ULTRAFILTER EXTENSIONS OF ASYMPTOTIC DENSITY

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ABSTRACT. We characterize for which ultrafilters on ω is the ultrafilter extension of the asymptotic density on natural numbers σ-additive on the quotient boolean algebra $\mathcal{P}(\omega)/d_{\mathcal{U}}$ or satisfies similar additive condition on $\mathcal{P}(\omega)/\text{fin}$. These notions were defined in [2] under the name $\boldsymbol{AP}(\text{null})$ and $\boldsymbol{AP}(*)$. We also present a characterization of a P- and semiselective ultrafilters using the ultraproduct of σ-additive measures.

This paper is based on the author's Bachelor thesis that was supervised by Bohuslav Balcar and defended in 2014. We investigate additive properties of measures on $\mathcal{P}(\omega)$ that are extensions of asymptotic density as defined in [2]. More concretely in Section 2 we give a necessary and sufficient combinatorial condition for an ultrafilter $\mathcal{U}$ on ω for the extension of asymptotic density given by $\mathcal{U}$ to satisfy $\boldsymbol{AP}(\text{null})$ or $\boldsymbol{AP}(*)$. In Section 3 we characterize P- and semiselective ultrafilters by a relations between some ideals in an ultraproduct of measures.

We note that since 2014 there has been made some progress in similar direction of density measures and additivity properties (see [4]).

1. Introduction

Let B be a boolean algebra and $m : B \to [0,1]$. We say that m is

- *monotone* if $m(a) \leq m(b)$ whenever $a \leq b \in B$,
- *strictly positive* if $m(a) = 0$ implies that $a = 0$,
- a *measure* if m is monotone, $m(1) = 1$ and $m\left(\bigvee_{i<n} a_i\right) = \sum_{i<n} m(a_i)$ for every finite antichain $\{a_i\}_{i<n} \subseteq B$,
- *σ-additive* if m is a measure and $m\left(\bigvee_{i<\omega} a_i\right) = \sum_{i<\omega} m(a_i)$ for every antichain $\{a_i\}_{i<\omega} \subseteq B$.

If m is a measure on B, then define $\mathcal{N}(m) = \{a \in B : m(a) = 0\}$. The quotient boolean algebra $B/\mathcal{N}(m)$ carries a unique strictly positive measure that is naturally derived from m. We will abuse the notation and write B/m for the quotient algebra, m for the unique induced measure on B/m and $[a]$ for the equivalence class of $a \in B$. The following theorem is in fact a corollary of a stronger statement from [5] but this version is sufficient for our purposes. Recall that a boolean algebra B is σ-complete if every countable subset of B has a supremum in B.

Theorem 1.1 (Smith–Tarski [5]). *Let m be a measure on a σ-complete boolean algebra B. Then B/m is a c.c.c. complete boolean algebra.*

The author was supported by the GACR project 17-33849L and RVO: 67985840.

We use ω for the set of natural numbers. We write n for the set $\{0,1,....,n-1\}$ and $[r,s]$ for the set $\{n\in\omega : r\le n\le s\}$ where $r,s\in\mathbb{R}$. Recall that a set $A\subset\omega$ has an asymptotic density if

$$\lim_{n\to\infty}\frac{|A\cap n|}{n}$$

exists, and in that case we denote the value of the limit as $d(A)$. We say that a measure m on $\mathcal{P}(\omega)$ is a *density* if it extends the asymptotic density, i.e. $m(A)=d(A)$ for every $A\subseteq\omega$ for which the asymptotic density exists. Note that a density m cannot be σ-additive on $\mathcal{P}(\omega)$ because it has the value 0 on each atom. Since the algebra $\mathcal{P}(\omega)/m$ is σ-complete by Theorem 1.1, it is natural to ask whether the density m is σ-additive on $\mathcal{P}(\omega)/m$. This question was considered in [2] where the authors define two additive properties for measures on $\mathcal{P}(\omega)$.

Definition 1.2. [2] *A measure m on $\mathcal{P}(\omega)$ satisfies $\boldsymbol{AP}(\text{null})$ if for every inclusion increasing sequence $\{A_n\}_{n<\omega}$ of subsets of ω there is $B\subseteq\omega$ such that*

- $\lim_{n\to\infty} m(A_n)=m(B)$,
- $m(A_n\setminus B)=0$ *for every* $n<\omega$.

If we can moreover find such B that also satisfies

- $|A_n\setminus B|<\omega$ *for every* $n<\omega$,

then we say that m satisfies $\boldsymbol{AP}(*)$.

One can easily check that $\boldsymbol{AP}(\text{null})$ is equivalent with the σ-additivity of m on $\mathcal{P}(\omega)/m$. It is known (see [2]) that there are densities that satisfy $\boldsymbol{AP}(\text{null})$ but there are also densities that fail to have $\boldsymbol{AP}(\text{null})$. The question about $\boldsymbol{AP}(*)$ is more complicated since there is a model of ZFC in which no density satisfies $\boldsymbol{AP}(*)$. On the other hand it is also consistent that densities satisfying $\boldsymbol{AP}(*)$ do exist, for example the existence of a P-ultrafilter is sufficient.

