DICHOTOMY FOR TSI POLISH GROUPS I: CLASSIFICATION BY COUNTABLE STRUCTURES

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Abstract. We introduce a property of orbit equivalence relation that we call *property (IC)* and show that a Borel orbit equivalence relation E^X_G induced by a continuous action of a tsi Polish group G on a Polish space X satisfies property (IC) if and only if it is classifiable by countable structures. Moreover, we describe a class of Borel equivalence relations that serve as a base for non-classification by countable structures for such Borel orbit equivalence relations.

The orbit equivalence relation E^X_G induced by a group action $G \curvearrowright X$ is defined as

$$(x,y) \in E^X_G \ \Leftrightarrow \ \exists g \in G \ g \cdot x = y.$$

We only work in the setting when X is a Polish space, G is a Polish group, $G \curvearrowright X$ is a continuous action and E^X_G is a Borel subset of $X \times X$.

We say that an equivalence relation E on a Polish space X is *classifiable by countable structures* if it admits a Borel reduction to an isomorphism relation of countable structures in some countable language. This is equivalent, see [7, Section 6, Theorem 6.1], with E being Borel reducible to $E^Y_{S_\infty}$ where Y is a Polish S_∞-space and S_∞ is the Polish group of all permutations of natural numbers $\mathbb{N}$. In fact, we use the latter as a definition of classification by countable structures.

In this note we introduce a property for orbit equivalence relation that we call *property (IC)*, see Section 3 for the definition. Informally, property (IC) gives a countable Borel decomposition of a Polish G-space X into arbitrarily small independent clusters within each orbit. Next we state our main result.

Theorem. *Let G be a* **tsi** *Polish group and X be a Polish G-space such that E^X_G is a Borel equivalence relation. Then the following are equivalent*

- *X satisfies property (IC),*
- *E^X_G is classifiable by countable structures.*

Our result follows immediately from much refined Theorem 7.1. In the proof we use a version of the $\mathbb{G}_0$-dichotomy, see [9], [12] and a certain class $\mathcal{B}$ of Borel equivalence relations as a base for non-classification by countable structures. Informally, $\mathcal{B}$ consists of all turbulent c_0-equalities, equivalence relations that are induced by canonical actions of Polishable tall ideals on $\mathbb{N}$ and Borel equivalence relation that contain one of these and are meager in the corresponding topology, see Section 5 for precise definition.

In [4] we use this characterization of classification by countable structures to show the following. Let G be tsi Polish group and X be a Polish G-space such that E^X_G is Borel and classifiable by countable structures. Then either E^X_G is essentially countable or $\mathbb{E}_3 \leq_B E^X_G$ where $\mathbb{E}_3 = \mathbb{E}_0^{\mathbb{N}}$.

1. Notation

For a set X we write $X^{<\mathbb{N}}$ for the set of all nonempty finite sequences of X. Let $\overline{x} \in X^{<\mathbb{N}}$. We define $\mathfrak{s}(\overline{x}) \in X$, $\mathfrak{t}(\overline{x}) \in X$ and $l(\overline{x}) \in \mathbb{N}$ to be the first element of $\overline{x}$, last element of $\overline{x}$ and the length of $\overline{x}$. When $X = \mathbb{N}$ then we use $|s|$ instead of $l(s)$ where $s \in \mathbb{N}^{<\mathbb{N}}$. For a natural number $i < l(\overline{x})$ we define $\overline{x}_i$ to be the i-th element of $\overline{x}$. Given a map $\varphi : X \to Y$ we abuse the notation and extend it to a map $\varphi : X^{<\mathbb{N}} \to Y^{<\mathbb{N}}$ coordinate-wise, i.e.,

$$\varphi(\overline{x})_i = \varphi(\overline{x}_i)$$

for every $i < l(\overline{x})$. Define

$$\Delta_X = \left\{\overline{x} \in X^{<\mathbb{N}} : \exists i < j < l(x)\ \overline{x}_i = \overline{x}_j\right\}.$$

