epsilon$. Set $d_0 = c$ and $E_0 = E$. We construct by induction on all countable ordinals a sequence $\{d_\kappa\}_{\kappa<\aleph_1}$ of proper partial edge colorings and a sequence $\{E_\kappa\}_{\kappa<\aleph_1}$ of Borel η-conull $I_{\mathcal{G}}$-invariant subsets of E such that

(1) d_κ is an improvement of c,
(2) $E_\kappa \subseteq E_\lambda$ whenever $\lambda \leq \kappa < \aleph_1$,
(3) if $\kappa < \aleph_1$ is a limit ordinal, then $\lim_{\lambda\to\kappa} d_\lambda(e) = d_\kappa(e)$ whenever $e \in E_\kappa \cap \mathrm{dom}(c)$,
(4) $E_\kappa \cap B_\kappa \subseteq E_\lambda \cap B_\lambda$ whenever $\lambda \leq \kappa < \aleph_1$,
(5) if $\eta(A_\kappa) > 0$, then $\eta(B_\kappa \setminus B_{\kappa+1}) > 0$

where $B_\kappa = d_\kappa^{-1}(\mathbf{a})$ and $A_\kappa = \left\{e \in B_\kappa : \sum_{f \in P^{d_\kappa}(e)} \rho(f, e) < 2\Delta(\mathcal{G})L\right\}$.

Once we have this then we take the minimal $\kappa_0 < \aleph_1$ such that $\eta(A_{\kappa_0}) = 0$ and define $d = d_{\kappa_0}$. Note that such $\kappa_0 < \aleph_1$ exists by (4) and (5). By the Claim we have $\eta(d^{-1}(\mathbf{a})) \leq \frac{1}{L} < \epsilon$. Hence, d works as required.

Let $\kappa < \aleph_1$ and assume that d_κ is defined and $\eta(A_\kappa) > 0$. There is a pair $\beta, \gamma \in (\Delta(\mathcal{G})+1) \setminus \{\mathbf{a}\}$ and $k \in \mathbb{N} \cup \{+\infty\}$ such that the Borel set A' of those $e \in A_\kappa$ such that $P^{d_\kappa}(e) \setminus \{e\}$ is an alternating β/γ path of length k satisfies $\eta(A') > 0$. If $k = +\infty$, then find 3-sparse Borel set A that is a subset of A' and $\eta(A) > 0$. This can be done by [8, Proposition 4.6]. Note that $\left\{P^{d_\kappa}(e)\right\}_{e\in A}$ is a collection of pairwise vertex disjoint paths. If $k < +\infty$, then find a $2k$-sparse Borel set A that is a subset of A' and $\eta(A) > 0$, again by [8, Proposition 4.6]. Then $\left\{P^{d_\kappa}_e\right\}_{e\in A}$ is a collection of pairwise vertex disjoint paths.

Define $d_{\kappa+1}$ as in Claim 2.1 applied for A and d_κ. Observe that $C_\kappa = \{e \in \operatorname{dom}(c) : d_\kappa(e) \neq d_{\kappa+1}(e)\}$ satisfies $\eta(C_\kappa) \leq 2\Delta(\mathcal{G})L\eta(B_\kappa \setminus B_{\kappa+1})$.

Let $\kappa < \aleph_1$ be a limit ordinal and d_λ be defined for every $\lambda < \kappa$. We have

$$\sum_{\lambda<\kappa} \eta(C_\lambda) \leq 2\Delta(\mathcal{G})L \sum_{\lambda<\kappa} \eta(B_\lambda \setminus B_{\lambda+1}) \leq 2\Delta(\mathcal{G})L$$

because $\{B_\lambda \setminus B_{\lambda+1}\}_{\lambda<\kappa}$ is a pairwise disjoint collection of sets when restricted to the η-conull set $\bigcap_{\lambda<\kappa} E_\lambda$ by (2) an (4). The Borel-Cantelli lemma implies that there is a η-conull $I_\mathcal{G}$-invariant set $H \subseteq E$ such that for every $e \in H \cap \operatorname{dom}(c)$ there is $\lambda_e < \kappa$ such that $e \notin C_\lambda$ for every $\lambda_e \leq \lambda < \kappa$. Define $E_\kappa = H \cap \bigcap_{\lambda<\kappa} E_\lambda$. It follows from (2) and (3), that if $e \in E_\kappa \cap \operatorname{dom}(c)$, then $d_\lambda(e)$ is constant for every $\lambda_e \leq \lambda < \kappa$, i.e., $d'(e) = \lim_{\lambda\to\kappa} d_\lambda(e)$ exists for every $e \in E_\kappa \cap \operatorname{dom}(c)$. Define $d_\kappa = d'$ on E_κ and $d_\kappa = c$ outside of E_κ. □