Definition 1.3. *Let $\mathcal{U}$ be an ultrafilter on ω. Define*

$$d_{\mathcal{U}}(A)=\mathcal{U}\text{-}\lim\frac{|A\cap n|}{n}$$

for every $A\subseteq\omega$.

We call densities of the form $d_{\mathcal{U}}$ ultrafilter densities. All examples presented in [2] are in fact ultrafilter densities. The aim of this paper is to give a complete combinatorial characterization of ultrafilters for which the ultrafilter densities satisfy $\boldsymbol{AP}(\text{null})$ or $\boldsymbol{AP}(*)$. Let us state here the case of $\boldsymbol{AP}(\text{null})$ and postpone the more technical case of $\boldsymbol{AP}(*)$ until the end of Section 2.

Definition 1.4. *We say that an ultrafilter $\mathcal{U}$ on ω is* $\times$-invariant *if for all $U\in\mathcal{U}$ there is $1<k\in\omega$ such that*

$$kU=\bigcup_{n\in U}[kn,(k+1)n]\in\mathcal{U}.$$

The following is the main result of this paper and Section 2 is devoted to the proof of this statement.

Theorem 1.5. *Let $\mathcal{U}$ be an ultrafilter on ω. The following are equivalent*

- $d_{\mathcal{U}}$ *is σ-additive on $\mathcal{P}(\omega)/d_{\mathcal{U}}$ (i.e. satisfies $\boldsymbol{AP}(\mathrm{null})$),*
- $\mathcal{U}$ *is not $\times$-invariant.*

2. Ultrafilter Densities

In this section we present the proof of Theorem 1.5. We start with some general facts about ultrafilters on ω. All ultrafilters considered in this section are non-principal.

Claim 2.1. *Let $\mathcal{U}$ be a $\times$-invariant ultrafilter (see Definition 1.4). Then for every $U \in \mathcal{U}$ there are infinitely many $k < \omega$ such that*

$$kU = \bigcup_{n \in U} [kn, (k+1)\, n] \in \mathcal{U}.$$

Proof. Assume that for a given $U \in \mathcal{U}$ there is some maximal k such that $kU \in \mathcal{U}$. Then there must be some $2 \leq l < \omega$ such that

$$l\,(kU) = \bigcup_{m \in kU} [lm, (l+1)\, m] \subseteq \bigcup_{n \in U} [lkn, (l+1)\,(k+1)\, n] \in \mathcal{U}.$$

Because $\mathcal{U}$ is an ultrafilter, there must be some $p < \omega$ such that $lk \leq p \leq (l+1)\,(k+1) - 1$ and $pU \in \mathcal{U}$. Now $2k \leq lk \leq p$ contradicts the maximality of k. □

In order to prove our main result we need to investigate which ultrafilters give rise to the same ultrafilter densities.

Definition 2.2. *Let $\mathcal{U}, \mathcal{V}$ be ultrafilters. We say that $\mathcal{U}$* is close to *$\mathcal{V}$ if for every $U \in \mathcal{U}$ and for every $\epsilon > 0$ there is $V \in \mathcal{V}$ such that*

- *for all $x \in U$ there is $y \in V$ such that* $\max\left\{\left|1 - \frac{x}{y}\right|, \left|1 - \frac{y}{x}\right|\right\} < \epsilon$,
- *for all $x \in V$ there is $y \in U$ such that* $\max\left\{\left|1 - \frac{x}{y}\right|, \left|1 - \frac{y}{x}\right|\right\} < \epsilon$.

Claim 2.3. *Let $\mathcal{U}, \mathcal{V}$ be ultrafilters. Then $\mathcal{U}$ is close to $\mathcal{V}$ if and only if*

$$U_\epsilon = \left\{x < \omega : \exists n \in U \;\; \max\left\{\left|1 - \frac{n}{x}\right|, \left|1 - \frac{x}{n}\right|\right\} < \epsilon\right\} \in \mathcal{V}$$

for every $\epsilon > 0$.

Proposition 2.4. *The relation of being close is an equivalence relation on the set of ultrafilters.*

Proof. Suppose that $\mathcal{U}$ is close to $\mathcal{V}$ but $\mathcal{V}$ is not close to $\mathcal{U}$. Then there is $\delta > 0$ and $V \in \mathcal{V}$ such that $V_\delta \notin \mathcal{U}$. Therefore $B = \omega \setminus V_\delta \in \mathcal{U}$. Then $B_\delta \cap V = \emptyset$ because if $x \in B_\delta \cap V$, then there exists $y \in B$ such that $\max\left\{\left|1 - \frac{x}{y}\right|, \left|1 - \frac{y}{x}\right|\right\} < \delta$ and also $x \in V$ implies $y \in \omega \setminus B$. Claim 2.3 gives us that $B_\delta \cap V = \emptyset \in \mathcal{V}$, a contradiction.