Let X and Y be sets, I some index set and $(A_j)_{j\in I}$ and $(B_j)_{j\in I}$ be sequences of subsets of $X^{<\mathbb{N}}$ and $Y^{<\mathbb{N}}$, respectively. We say that a map $\varphi : X \to Y$ is a *homomorphism from* $(A_j)_{j\in I}$ *to* $(B_j)_{j\in I}$ if

$$\overline{x} \in A_j \ \Rightarrow\ \varphi(\overline{x}) \in B_j$$

for every $\overline{x} \in X^{<\mathbb{N}}$ and $j \in I$. It is a *reduction* if

$$\overline{x} \in A_j \ \Leftrightarrow\ \varphi(\overline{x}) \in B_j$$

for every $\overline{x} \in X^{<\mathbb{N}}$ and $j \in I$.

A *(finite-dimensional) dihypergraph on* X is any subset of $X^{<\mathbb{N}} \setminus (\Delta_X \cup X)$. If $\mathcal{H}$ is a dihypergraph on X and $A \subseteq X$, then we say that A is $\mathcal{H}$-independent if $\mathcal{H} \cap A^{<\mathbb{N}} = \emptyset$.

A topological space X is a *Polish space* if the underlying topology is separable and completely metrizable. A topological group G is a *Polish group* if the underlying topology is Polish. We denote the σ-ideal of meager sets on G as $\mathcal{M}_G$. We use the category quantifiers $\exists^*$, $\forall^*$ in the standard meaning, i.e.,

$$\forall^* g \in U\ P(g) \ \Leftrightarrow\ \{g \in U : \neg P(g)\} \in \mathcal{M}_G$$

$$\exists^* g \in U\ P(g) \ \Leftrightarrow\ \{g \in U : P(g)\} \notin \mathcal{M}_G$$

where $U \subseteq G$ is open set and P is some property.

A Polish group G is *tsi (two-sided invariant)* if there is an open basis at 1_G made of conjugacy invariant open sets. Equivalently, see [2, Exercise 2.1.4], there is a compatible metric d on G that is two sided invariant, i.e., $d(g,h) = d(h^{-1}\cdot g, 1_G) = d(g \cdot h^{-1}, 1_G)$ for every $g,h \in G$. It follows from [2, Exercise 2.2.4] that such a metric d is necessarily complete. We fix such a metric d on G and put $V_\epsilon = \{g \in G : d(g,1_G) < \epsilon\}$. Note that $h \cdot V_\epsilon \cdot h^{-1} = V_\epsilon$ for every $\epsilon > 0$ and $h \in G$. We abuse the notation and put $V_k = V_{\frac{1}{2^k}}$. In some cases we do not require G to be tsi and in that cases we assume that $\{V_k\}_{k\in\mathbb{N}}$ is some open neighborhood base at 1_G such that $V_{k+1} \cdot V_{k+1} \subseteq V_k$ and $V_k = V_k^{-1}$ for every $k \in \mathbb{N}$.

If there is a fixed continuous action of a Polish group G on a Polish space X, then we say that X is a *Polish G-space*. The orbit equivalence relation E_G^X is defined as

$$(x,y) \in E_G^X \;\Leftrightarrow\; \exists g \in G \; g \cdot x = y.$$

where $x, y \in X$.

Let X be a Polish G-space, $V \subseteq G$, $U \subseteq X$ and $x \in X$. We define

$$\mathcal{J}(V) = \{\overline{x} \in X^{<\mathbb{N}} \setminus \Delta_X : (\forall i < l(\overline{x}) - 1) \; \overline{x}_{i+1} \in V \cdot \overline{x}_i\},$$

$$\mathcal{J}(x,V) = \{\overline{x} \in \mathcal{J}(V) : \mathfrak{s}(\overline{x}) = x\},$$

$$\mathcal{J}(U,V) = \mathcal{J}(V) \cap U^{<\mathbb{N}},$$

$$\mathcal{J}(x,U,V) = \mathcal{J}(x,V) \cap \mathcal{J}(U,V).$$

If we assume that U and V are open neighborhoods of x and 1_G, then the local orbit $\mathcal{O}(x,U,V)$ is defined as

$$\mathcal{O}(x,U,V) = \{\mathfrak{t}(\overline{x}) : \overline{x} \in \mathcal{J}(x,U,V)\}$$

(see [2, Section 10.2]).