We remark that for a $\mathcal{E}$-invariant measures the proof can be modified as follows. Fix a sequence of $4\Delta(\mathcal{G})L$-sparse Borel sets $\{C_l\}_{l\in\mathbb{N}} \subseteq \mathcal{C}$ such that every $e \in E$ appears infinitely often. The induction runs over all natural numbers in the same spirit. Namely, define $A_l = \{e \in B_l : |P^{d_l}(e)| < 2\Delta(\mathcal{G})L\}$ and note that $\left\{P^{d_l}(e)\right\}_{e\in A_l\cap C_l}$ is a collection of pairwise vertex disjoint paths. Use Claim 2.1 to define d_{l+1}. Define $d(e) = \lim_{l\to\infty} d_l(e)$. It is easy to see that d is defined for every $e \in \operatorname{dom}(c)$ and $|P^d(e)| \geq 2\Delta(\mathcal{G})L$ for every $e \in d^{-1}(\mathbf{a})$. This implies that d improves c and satisfies $\eta(d^{-1}((\mathbf{a}))) < \frac{1}{L}$ for every $\mathcal{E}$-invariant measure η.

Before we formulate corollaries of the preceding result we would like to point out that the proof of the approximate version of Vizing's Theorem, see [5, Theorem 1.8, Section 5], follows the same strategy as the proof of Proposition 2.2. Namely, modify a given coloring such that it does not contain short weighted Vizing chains, see [5, Section 2.4]. Similarly as in the preceding paragraph, there is an improvement that works simultaneously for every $\mathcal{G}$-invariant measure.

Proof of Theorem 0.2 (I). Since we consider any Borel probability measure and there is $v \in V$ of degree $\Delta(\mathcal{G})$ by the definition of $\Delta(\mathcal{G})$ we have that $\chi'_{AP}(\mathcal{G}) \geq \Delta(\mathcal{G})$.

Let μ be a Borel probability measure on $(V, \mathcal{B})$. By the approximate measurable version of Vizing's Theorem, see [5, Theorem 1.8], we find a proper partial edge coloring $c; E \to (\Delta(\mathcal{G}) + 1)$ such that

$$\mu\left(\{v \in V : N(v) \subseteq \operatorname{dom}(c)\}\right) > 1 - \frac{\epsilon}{2}.$$

Consider the $\mathcal{E}$-quasi-invariant Borel probability measure η on $(E, \mathcal{C})$ that is given by a consecutive application of Claims 1.1, 1.2 and apply Proposition 2.2. This yields a proper partial edge coloring $d'; E \to (\Delta(\mathcal{G}) + 1)$ that improves c and $\eta(d^{-1}(\mathbf{a})) < \frac{\epsilon}{4\Delta(\mathcal{G})}$. Consider an edge coloring $d : E \to \Delta(\mathcal{G})$ that agrees with d' on the set $\bigcup_{\beta\in(\Delta(\mathcal{G})+1)\setminus\{\mathbf{a}\}} d'^{-1}(\beta)$.

Put $X = \{v \in V : N(v) \subseteq \operatorname{dom}(c)\}$ and $Y = \{v \in V : N(v) \cap d'^{-1}(\mathbf{a}) = \emptyset\}$. Let $v \in X \cap Y$. Then it is easy to see that $v \in \mathbf{Corr}(d)$ and we have

$$\mu(V \setminus \mathbf{Corr}(d)) \leq \mu(V \setminus X) + \mu(V \setminus Y) < \frac{\epsilon}{2} + 2\Delta(\mathcal{G})\frac{\epsilon}{4\Delta(\mathcal{G})} < \epsilon$$

by the definition of η. That finishes the proof. □

Corollary 2.3. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a Borel graph that is Δ-regular and that does not contain odd cycles. Then for every Borel probability measure μ on $(V, \mathcal{B})$ and $\epsilon > 0$ there is a Borel matching $M \subseteq E$ such that*

$$\mu(\{v \in V : M \cap N(v) = \emptyset\}) < \epsilon.$$

Proof. By Theorem 0.2 (I) we find an edge coloring $c : E \to \Delta$ such that $\mu(\mathbf{Corr}(c)) > 1-\epsilon$. Let $\beta \in \Delta$. Then $M = c^{-1}(\beta)$ works as required. □

Note the next result needs the full measurable Vizing's Theorem for graphings.