In order to prove that the relation is transitive first notice that

$$U_\epsilon = \bigcup_{n \in U} \left[n(1-\epsilon), \frac{n}{(1-\epsilon)} \right].$$

Assume now that $\mathcal{U}$ is close to $\mathcal{V}$, $\mathcal{V}$ is close to $\mathcal{W}$ and take $U \in \mathcal{U}$. We know that $U_\epsilon \in \mathcal{V}$ and $(U_\epsilon)_\epsilon \in \mathcal{W}$ but

$$U_{2\epsilon-\epsilon^2} = \bigcup_{n \in U} \left[n(1-\epsilon)^2, \frac{n}{(1-\epsilon)^2} \right] \supseteq (U_\epsilon)_\epsilon \in \mathcal{W}.$$

Since $\epsilon > 0$ was arbitrary we see that $\mathcal{U}$ is close to $\mathcal{W}$. □

Once we have established Proposition 2.4 we can write that a pair of ultrafilters $\mathcal{U}, \mathcal{V}$ *is close* since the relation $\mathcal{U}$ is close to $\mathcal{V}$ is symmetric. Note also that $\mathcal{U}, \mathcal{V}$ are close if and only if

$$\langle \{U_\epsilon : U \in \mathcal{U},\ \epsilon > 0\} \rangle = \langle \{V_\epsilon : V \in \mathcal{V},\ \epsilon > 0\} \rangle,$$

where $\langle \mathcal{A} \rangle$ denotes the filter generated by $\mathcal{A} \subseteq \mathcal{P}(\omega)$.

Theorem 2.5. *Let $\mathcal{U}, \mathcal{V}$ be close ultrafilters. Then $d_\mathcal{U} = d_\mathcal{V}$ and $\mathcal{U}$ is $\times$-invariant if and only if $\mathcal{V}$ is $\times$-invariant.*

Proof. Let $A \subseteq \omega$ and $\epsilon > 0$ be given. Find a set $U \in \mathcal{U}$ such that

$$\left| d_\mathcal{U}(A) - \frac{|A \cap n|}{n} \right| < \epsilon$$

holds for every $n \in U$. Since $\mathcal{U}, \mathcal{V}$ are close, we have that $U_\epsilon \in \mathcal{V}$. Let $x \in U_\epsilon$ and $n \in U$ such that $\max\left\{ \left|1 - \frac{n}{x}\right|, \left|1 - \frac{x}{n}\right| \right\} < \epsilon$. We have

$$\left| d_\mathcal{U}(A) - \frac{|A \cap x|}{x} \right| \le \left| d_\mathcal{U}(A) - \frac{|A \cap n|}{n} \right| + \left| \frac{|A \cap n|}{n} - \frac{|A \cap x|}{x} \right| < 3\epsilon$$

because if for example $n \le x$, then

$$\left| \frac{|A \cap n|}{n} - \frac{|A \cap x|}{x} \right| \le \frac{|A \cap n|}{n} \left|1 - \frac{n}{x}\right| + \frac{x-n}{x} < \epsilon + \epsilon < 2\epsilon.$$

We may conclude that $d_\mathcal{V}(A) = d_\mathcal{U}(A)$.

Next suppose that $\mathcal{U}$ is $\times$-invariant and let $V \in \mathcal{V}$ be given. We know from Claim 2.3 that $V_{\frac{1}{4}} = \left\{ y : \exists n \in V \ \max\left\{ \left|1 - \frac{n}{y}\right|, \left|1 - \frac{y}{n}\right| \right\} < \frac{1}{4} \right\} \in \mathcal{U}$. Therefore using Claim 2.1 there exists $4 \le k < \omega$ such that $kV_\epsilon \in \mathcal{U}$. We show that there are $\delta > 0$ and $3 \le b < \omega$ such that

$$(kV_\epsilon)_\delta \subseteq \bigcup_{n \in V} [2n, bn].$$

Once we have this the proof is finished because $(kV_\epsilon)_\delta \in \mathcal{V}$. We describe how to find δ and $3 \le b < \omega$. By a simple computation it follows that

$$V_\epsilon = \bigcup_{n \in V} \left[n\left(1 - \frac{1}{4}\right), \frac{n}{\left(1 - \frac{1}{4}\right)} \right],$$

therefore

$$(kV_\epsilon)_\delta = \bigcup_{n\in V}\left[kn\left(1-\frac{1}{4}\right)(1-\delta),\frac{(k+1)\,n}{\left(1-\frac{1}{4}\right)(1-\delta)}\right].$$

We see that if we choose $\delta < \frac{1}{3}$ and $b \geq \frac{(k+1)}{\left(1-\frac{1}{4}\right)(1-\delta)}$, we have the desired conclusion. □

Next we show that close to any given ultrafilter there is a thin ultrafilter. Recall that an ultrafilter $\mathcal{V}$ is thin if

$$\inf_{V\in\mathcal{V}}\left\{\limsup_{n\to\infty}\frac{F_V(n)}{F_V(n+1)}\right\} = 0,$$

where $F_A(n)$ is the n-th element of A, i.e. F_A is the enumerating function of A. Note that an ultrafilter $\mathcal{V}$ is thin if and only if there is a set $V \in \mathcal{V}$ such that

$$\limsup_{n\to\infty}\frac{F_V(n)}{F_V(n+1)} < 1.$$

Denote $I_n = [2^n, 2^{n+1})$ for every $n < \omega$.