Let X be a Polish G-space, $x \in X$ and $A \subseteq X$. We write $G(x,A) = \{g \in G : g \cdot x \in A\}$.

Definition 1.1. *Let X be a Polish G-space. We say that $C \subseteq X$ is a G-*lg comeager set *if $G \setminus G(x,C) \in \mathcal{M}_G$ for every $x \in X$. Equivalently,*

$$\forall^* g \in G \; g \cdot x \in C$$

holds for every $x \in X$

We say that a tree $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is *finitely uniformly branching* if there is a sequence $\{l_m^T\}_{m\in\mathbb{N}}$ of natural numbers such that $l_m^T \geq 2$ for every $m \in \mathbb{N}$ and

$$l_{|s|}^T = \{i \in \mathbb{N} : s^\frown(i) \in T\}$$

for every $s \in T$. If T is a tree and $s \in T$, then we define $T_s = \{t \in \mathbb{N}^{<\mathbb{N}} : s^\frown t \in T\}$. Note that $T_s = T_t$ whenever $t, s \in T$ and $|t| = |s|$. We denote as $[T] \subseteq \mathbb{N}^{\mathbb{N}}$ the set of all branches through T, i.e., $\alpha \in [T]$ if and only if $\alpha \upharpoonright m \in T$ for every $m \in \mathbb{N}$.

Definition 1.2. *Let T be a finitely uniformly branching tree and $s \in T$. The dihypergraph $\mathbb{G}_s^T$ on $[T]$ is defined as*

$$\mathbb{G}_s^T = \left\{ (s^\frown(i)^\frown\alpha)_{i<l_{|s|}^T} : \alpha \in [T_{s^\frown(0)}] \right\}.$$

The equivalence relation $\mathbb{E}_0^T$ on $[T]$ is defined as

$$(\alpha,\beta) \in \mathbb{E}_0^T \;\Leftrightarrow\; |\{n \in \mathbb{N} : \alpha(n) \neq \beta(n)\}| < \aleph_0$$

where $\alpha, \beta \in [T]$. In the case when $T = 2^{<\mathbb{N}}$ we write $\mathbb{E}_0$ instead of $\mathbb{E}_0^{2^{<\mathbb{N}}}$.

Let E be an equivalence relation on a Polish space X and F be an equivalence relation on a Polish space Y. Then we say that E is *Borel reducible* to F and write $E \leq_B F$ if there is a Borel map $\phi : X \to Y$ that is a reduction from E to F.

2. $\mathbb{G}_0$-like dichotomy

Recall that if G is a Polish group, then $\{V_k\}_{k\in\mathbb{N}}$ is an open neighborhood base at 1_G such that $V_{k+1}\cdot V_{k+1}\subseteq V_k$ and $V_k=V_k^{-1}$ for every $k\in\mathbb{N}$.

Let X be a Polish G-space. Define

$$\mathcal{H}_{k,m}=\left\{\overline{x}\in X^{<\mathbb{N}}:\overline{x}\in\mathcal{J}(V_m)\ \wedge\ \mathfrak{t}(\overline{x})\notin V_k\cdot\mathfrak{s}(\overline{x})\right\}$$

for every $k,m\in\mathbb{N}$. Note that if $A\subseteq X$ is $\mathcal{H}_{k,m}$-independent, then it is $\mathcal{H}_{k',m'}$-independent for every $m\leq m'\in\mathbb{N}$ and $k\geq k'\in\mathbb{N}$. This is because $\mathcal{H}_{k,m}\supseteq\mathcal{H}_{k',m'}$ whenever $m\leq m'\in\mathbb{N}$ and $k\geq k'\in\mathbb{N}$.