Corollary 2.4. [11, Theorem 3] *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a bounded degree Borel graph that does not contain odd cycles, μ be a $\mathcal{G}$-invariant Borel probability measure on $(V, \mathcal{B})$ and $\epsilon > 0$. Then there is a full μ-measurable proper edge coloring $c : E \to (\Delta(\mathcal{G}) + 1)$ such that*

$$\mu\left(\{v \in V : \mathbf{a} \notin m_c(v)\}\right) < \epsilon.$$

Proof. Combine measurable Vizing's Theorem for invariant measure μ, see [5, Theorem 1.6] or bipartite version [2, Theorem 1.5], and Proposition 2.2. □

3. Approximate balanced orientation

Recall that $(E, \mathcal{C})$ is a standard Borel space of edges of a Borel graph $\mathcal{G} = (V, \mathcal{B}, E)$ and $[E]^{<\infty}$ is the standard Borel space of all finite subsets of E. One can easily verify that the set of all finite paths $\mathfrak{T} \subseteq [E]^{<\infty}$ and cycles $\mathfrak{C} \subseteq [E]^{<\infty}$ of $\mathcal{G}$ are Borel sets.

Proposition 3.1. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a bounded degree Borel graph such that every vertex has even degree. Then there is a Borel set $\mathfrak{M} \subseteq \mathfrak{C}$ such that $C \cap D = \emptyset$ for every $C \neq D \in \mathfrak{M}$ that is maximal with this property. In particular, $\mathcal{H} = (V, \mathcal{B}, E \setminus \bigcup_{C \in \mathfrak{M}} C)$ is an acyclic Borel graph such that every vertex has even degree bounded by $\Delta(\mathcal{G})$.*

Proof. It follows from [7, Lemma 7.3] that the intersection graph on $\mathfrak{C}$ has a countable Borel chromatic number, i.e., there is a sequence of Borel sets $\{\mathfrak{C}_i\}_{i \in \mathbb{N}}$ such that $C \cap D = \emptyset$ whenever $C, D \in \mathfrak{C}_i$ and $\mathfrak{C} = \bigcup_{i \in \mathbb{N}} \mathfrak{C}_i$. Let $\mathfrak{D}_0 = \mathfrak{C}_0$ and define inductively

$$\mathfrak{D}_{i+1} = \mathfrak{D}_i \cup \{C \in \mathfrak{C}_{i+1} : \forall D \in \mathfrak{D}_n \ C \cap D = \emptyset\}.$$

It is easy to see that $\mathfrak{M} = \bigcup_{i \in \mathbb{N}} \mathfrak{D}_i$ works as required. □

Let $\mathfrak{P} \subseteq \mathfrak{T}$ be a collection of finite paths. Then we define $\mathrm{E}(\mathfrak{P})$ to be the set of endpoints of $T \in \mathfrak{P}$.

Proposition 3.2. *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be an acyclic bounded degree Borel graph such that every vertex has even degree. Then there is a sequence $\{\mathfrak{P}_n\}_{n \in \mathbb{N}}$ of Borel subsets of $\mathfrak{T}$ such that*

1. *$E \subseteq \mathfrak{P}_0$,*
2. *$T \cap T' = \emptyset$ for any $T \neq T' \in \mathfrak{P}_n$ and every $n \in \mathbb{N}$,*
3. *for every $n \in \mathbb{N}$ and $T \in \mathfrak{P}_n$ there is a unique $T' \in \mathfrak{P}_{n+1}$ such that $T \subseteq T'$,*
4. *$\bigcap_{n \in \mathbb{N}} \mathrm{E}(\mathfrak{P}_n) = \emptyset$.*

Proof. If $e,f\in E$, then we define $d(e,f)$ as the minimal size of a path $P\in \mathfrak{T}$ that connects e and f. For $T\in\mathfrak{T}$ we put $d(e,T)=\max\{d(e,f):f\in T\}$. Because $\mathcal{G}$ is of bounded degree we find a sequence $\{A_n\}_{n\in\mathbb{N}}$ of Borel subsets of E such that $|\{n\in\mathbb{N}:e\in A_n\}|=\infty$ for every $e\in E$ and

$$\mathfrak{s}(n)=\min\{d(e,f):e\neq f\in A_n\}\to\infty,$$

see [8, Proposition 4.6].