Proposition 2.6. *Let $\mathcal{U}$ be an ultrafilter. For every $\epsilon,\delta > 0$ there is a set $U \in \mathcal{U}$ such that for every $x < y \in U$*

$$\frac{x}{y} < \epsilon \textit{ or } \frac{x}{y} > 1-\delta.$$

Proof. Let $\alpha : \omega \to \{0,1\}$. Inductively define intervals $I_n^{\alpha\restriction k}$ for $k \in \omega$ as

- $I_n^{\alpha\restriction 0} := I_n$,
- for $0 < k \leq n$ if $\alpha(k-1) = 0$ put $I_n^{\alpha\restriction k}$ to be the left half of the interval $I_n^{\alpha\restriction k-1}$,
- for $0 < k \leq n$ if $\alpha(k-1) = 1$ put $I_n^{\alpha\restriction k}$ to be the right half of the interval $I_n^{\alpha\restriction k-1}$,
- for $k > n$ put $I_n^{\alpha\restriction k} := I_n^{\alpha\restriction n}$.

There exists $\alpha_{\mathcal{U}} : \omega \to \{0,1\}$ such that for every $k \in \omega$

$$\bigcup_{n\in\omega} I_n^{\alpha_{\mathcal{U}}\restriction k} \in \mathcal{U}.$$

Let $x < y \in I_n^{\alpha_{\mathcal{U}}\restriction k}$. Since $\left|I_n^{\alpha_{\mathcal{U}}\restriction k}\right| = 2^{\max\{n-k,0\}}$ we have that

$$\frac{x}{y} > \frac{2^n}{2^n + \left|I_n^{\alpha_{\mathcal{U}}\restriction k}\right|} = \frac{2^n}{2^n+2^{n-k}} = 1 - \frac{2^{n-k}}{2^n+2^{n-k}} > 1-\frac{1}{2^k}.$$

Finally it is enough to observe that for every $k < \omega$ and $\mathcal{U}$ there is $A \subseteq \omega$ such that $\bigcup_{n\in A} I_n \in \mathcal{U}$ and $(A+j)\cap A = \emptyset$ for every $j < k$. If $n < m \in A$, $x \in I_n$ and $y \in I_m$, then

$$\frac{x}{y} < \frac{2^{n+1}}{2^m} \leq \frac{2^{n+1}}{2^{n+k}} \leq \frac{1}{2^{k+1}}.$$

To finish the proof it is enough to combine the two estimates. □

We use the function $\alpha_{\mathcal{U}}$ that was defined in the proof of Proposition 2.6 for the next definition.

Definition 2.7. *Let $\mathcal{U}$ be an ultrafilter on ω. Define the function $\alpha_{\mathcal{U}}$ as in the proof of Proposition 2.6. Let*

$$A_{\mathcal{U}} = \bigcap_{k<\omega} \bigcup_{n<\omega} I_n^{\alpha_{\mathcal{U}} \restriction k}.$$

The ultrafilter $G(\mathcal{U})$ is defined by $U \in G(\mathcal{U})$ if

$$\bigcup \{I_n : I_n \cap U \cap A_{\mathcal{U}} \neq \emptyset\} \in \mathcal{U}.$$

Proposition 2.8. *Let $\mathcal{U}$ be an ultrafilter. Then $G(\mathcal{U})$ is a thin ultrafilter and $\mathcal{U}, G(\mathcal{U})$ are close.*

Proof. From the definition it follows that $G(\mathcal{U})$ is a non-principal ultrafilter and we have $\limsup_{n\to\infty} \frac{F_{A_{\mathcal{U}}}(n)}{F_{A_{\mathcal{U}}}(n+1)} < 1$. Since $A_{\mathcal{U}} \in G(\mathcal{U})$, it follows that $G(\mathcal{U})$ is thin.

Let $\epsilon > 0$ and $V \in G(\mathcal{U})$ be given. We may assume that $V \subseteq A_{\mathcal{U}}$. Find $k < \omega$ such that $\max\left\{\left|1 - \frac{x}{y}\right|, \left|1 - \frac{y}{x}\right|\right\} < \epsilon$ for every $n < \omega$ and every $x, y \in I_n^{\alpha \restriction k}$. Then

$$V_\epsilon \supseteq U = \bigcup \left\{I_n^{\alpha\restriction k} : V \cap I_n^{\alpha \restriction k} \neq \emptyset\right\} \in \mathcal{U}.$$

□

Corollary 2.9. *Let $\mathcal{U}$ be an ultrafilter. Then $d_{\mathcal{U}} = d_{G(\mathcal{U})}$ and $\mathcal{U}$ is $\times$-invariant if and only if $G(\mathcal{U})$ is $\times$-invariant.*

The last ingredient needed for the proof of Theorem 1.5 is the ultraproduct of measures. Let us define for a non-principal ultrafilter $\mathcal{U}$ a measure $m_{\mathcal{U}}$ on the set $\prod_{n\in\omega} \mathcal{P}(n)$ by putting

$$m_{\mathcal{U}}(f) = \mathcal{U}\text{-}\lim_{n\to\infty} \frac{|f(n)|}{n},$$

i.e. we are taking the measure ultraproduct of the sequence $(\mathcal{P}(n))_{n<\omega}$ where each $\mathcal{P}(n)$ is endowed with the normalized counting measure. Next we consider the embedding $e \colon \mathcal{P}(\omega) \to \prod_{n\in\omega} \mathcal{P}(n)$ defined for $A \subseteq \omega$ as $e(A)(n) = A \cap n$. Immediately from the definitions we have $m_{\mathcal{U}}(e(A)) = d_{\mathcal{U}}(A)$. Therefore the embedding e lifts to the quotients, i.e.

$$e \colon \mathcal{P}(\omega)/d_{\mathcal{U}} \to \prod_{n\in\omega} \mathcal{P}(n)/m_{\mathcal{U}}.$$

It is well-known that the measure $m_{\mathcal{U}}$ on $\prod_{n\in\omega} \mathcal{P}(n)/m_{\mathcal{U}}$ is σ-additive (see [3]).