Proposition 2.1. *Let X be a Polish G-space such that E_G^X is Borel. Then $\mathcal{H}_{k,m}$ is a Borel subset of $X^{<\mathbb{N}}$ for every $k,m\in\mathbb{N}$.*

Proof. Let $V\subseteq G$ be an open neighborhood of 1_G. Define a binary relation R_V on X as

$$(x,y)\in R_V\ \Leftrightarrow\ \exists g\in V\ g\cdot x=y.$$

Then it follows from the assumption that E_G^X is Borel together with [1, Theorem 7.1.2] that R_V is Borel.

Let $k,m\in\mathbb{N}$. We have

$$\overline{x}\in\mathcal{H}_{k,m}\ \Leftrightarrow\ \overline{x}\notin\Delta_X\ \wedge\ \forall i<(l(\overline{x})-1)\ (\overline{x}_i,\overline{x}_{i+1})\in R_{V_m}\ \wedge\ (\mathfrak{s}(\overline{x}),\mathfrak{t}(\overline{x}))\notin R_{V_k}$$

and that shows that $\mathcal{H}_{k,m}$ is a Borel subset of $X^{<\mathbb{N}}$ by the previous paragraph. □

Theorem 2.2 ($\mathbb{G}_0$-like dichotomy)**.** *Let G be a Polish group, X be a Polish G-space such that E_G^X is Borel and $A\subseteq X$ be a $\mathbf{\Sigma}_1^1$ set. Then one of the following holds*

- **(A)** *there is a sequence $\{A_{k,l}\}_{l\in\mathbb{N}}$ of $\mathbf{\Sigma}_1^1$ subsets of X such that $A=\bigcup_{l\in\mathbb{N}}A_{k,l}$ for every $k\in\mathbb{N}$ and for every $k,l\in\mathbb{N}$ there is $m(k,l)\in\mathbb{N}$ such that $A_{k,l}$ is $\mathcal{H}_{k,m(k,l)}$-independent,*
- **(B)** *there is $\mathbf{k}\in\mathbb{N}$, a finitely uniformly branching tree T, a dense set $\{s_m\}_{m\in\mathbb{N}}\subseteq T$ such that $s_m\in\mathbb{N}^m$ and a continuous map $\varphi:[T]\to A$ that is a homomorphism from $(\mathbb{G}_{s_m}^T)_{m\in\mathbb{N}}$ to $(\mathcal{H}_{\mathbf{k},m})_{m\in\mathbb{N}}$.*

Proof. It follows from Proposition 2.1 that $\mathcal{H}_{k,m}^A=\mathcal{H}_{k,m}\cap A^{<\mathbb{N}}$ is a $\mathbf{\Sigma}_1^1$ dihypergraph on an analytic Hausdorff space A. Fix $k\in\mathbb{N}$ and apply a version of the $\mathbb{G}_0$-dichotomy, see [11, Theorem 2.2.12], for sequence $(\mathcal{H}_{k,m}^A)_{m\in\mathbb{N}}$. Then either there is a sequence $\{A_{k,l}\}_{l\in\mathbb{N}}$ of relative Borel subsets of A such that $\bigcup_{l\in\mathbb{N}}A_{k,l}=A$ and $A_{k,l}$ is $\mathcal{H}_{k,m(k,l)}^A$-independent for some $m(k,l)\in\mathbb{N}$, or **(B)** holds with $\mathbf{k}=k$. It is easy to see that if the first case occurs for every $k\in\mathbb{N}$, then $\{A_{k,l}\}_{k,l\in\mathbb{N}}$ is the desired sequence in **(A)**. □

3. Property (IC)

Definition 3.1. *Let X be a Polish G-space and $B\subseteq X$ be a G-invariant Borel set. We say that B satisfies* property (IC) *if there is a sequence of Borel sets $\{A_{k,l}\}_{k,l\in\mathbb{N}}$ such that for every $k,l\in\mathbb{N}$ there is $m(k,l)\in\mathbb{N}$ such that $A_{k,l}$ is $\mathcal{H}_{k,m(k,l)}$-independent and $B=\bigcup_{l\in\mathbb{N}}A_{k,l}$ for every $k\in\mathbb{N}$.*

We say that Polish G-space X satisfies property (IC) if X satisfies property (IC).