As a first step we define inductively a collection $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ that satisfies (1)–(3) and a relaxation of (4). Set $\mathcal{P}_0=E$. Suppose that $\mathcal{P}_n$ satisfies (2) and (3) and let $P\in\mathcal{P}_n$ be such that $A_n\cap P\neq\emptyset$. Choose in a Borel way a $\subseteq$-maximal path extension $\widetilde{P}$ of P that consists of elements from $\mathcal{P}_n$, say

$$\widetilde{P}=R_1{}^\frown\dots{}^\frown R_l{}^\frown P^\frown R_{l+1}{}^\frown\dots{}^\frown R_k$$

where $R_i\in\mathcal{P}_n$, such that for every $i\leq k$ there is $e\in P\cap A_n$ such that $d(e,R_i)<\frac{\mathfrak{s}(n)}{3}$.

Let $P,Q\in\mathcal{P}_n$ be such that $P\cap A_n\neq\emptyset\neq Q\cap A_n$ and $\widetilde{P}\cap\widetilde{Q}\neq\emptyset$. We show that $P=Q$. It follows from the inductive assumption that there is $R\in\mathcal{P}_n$ such that $R\subseteq\widetilde{P},\widetilde{Q}$. If $R=P=Q$, then we are done. Suppose that $R\neq P$ and $R=Q$. By the definition of $\widetilde{P}$ there is $e\in A_n\cap P$ such that $d(e,f)\leq\frac{\mathfrak{s}(n)}{3}$ for every $f\in R=Q$. In particular, there is $f\in A_n\cap Q$ such that $d(e,f)\leq\frac{\mathfrak{s}(n)}{3}$ and that contradicts the definition of $\mathfrak{s}(n)$ because $e\neq f$. Suppose now that $R\neq P$ and $R\neq Q$. By the definition of $\widetilde{P}$ and $\widetilde{Q}$ we find $e\in A_n\cap P$ and $f\in A_n\cap Q$ such that $d(e,h),d(f,h)\leq\frac{\mathfrak{s}(n)}{3}$ for every $h\in R$. Consequently, $d(e,f)\leq d(e,h)+d(f,h)\leq\frac{2\mathfrak{s}(n)}{3}$ for any $h\in R$ and we must have $e=f$ by the definition of $\mathfrak{s}(n)$. By the inductive assumption (2) we conclude that $P=Q$.

Let

$$\mathcal{Q}_{n+1}=\{\widetilde{P}:P\in\mathcal{P}_n\ \wedge\ A_n\cap P\neq\emptyset\}$$

and define

$$\mathcal{P}_{n+1}=\mathcal{Q}_{n+1}\cup\{P\in\mathcal{P}_n:\forall Q\in\mathcal{Q}_{n+1}\ P\cap Q=\emptyset\}.$$

It follows from the previous argument that $\mathcal{P}_{n+1}$ satisfies (2) and (3).

The construction guarantees the following relaxation of (4). Let $e\in E$ and $n\in\mathbb{N}$. It follows from (1)–(3) that there is a unique $P_{e,n}\in\mathcal{P}_n$ such that $e\in P_{e,n}$. Note that if $f\in P_{e,n}$, then $P_{e,n}=P_{f,n}$. Define $P_e=\bigcup_{n\in\mathbb{N}}P_{e,n}$. Then P_e is a path by (1)–(3) and we show that it is infinite. Suppose for a contradiction that P_e is finite, i.e., $|P_e|=m\in\mathbb{N}$. Let $v\in V$ be an end point of P_e. By the assumption that the degree of v is even we find $f\in N(v)\setminus P_e$ such that v is an endpoint of P_f. Since $\mathcal{G}$ is acyclic we have $P_f\neq P_e$. Let $n\in\mathbb{N}$ be such that $f\in A_n$, $3m<\mathfrak{s}(n)$ and $P_e=P_{e,n}$. By the definition we have that $P_{f,n+1}\in\mathcal{P}_{n+1}$ is a $\subseteq$-maximal extension of $P_{f,n}\in\mathcal{P}_n$ that satisfies the condition above. However, $Q=P_{f,n+1}{}^\frown P_{e,n}$ is an extension of $\mathcal{P}_{f,n+1}$ and also satisfies the condition because $d(f,P_{e,n})\leq m<\frac{\mathfrak{s}(n)}{3}$. This shows that P_e is infinite for every $e\in E$.