Proposition 2.10. *Let $\mathcal{U}$ be a thin ultrafilter. Then the density $d_{\mathcal{U}}$ is σ-additive if and only if the embedding e is isomorphism.*

Proof. Let $f \in \prod_{n\in\omega} \mathcal{P}(n)$ and $\epsilon > 0$ be given. We show that there is $A \subseteq \omega$ such that $|m_{\mathcal{U}}(e(A) \triangle f)| < \epsilon$. Because $\mathcal{U}$ is thin, there is $U \in \mathcal{U}$ such that

$$\frac{F_U(n)}{F_U(n+1)} < \epsilon.$$

We define

$$A := \bigcup_{n<\omega} \left(\left[F_{U(n)}, F_{U(n+1)} \right] \cap f \left(F_U \left(n+1 \right) \right) \right).$$

We have for every $n < \omega$ that

$$\left| \frac{\left| \left(e\left(A\right) \left(F_U\left(n+1\right)\right)\right) \triangle f\left(F_U\left(n+1\right)\right) \right|}{F_U\left(n+1\right)} \right| \leq \frac{F_U\left(n\right)}{F_U\left(n+1\right)} < \epsilon.$$

This implies that $e\left(\mathcal{P}\left(\omega\right)/d_{\mathcal{U}}\right)$ is dense in $\prod_{n\in\omega} \mathcal{P}\left(n\right)/m_{\mathcal{U}}$, therefore $d_{\mathcal{U}}$ is σ-additive if and only if e is surjective. □

We are now ready to prove our main result.

Proof of Theorem 1.5. Assume first that $\mathcal{U}$ is thin and not $\times$-invariant. We show that e is onto. Let $f \in \prod_{n\in\omega} \mathcal{P}\left(n\right)$. We find $A \subseteq \omega$ such that $\left|m_{\mathcal{U}}\left(e\left(A\right) \triangle f\right)\right| = 0$. Let $U \in \mathcal{U}$ such that for every $3 \leq k < \omega$ is

$$U_k = \left(\omega \setminus \bigcup_{n\in U} \left[2n, kn\right] \right) \cap U \in \mathcal{U}$$

and $\frac{F_U(n)}{F_U(n+1)} < \frac{1}{2}$. Define

$$A = \bigcup_{n<\omega} \left(\left[F_U\left(n\right), F_U\left(n+1\right)\right] \cap f\left(F_U\left(n+1\right)\right)\right).$$

Let $m \in U_k$. Choose the largest $n \in U$ such that $n < m$. Then by definition of U_k we have that $\frac{n}{m} < \frac{1}{k}$. Note that $m \in U$. Therefore by the definition of A we have the estimate

$$\frac{\left|e\left(A\right)\left(m\right) \triangle f\left(m\right)\right|}{m} \leq \frac{n}{m} < \frac{1}{k},$$

and the claim follows.

Assume on the other hand that $\mathcal{U}$ is thin and $\times$-invariant. There is a decreasing sequence $\left\{U_k\right\}_{k<\omega} \subseteq \mathcal{U}$ such that $\frac{F_{U_k}(n)}{F_{U_k}(n+1)} < \frac{1}{2^{k+1}}$. Define

$$A_k = \bigcup_{n\in U_k} \left[\frac{n}{2^{k+1}}, \frac{n}{2^k} \right].$$

We have $d_{\mathcal{U}}\left(A_k\right) < \frac{1}{2^k}$. Assume that there is $A \subseteq \omega$ such that $d_{\mathcal{U}}\left(A_k \setminus A\right) = 0$ and $d_{\mathcal{U}}\left(A\right) < \frac{1}{8}$ for every $3 < k < \omega$, i.e. A is a candidate for the upper bound of the sequence $\left\{A_k\right\}_{3<k<\omega}$. Let $U = \left\{ n : \frac{|A\cap n|}{n} \leq \frac{1}{8} \right\}$. There must be $16 \leq l < \omega$ such that

$$W = \bigcup_{n\in U} \left[ln, \left(l+1\right)n\right] \in \mathcal{U}.$$

Consider now the smallest $k < \omega$ such that $l + 1 < 2^k$. Define $V = U_k \cap W \in \mathcal{U}$. Since for $n \in V$ there is $m \in U$ such that $lm \leq n \leq \left(l+1\right)m < 2^k m$ and $\left[\frac{n}{2^{k+1}}, \frac{n}{2^{k-1}}\right] \subseteq A_{k-1} \cup A_k$, we have

$$\frac{n}{2^{k+1}} \leq \frac{m}{2},\ m \leq \frac{n}{2^{k-1}}.$$

Therefore $\left[\frac{m}{2}, m\right] \subseteq A_{k-1} \cup A_k$. Since $m \in U$, we must have

$$\frac{|A \cap m|}{m} \leq \frac{1}{8},$$

and therefore

$$\left|\left[\frac{m}{2}, m\right] \setminus A\right| \leq \frac{3m}{8}.$$

Finally we can conclude that

$$\frac{|((A_{k-1} \cup A_k) \setminus A) \cap n|}{n} \geq \frac{3m}{8n} \geq \frac{3}{8(l+1)}$$

for $n \in V$. This is a contradiction with the properties of A. We conclude that there is no upper bound for $\{A_k\}_{3<k<\omega}$ such that its measure is less than $\frac{1}{8}$, consequently $d_{\mathcal{U}}$ is not σ-additive. □