Note that if $V_k \subseteq G$ is a subgroup, then X is $\mathcal{H}_{k,k}$-independent. Therefore property (IC) holds for X whenever G contain an open basis at 1_G made of clopen subgroups, i.e., whenever G is a closed subgroup of S_∞.

Let X be a Polish G-space. Recall that the action $G \curvearrowright X$ is *turbulent* if

(1) every orbit is dense and meager in X,
(2) $\mathcal{O}(x, U, V)$ is somewhere dense for every $x \in X$ and every open sets $U \subseteq X$, $V \subseteq G$ such that $x \in U$, $1_G \in V$,

see [2, Section 10].

Theorem 3.2. *Let X be a Polish G-space that satisfies property (IC). Then the action is not turbulent.*

Proof. Suppose that the action is turbulent. Let $D \subseteq X$ be a Borel comeager set such that $A_{k,l} \cap D$ is relatively open in D for every $k, l \in \mathbb{N}$. This can be done using [8, Proposition 8.26]. It follows from [8, Theorem 16.1] and [8, Theorem 8.41] that

$$D' = \{x \in D : \forall^* g \in G \ g \cdot x \in D\}$$

is a Borel comeager subset of X.

Pick $x \in D'$. Note that $G(x, D')$ is comeager in G. We show that $G \cdot x = [x]_{E^X_G}$ is nonmeager. Suppose that $G \cdot x$ is meager. Then there are closed nowhere dense sets $\{F_r\}_{r\in\mathbb{N}}$ such that $G \cdot x \subseteq \bigcup_{r\in\mathbb{N}} F_r$. Note that $G(x, F_r)$ is closed for every $r \in \mathbb{N}$ and $G = \bigcup_{r\in\mathbb{N}} G(x, F_r)$. By [8, Proposition 8.26] there is an index $r \in \mathbb{N}$ such that $G(x, F_r)$ contains an open set. This implies that there is $g \in G$ and $k \in \mathbb{N}$ such that $V_k \cdot g \subseteq G(x, F_r)$ and $y = g \cdot x \in D'$. Let $l \in \mathbb{N}$ such that $y \in A_{k,l}$. Note that

$$\overline{V_k \cdot y} = \overline{V_k \cdot g \cdot x} \subseteq F_r$$

because F_r is closed.

Use the definition of D to find an open set U such that $U \cap D' = A_{k,l} \cap D'$. Consider the local orbit $\mathcal{O}(y, U, V_{m(k,l)})$ and pick $z \in \mathcal{O}(y, U, V_{m(k,l)})$. By the definition, there is $w \in U^{<\mathbb{N}}$ such that $w_0 = y$, $w_{l(w)-1} = z$ and $w_{i+1} \in V_{m(k,l)} \cdot z_i$ for every $i < l(z) - 1$. Let $P \subseteq X$ be an open neighborhood of z. Note that $G(y, U)$, $G(y, P)$ are open and $G(y, D')$ is comeager, in particular, dense in $G(y, U)$. Therefore we can find a sequence $z' \in U^{<\mathbb{N}}$ such that $l(z) = l(z')$, $z'_0 = y$, $z'_i \in U \cap D'$ for every $i < l(z')$, $z'_{i+1} \in V_{m(k,l)} \cdot z'_i$ for every $i < l(z') - 1$ and $z'_{l(z')-1} \in P$. Note that we have

$$z'_i \in U \cap D' = A_{k,l} \cap D' \subseteq A_{k,l}$$

for every $i < l(z')$. The set $A_{k,l}$ is $\mathcal{H}_{k,m(k,l)}$-independent and therefore $z'_{l(z')-1} \in V_k \cdot y$. This implies that $V_k \cdot y \cap P \neq \emptyset$ and consequently that

$$\mathcal{O}(y, U, V_{m(k,l)}) \subseteq \overline{V_k \cdot y}.$$

Therefore F_r contains an open set by the assumption that the action is turbulent, i.e., $\mathcal{O}(y, U, V_{m(k,l)})$ is somewhere dense. This shows that $[x]_{E^X_G}$ is nonmeager and that contradicts the definition of turbulence. $\square$