Now we pair the one ended rays of $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ to obtain $\{\mathfrak{P}_n\}_{n\in\mathbb{N}}$ that satisfies (1)–(4). Let $v\in V$ and $E(v)\subseteq N(v)$ be the set of edges $e\in N(v)$ such that v is an endpoint of P_e. Since the degree of v is even it follows that $|E(v)|$ is even for every $v\in V$. Note that $E(v)\cap E(w)=\emptyset$ for any $v\neq w\in V$ because P_e has at most one endpoint for every $e\in E$.

Let $I : E \to E$ be an involution such that $I(e) \neq e$ if and only if there is $v \in V$ such that $e \in E(v)$ and in that case $I(e) \in E(v)$, i.e., I is a pairing when restricted to any $E(v)$.

Let $M = \bigcup_{v \in V} E(v)$ and define

$$\mathfrak{P}_n = \{P_{e,n} \cup P_{I(e),n} : e \in M\} \cup \{P \in \mathcal{P}_n : P \cap M = \emptyset\}.$$

Property (1) is trivially satisfied. Note that $|M \cap P_{e,n}| \leq 1$ for every $e \in E$ and $n \in \mathbb{N}$ since P_e is infinite. This implies that if $P, Q \in \mathfrak{P}_n$ and $P \cap Q \neq \emptyset$, then $P = Q$. Consequently we have (2). Similarly we get that $P_{e,n} \cup P_{I(e),n} \subseteq P_{e,n+1} \cup P_{I(e),n+1}$ for every $e \in M$ and that gives (3).

Let $v \in \bigcap_{n \in \mathbb{N}} E(\mathfrak{P}_n)$. Then there is a sequence $T_n \in \mathfrak{P}_n$ such that v is an endpoint of T_n. By (3) we may assume that $T_n \subseteq T_{n+1}$ and there is $e \in N(v)$ such that $e \in T_n$ for every $n \in \mathbb{N}$. Note that $P_{e,n} \subseteq T_n$ by the definition of $\mathfrak{P}_n$. Consequently, v is an endpoint of P_e. We have $T_0 = \{e, I(e)\}$ by the definition of $\mathfrak{P}_0$. That is a contradiction because v is not an endpoint of T_0. This shows (4) and finishes the proof. □

Theorem 3.3 (Theorem 0.2 (II))**.** *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a bounded degree Borel graph such that every vertex has an even degree. Then there is a sequence $\{S_n\}_{n \in \mathbb{N}}$ of orientations of $\mathcal{G}$ such that*

$$\mu\left(\{v \in \mathbf{Corr}(S_n) : N(v) \subseteq \mathbf{Corr}(S_n)\}\right) \to 1$$

for every Borel probability measure μ on $(V, \mathcal{B})$. In particular, $\mathcal{G}$ admits approximate balanced orientation.

Proof. Proposition 3.1 produces a maximal Borel set $\mathfrak{M}$ of pairwise disjoint cycles and Proposition 3.2 applied to the acyclic Borel graph $\mathcal{H} = (V, \mathcal{B}, E \setminus \bigcup_{C \in \mathfrak{M}} C)$ produces a sequence $\{\mathfrak{P}_n\}_{n \in \mathbb{N}}$.

Note that since $\mathfrak{M} \cup \mathfrak{P}_n$ is a collection of finite paths and cycles that cover E it is easy to produce an orientation S_n of $\mathcal{G}$ such that $v \notin \mathbf{Corr}(S_n)$ only if v is an endpoint of some $T \in \mathfrak{P}_n$. This implies that

$$\bigcap_{n \in \mathbb{N}} V \setminus \{v \in \mathbf{Corr}(S_n) : N(v) \subseteq \mathbf{Corr}(S_n)\} =$$

$$= \bigcap_{n \in \mathbb{N}} \{v \in V : v \in E(\mathfrak{P}_n) \ \vee \ N_0(v) \cap E(\mathfrak{P}_n) \neq \emptyset\} = \emptyset$$

where $N_0(v) = \{w \in V : (v, w) \in E\}$. That finishes the proof. □

4. Approximate Schreier decoration

Before proving the remaining part of Theorem 0.2 we remind the reader how to use (I) and (II) in the finite setting. Suppose that $G = (V, E)$ is a finite 2Δ-regular graph. Then by (II) we find a balanced orientation $S \subseteq E$ of G. Consider now a bipartite graph $H = (V_0 \sqcup V_1, F)$ where the bipartition is formed by two disjoint copies of V and there is an edge $(v, w) \in F$, where $v \in V_0$ and $w \in V_1$, if and only if there is an oriented edge pointing from v to w in S. The fact that S is balanced implies that H is Δ-regular. By (I), i.e., König's Theorem, we find a proper coloring $c' : F \to \Delta$. This induces a coloring $c : E \to \Delta$ such that (S, c) is a Schreier decoration of G.