Corollary 2.11 ([2]). *Let $\mathcal{U}$ be an ultrafilter that contains a thin set, i.e. a set A such that $\lim_{n\to\infty} \frac{F_A(n)}{F_A(n+1)} = 0$. Then $d_{\mathcal{U}}$ satisfies $\boldsymbol{AP}$ (null).*

An example of an ultrafilter $\mathcal{U}$ such that $d_{\mathcal{U}}$ does not satisfy $\boldsymbol{AP}$ (null) was presented in [2] (the construction is due to Fremlin).

Our aim is now to characterize those ultrafilters $\mathcal{U}$ such that $d_{\mathcal{U}}$ satisfies $\boldsymbol{AP}$ (*). For that we need the following observation. Recall that an ultrafilter $\mathcal{U}$ is a *P-ultrafilter* if every decresaing sequence $\{U_i\}_{i<\omega} \subseteq \mathcal{U}$ has a pseudointersection $U \in \mathcal{U}$,

Proposition 2.12 ([2]). *Let $\mathcal{U}$ be an ultrafilter that contains a thin set. Then $d_{\mathcal{U}}$ has $\boldsymbol{AP}$ (*) if and only if $\mathcal{U}$ is a P-ultrafilter.*

Claim 2.13. *Let $\mathcal{U}$ be a thin P-ultrafilter. Then $\mathcal{U}$ contains a thin set.*

Proof. Let $\{U_k\}_{k<\omega} \subseteq \mathcal{U}$ be a decreasing sequence such that $\frac{F_{U_k}(n)}{F_{U_k}(n+1)} < \frac{1}{k}$ for every $k < \omega$. Take the pseudointersection U of $\{U_k\}_{k<\omega}$. Then for every $k < \omega$ there is $n_0 < \omega$ such that for every $n > n_0$

$$\frac{F_U(n)}{F_U(n+1)} < \frac{1}{k}$$

□

Proposition 2.14. *Let $\mathcal{U}$ be an ultrafilter. Then the following are equivalent*

- *$G(\mathcal{U})$ is a P-ultrafilter,*
- *$d_{\mathcal{U}}$ has $\boldsymbol{AP}$ (*).*

Proof. Assume that $G(\mathcal{U})$ is a P-ultrafilter. By the Claim 2.13 it must contain a thin set and by Proposition 2.13 $d_{\mathcal{U}}$ has $\boldsymbol{AP}$ (*).

Assume that $d_{\mathcal{U}}$ has $\boldsymbol{AP}$ (*). Again by Proposition 2.13 it is enough to show that $G(\mathcal{U})$ contains a thin set. Fix a decreasing sequence of $\{U_k\}_{k<\omega} \subseteq G(\mathcal{U})$ such that

$$\frac{F_{U_k}(n)}{F_{U_k}(n+1)} < \frac{1}{k+1}$$

and define

$$A_k = \bigcup_{n \in U_k} \left[\frac{n}{2}, n\right].$$

One can easily verify that $\{A_k\}_{k<\omega}$ is a decreasing sequence such that $\lim_{k\to\infty} d_{\mathcal{U}}(A_k) = \frac{1}{2}$. By the property $\boldsymbol{AP}(*)$ there is a set $A \subseteq \omega$ such that $|A \setminus A_k| < \omega$ and $d_{\mathcal{U}}(A) = \frac{1}{2}$ (here we use the property $\boldsymbol{AP}(*)$ for decreasing rather than increasing sequences). Define

$$U = \left\{n \in U_3 : \left[\frac{n}{2}, n\right] \cap A \neq \emptyset\right\}.$$

We must show that $U \in G(\mathcal{U})$ and U is thin. Assume that $U \notin G(\mathcal{U})$. Then $U_3 \setminus U \in G(\mathcal{U})$. For $n \in U_3 \setminus U$ we have

$$\frac{|A \cap n|}{n} \leq \frac{1}{4},$$

which is a contradiction with $d_{\mathcal{U}}(A) = \frac{1}{2}$. To prove that U is thin it is enough to observe that $|A \setminus A_k| < \omega$ implies $|U \setminus U_k| < \omega$. □

Definition 2.15. *We say that ultrafilter* $\mathcal{U}$ *is* close to a P-ultrafilter *if for every decreasing sequence* $\{U_k\}_{k\in\mathbb{N}} \subseteq \mathcal{U}$ *and every* $\epsilon > 0$ *there is* $U \in \mathcal{U}$ *such that* $|U \setminus (U_k)_\epsilon| < \omega$ *for all* $k \in \mathbb{N}$.