Recall that if G is a tsi Polish group, then there is a fixed compatible complete two-sided invariant metric d on G and the sequence $\{V_k\}_{k\in\mathbb{N}}$ is defined as $V_k = \{g \in G : d(g, 1_G) < \frac{1}{2^k}\}$.

Proposition 3.3. *Let G be a tsi Polish group, X be a Polish G-space and A be a $\mathcal{H}_{k+2,m}$-independent $\mathbf{\Sigma}^1_1$ subset of X. Then there is a Borel G-invariant set $B \subseteq X$ such that $A \subseteq B$ and a sequence $\{B_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}_{k,m+2}$-independent Borel subsets of X such that $\bigcup_{n\in\mathbb{N}} B_n = B$.*

Proof. We may assume that $k + 2 \leq m$. Define

$$A' = \{x \in X : \exists g \in V_{m+2}\ g \cdot x \in A\}.$$

Then it is easy to see that A' is a $\mathbf{\Sigma}^1_1$ subset of X. Let $\overline{x} \in \mathcal{J}(A', V_{m+2})$ and pick any $\overline{y} \in A^{<\mathbb{N}}$ such that $l(\overline{x}) = l(\overline{y})$ and $\overline{x}_i \in V_{m+2} \cdot \overline{y}_i$ for every $i < l(\overline{x})$. Then we have

$$\overline{y}_{i+1} \in V^{-1}_{m+2} \cdot \overline{x}_{i+1} \subseteq V^{-1}_{m+2} \cdot V_{m+2} \cdot \overline{x}_i \subseteq V^{-1}_{m+2} \cdot V_{m+2} \cdot V_{m+2} \cdot \overline{y}_i \subseteq V_m \cdot \overline{y}_i$$

for every $i < l(\overline{y}) - 1$. The set A is $\mathcal{H}_{k+2,m}$-independent and that gives $\mathfrak{t}(\overline{y}) \in V_{k+2} \cdot \mathfrak{s}(\overline{y})$. We have

$$\mathfrak{t}(\overline{x}) \in V_{m+2} \cdot \mathfrak{t}(\overline{y}) \subseteq V_{m+2} \cdot V_{k+2} \cdot \mathfrak{s}(\overline{y}) \subseteq V_{m+2} \cdot V_{k+2} \cdot V^{-1}_{m+2} \cdot \mathfrak{s}(\overline{x}) \subseteq V_{k+1} \cdot \mathfrak{s}(\overline{x})$$

and that shows that A' is $\mathcal{H}_{k+1,m+2}$-independent.

By [8, Theorem 28.5] there is a Borel set $D' \subseteq X$ that is $\mathcal{H}_{k+1,m+2}$-independent and $A' \subseteq D'$. Define

$$D = \{x \in X : \exists r \in \mathbb{N}\ \forall^* g \in V_r\ g \cdot x \in D'\}.$$

It follows from [8, Theorem 16.1] that D is a Borel set and the definition of A' together with $A' \subseteq D'$ implies that $A \subseteq D$. Similar argument as in previous paragraph shows that D is $\mathcal{H}_{k,m+2}$-independent. Moreover it is easy to see that if $G(x, D')$ is comeager in V_r, then $y \in D$ for every $y \in V_{r+1} \cdot x$. This shows that $G(x, D)$ is open in G for every $x \in X$.