Theorem 4.1 (Theorem 0.2 (III))**.** *Let $\mathcal{G} = (V, \mathcal{B}, E)$ be a 2Δ-regular Borel graph. Then $\mathcal{G}$ admits approximate Schreier decoration.*

Proof. Let μ be a Borel probability measure on $(V, \mathcal{B})$ and $\epsilon > 0$. Use Theorem 3.3 to find an orientation S of $\mathcal{G}$ such that

$$\mu(\{v \in \mathbf{Corr}(S) : N(v) \subseteq \mathbf{Corr}(S)\}) > 1 - \frac{\epsilon}{2}.$$

Define a bipartite Borel graph $\mathcal{H} = (C^0 \sqcup C^1, \mathcal{D}, H)$ where C^0, C^1 are disjoint copies of $\mathbf{Corr}(S)$, $\mathcal{D}$ is the corresponding σ-algebra and $(v^0, w^1), (w^1, v^0) \in H$ if and only if $(v, w) \in S$ where $v^0 \in C^0$, $w^1 \in C^1$ are the copies of $v, w \in \mathbf{Corr}(S)$. It follows from the definition of $\mathbf{Corr}(S)$ that the maximum degree of $\mathcal{H}$ is bounded by Δ. Define a Borel probability measure ν on $C^0 \sqcup C^1$ as

$$\nu(A \sqcup B) = \frac{\mu(A) + \mu(B)}{2\mu(\mathbf{Corr}(S))}$$

whenever $A \subseteq C^0$ and $B \subseteq C^1$ are Borel sets. Theorem 0.2 (I) gives an edge coloring $c' : H \to \Delta$ such that $\nu(\mathbf{Corr}(c')) > 1 - \frac{\epsilon}{4\mu(\mathbf{Corr}(S))}$. Let $c : E \to \Delta$ be an edge coloring that extends c', i.e., $c(v, w) = c(w, v) = c'(v^0, w^1)$ whenever $(v, w) \in S$ and $v, w \in \mathbf{Corr}(S)$.

Write $X = \{v \in \mathbf{Corr}(S) : N(v) \subseteq \mathbf{Corr}(S)\}$ and $Y_i = \{v \in V : v^i \in \mathbf{Corr}(c) \cap C^i\}$ where $i < 2$. Let $v \in X \cap Y_0 \cap Y_1$. Then it is easy to see that $v \in \mathbf{Corr}(S, c)$. We have

$$\mu(V \setminus X \cap Y_0 \cap Y_1) \leq \mu(V \setminus X) + \mu(V \setminus Y_0) + \mu(V \setminus Y_1) < \epsilon$$

because $\mu(V \setminus Y_0) + \mu(V \setminus Y_1) \leq 2\mu(\mathbf{Corr}(S))\nu(C^0 \sqcup C^1 \setminus \mathbf{Corr}(c')) < \frac{\epsilon}{2}$. This finishes the proof. □

5. Remarks

The ultraproduct technique for graphings, see [3] or [1], implies that we can find a locally-globally equivalent graphing that is an extension of the original one and satisfies fully (not just approximately) the corresponding conditions in Theorem 0.2 or Corollary 2.3. We refer the reader to [10, Chapter 19] for the corresponding definitions.

For example, if $(\mathcal{G}, \mu)$ is a graphing where $\mathcal{G} = (V, \mathcal{B}, E)$ is a 2Δ-regular Borel graph, then there is a 2Δ-regular Borel graph $\mathcal{G}' = (V', \mathcal{B}', E')$, a $\mathcal{G}'$-invariant Borel probability measure μ' and a Borel map $\varphi : V' \to V$ such that φ is a local isomorphisms, $\varphi^*\mu' = \mu$, $(\mathcal{G}, \mu)$, $(\mathcal{G}', \mu')$ are locally-globally equivalent and $\mathcal{G}$ admits a Schreier decoration, i.e., it is induced by a pmp action of the free group F_Δ. This extends [11, Corollary 4] and implies [11, Theorem 1].

Similar statement hold in the case of quasi-invariant probability measures when the notion of local-global equivalence is extended appropriately.

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