Note that the ambiguity in the Definition 2.15 with respect to the Definition 2.2 is justified by the following claims. It follows that if $\mathcal{U}$ is close to a P-ultrafilter, then we can find a P-ultrafilter $\mathcal{V}$ such that $\mathcal{U}$ is close to $\mathcal{V}$, in particular we can take $\mathcal{V} = G(\mathcal{U})$.

Claim 2.16. *Let* $\mathcal{U}$ *be thin and close to a* P*-ultrafilter. Then* $\mathcal{U}$ *is a* P*-ultrafilter.*

Proof. Let $\{U_k\}_{k<\omega} \subseteq \mathcal{U}$ be a decreasing sequence and assume that $\frac{F_{U_0}(n)}{F_{U_0}(n+1)} < \frac{1}{2}$. Find a pseudointersection U of $\left\{(U_k)_{\frac{1}{4}}\right\}_{k<\omega}$. We claim that $V = U \cap U_0$ is a pseudointersection of $\{U_k\}_{k<\omega}$. To see this fix $k < \omega$. We know that there is some m such that $U \setminus m \subseteq (U_k)_\epsilon$. Let $x \in U_0 \cap (U \setminus m)$. There is $y \in (U_k)_{\frac{1}{4}}$ such that $\max\left\{\left|1 - \frac{x}{y}\right|, \left|1 - \frac{y}{x}\right|\right\} < \frac{1}{4}$. Note that $y \in U_0$ because the sequence is decreasing. From the properties of U_0 we have that $x = y$. This implies that $V \setminus m \subseteq U_k$ which finishes the proof. □

Claim 2.17. *Let* $\mathcal{U}, \mathcal{V}$ *be close ultrafilters. Then* $\mathcal{U}$ *is close to a* P*-ultrafilter if and only if* $\mathcal{V}$ *is close to a* P*-ultrafilter.*

Proof. Assume that $\mathcal{U}, \mathcal{V}$ are close and $\mathcal{U}$ is close to a P-ultrafilter. Let $\{V_k\}_{k<\omega} \subseteq \mathcal{V}$ and $\epsilon > 0$ are given. Choose $\delta_0, \delta_1, \delta_2 > 0$ such that $1 - \epsilon < (1 - \delta_0)(1 - \delta_1)(1 - \delta_2)$. Then by simple computation we have for every $A \subseteq \omega$

$$\left((A_{\delta_0})_{\delta_1}\right)_{\delta_2} = \bigcup_{n\in A} \left[(1-\delta_0)(1-\delta_1)(1-\delta_2)\, n, \frac{n}{(1-\delta_0)(1-\delta_1)(1-\delta_2)}\right] \subseteq$$

$$\subseteq \bigcup_{n\in A} \left[(1-\epsilon)\, n, \frac{n}{(1-\epsilon)}\right] = A_\epsilon.$$

Because $\mathcal{U},\mathcal{V}$ are close, we have $\left\{(V_k)_{\delta_0}\right\}_{k<\omega}\subseteq\mathcal{U}$. By the assumption on $\mathcal{U}$ there is a pseudointersection V of $\left\{\left((V_k)_{\delta_0}\right)_{\delta_1}\right\}_{k<\omega}$. One can easily check that V_{δ_2} is a pseudointersection of $\left\{\left(\left((V_k)_{\delta_0}\right)_{\delta_1}\right)_{\delta_2}\right\}_{k<\omega}$. Since $\mathcal{U},\mathcal{V}$ are close, $V_{\delta_2}\in\mathcal{V}$ and $\left\{\left(\left((V_k)_{\delta_0}\right)_{\delta_1}\right)_{\delta_2}\right\}_{k<\omega}\subseteq\mathcal{V}$. So V_{δ_2} is also a pseudointersection of $\{(V_k)_\epsilon\}_{k<\omega}\subseteq\mathcal{V}$. □

Theorem 2.18. *An ultrafilter $\mathcal{U}$ is close to a P-ultrafilter if and only if $d_{\mathcal{U}}$ has $\boldsymbol{AP}\,(*)$.*

Proof. Combine Proposition 2.14, Claim 2.16 and Claim 2.17. □

Corollary 2.19. *There is a P-ultrafilter if and only if there exists ultrafilter density that satisfies $\boldsymbol{AP}\,(*)$.*

Question 2.20. *Does the existence of a density that satisfies $\boldsymbol{AP}\,(*)$ imply the existence of a P-ultrafilter?*

3. Ultraproducts

In the last section we show how certain special properties of ultrafilters may affect properties of some ideals in the measure ultraproduct. Recall that for a sequence $(B_i,m_i)_{i<\omega}$ of σ-complete boolean algebras with measures (not necessarily strictly positive or σ-additive) and for $\mathcal{U}$ an ultrafilter on ω we define the ultraproduct measure $m_{\mathcal{U}}$ on $\prod_{i<\omega}B_i$ as

$$m_{\mathcal{U}}(f)=\mathcal{U}\text{-}\lim m_i(f(i))$$

for $f\in\prod_{i<\omega}B_i$.

There are several natural ideals that one may assign to the product. In order to keep the presentation as straightforward as possible we make the assumption that $(B_i,m_i)=(B,m)$ for every $i<\omega$ where B is a σ-complete boolean algebra with a measure m. Given an ultrafilter $\mathcal{U}$ on ω we define

- $\mathcal{N}_{\mathcal{U}}=\{f\in B^\omega : m_{\mathcal{U}}(f)=0\}$,
- $\mathcal{Z}=\{f\in B^\omega : \lim_{i<\omega}m(f(i))=0\}$,
- $\mathcal{M}_{\mathcal{U}}=\{f\in B^\omega : \{i : m(f(i))=0\}\in\mathcal{U}\}$,
- $\mathcal{I}_{\mathcal{U}}=\left\{f\in B^\omega : \bigwedge_{U\in\mathcal{U}}\bigvee_{i\in U}f(i)\right\}$.

We summarize basic relations between these ideals.

Proposition 3.1. *Let (B,m) be a σ-complete boolean algebra with a σ-additive and strictly positive measure. Then $\mathcal{Z},\mathcal{M}_{\mathcal{U}}\subseteq\mathcal{N}_{\mathcal{U}}$ and $\mathcal{M}_{\mathcal{U}}\subseteq\mathcal{I}_{\mathcal{U}}\subseteq\mathcal{N}_{\mathcal{U}}$.*

Proof. The only case that does not follow immediately from the definitions is $\mathcal{I}_{\mathcal{U}}\subseteq\mathcal{N}_{\mathcal{U}}$. Let $f\notin\mathcal{N}_{\mathcal{U}}$. Then

$$\inf_{U\in\mathcal{U}} m\left(\bigvee_{i\in U}f(i)\right)=c>0.$$

Take a decreasing sequence $\{U_k\}_{k<\omega} \subseteq \mathcal{U}$ such that

$$\lim_{k\to\infty} m\left(\bigvee_{i\in U_k} f(i)\right) = c.$$

Since the sequence $\left\{\bigvee_{i\in U_k} f(i)\right\}_{k<\omega}$ is also decreasing there must be some $b \in B$ such that $b \leq \bigvee_{i\in U_k} f(i)$ for every $k < \omega$ and $m(b) = c$. We show that $d \leq \bigvee_{i\in U} f(i)$ for every $U \in \mathcal{U}$, this finishes the proof. Assume that there is some $U \in \mathcal{U}$ such that $b \not\leq \bigvee_{i\in U} f(i) = a$. Then $m(b \setminus a) = \epsilon > 0$ and therefore

$$\lim_{k\to\infty} m\left(\bigvee_{i\in U_k\cap U} f(i)\right) = c - \epsilon$$

which is a contradiction. □

Let $\mathcal{U}$ be a non-principal ultrafilter on ω. We say that $\mathcal{U}$ is

- *semi-selective* if for every $\{a_n\}_{n<\omega}$ of positive real numbers such that $\mathcal{U}\text{-}\lim_{n\to\infty} a_n = 0$ there is $U \in \mathcal{U}$ such that $\sum_{n\in U} a_n < \infty$.

Theorem 3.2. *Let (B, m) be a σ-complete infinite boolean algebra with a σ-additive strictly positive measure and $\mathcal{U}$ an ultrafilter on ω. Then the following hold*

- *$\mathcal{U}$ is a P-ultrafilter if and only if $\mathcal{N}_\mathcal{U} = \mathcal{Z} + \mathcal{M}_\mathcal{U} = \{f \vee g : f \in \mathcal{Z},\ g \in \mathcal{M}_\mathcal{U}\}$,*
- *$\mathcal{U}$ is semi-selective if and only if $\mathcal{I}_\mathcal{U} = \mathcal{N}_\mathcal{U}$.*

Proof. To prove the first claim notice that it is enough for each $f \in \mathcal{N}_\mathcal{U}$ find a set $U \in \mathcal{U}$ such that $\lim_{i\in U} m(f(i)) = 0$. Under the assumption that B is infinite, this is possible if and only if $\mathcal{U}$ is P-ultrafilter.

Let $\mathcal{U}$ be a semi-selective ultrafilter and $f \in \mathcal{N}_\mathcal{U}$. Then there is $U \in \mathcal{U}$ such that $\sum_{i\in U} m(f(i)) < \infty$ and therefore

$$\bigwedge_{n<\omega} \bigvee_{i\in(U\setminus n)} f(i) = 0.$$

Let $\mathcal{U}$ be not semi-selective. There must be a sequence $\{a_i\}_{i<\omega}$ of positive real numbers such that $\mathcal{U}\text{-}\lim a_i = 0$ and for every $U \in \mathcal{U}$ is $\sum_{i\in U} a_i = \infty$. Take a sequence $\{b_i\}_{i<\omega} \subseteq B$ such that $m(b_i) = a_i$ and $\{b_i\}_{i<\omega}$ is independent (see for example [1]). We have for every $U \in \mathcal{U}$ that

$$m\left(1 - \bigvee_{i\in U} f(i)\right) = m\left(\bigwedge_{i\in U} (1 - f(i))\right) = \prod_{i\in U} m(1 - f(i)) = 0.$$

Therefore $\bigvee_{i\in U} f(i) = 1$ and $f \in \mathcal{N}_\mathcal{U} \setminus \mathcal{I}_\mathcal{U}$. □

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