Let $\{g_n\}_{n\in\mathbb{N}}$ be a dense subset of G such that $g_0 = 1_G$. Define $B_n = g_n \cdot D$ and $B = \bigcup_{n\in\mathbb{N}} B_n$. Then B is a G-invariant Borel set because $G(x, D)$ is nonempty open set whenever $x \in D$. Moreover, $A \subseteq D = B_0 \subseteq B$.

It remains to show that B_n is $\mathcal{H}_{k,m+2}$-invariant for every $n \in \mathbb{N}$. Let $g \in G$, V be a conjugacy invariant open neighborhood of 1_G and $x, y \in X$, then $y \in V \cdot x$ if and only if $g \cdot y \in V \cdot (g \cdot x)$. This shows that

$$g_n \cdot \mathcal{J}(D, V_{m+2}) = \mathcal{J}(B_n, V_{m+2})$$

where the action is extended coordinate-wise and consequently that B_n is $\mathcal{H}_{k,m+2}$-independent for every $n \in \mathbb{N}$. This finishes the proof. □

Corollary 3.4. *Let G be a tsi Polish group, X be a Polish G-space and A be a $\mathbf{\Sigma}^1_1$ subset of X such that* **(A)** *in Theorem 2.2 holds. Then there is a Borel G-invariant set $B \subseteq X$ that satisfies property (IC) and $A \subseteq B$.*

Proof. Let $k, l \in \mathbb{N}$. Apply Proposition 3.3 to $A_{k+2,l} \subseteq X$ to get a Borel G-invariant set $B^{k,l} \subseteq X$ together with a sequence $\{B^{k,l}_n\}_{n\in\mathbb{N}}$ of $\mathcal{H}_{k,m(k+2,l)+2}$-independent Borel subsets of X such that $B^{k,l} = \bigcup_{n\in\mathbb{N}} B^{k,l}_n$.

Define

$$B = \bigcap_{k\in\mathbb{N}} \left(\bigcup_{l\in\mathbb{N}} B^{k,l} \right).$$

Then it is easy to see that B is a Borel G-invariant subset of X that satisfies property (IC) and $A \subseteq B$. □

Next theorem shows that property (IC) is stronger condition than classification by countable structures for tsi Polish groups.

Theorem 3.5. *Let G be a tsi Polish group and X be a Polish G-space that satisfies property (IC) and E_G^X is Borel. Then E_G^X is classifiable by countable structures.*

Proof. An elementary proof of this statement follows from [3, Definition 3.3.6, Proposition 3.3.7, Theorem 3.3.8].**Maybe sketch**

Alternative approach that does not need the assumption that E_G^X is Borel is to appeal to [7, Theorem 13.18] and Theorem 3.2. □

Corollary 3.6. *Let G be a tsi Polish group, X be a Polish G-space such that E_G^X is Borel and A be a $\mathbf{\Sigma}^1_1$ subset of X such that* **(A)** *in Theorem 2.2 holds. Then there is a G-invariant Borel set $B \subseteq X$ such that $A \subseteq B$ and $E_G^X \restriction B \times B$ is classifiable by countable structures.*

In particular, if $A = X$, then **(A)** *implies that E_G^X is classifiable by countable structures.*

Proof. Corollary 3.4 produces a Borel G-invariant set $B \subseteq X$ such that $A \subseteq B$. There is a finer Polish topology on X such that B is clopen and the action is continuous, see [2, Corollary 4.3.4]. This turns B into a Polish G-space that satisfies (IC) and $E_G^B = E_G^X \restriction B \times B$ is Borel. The proof is finished by applying Theorem 3.5. □

4. Uniform Pseudometric

Definition 4.1. *Let T be a finitely uniformly branching tree. A function $\mathbf{d} : [T] \times [T] \to [0, +\infty]$ is called a* Borel pseudometric *if*

1. $\mathbf{d}$ *is pseudometric,*
2. $\mathbf{d}^{-1}([0, \epsilon))$ *is a Borel subset of $[T] \times [T]$ for every $\epsilon > 0$,*
3. $(\{\beta : \mathbf{d}(\alpha, \beta) < +\infty\}, \mathbf{d})$ *is a separable pseudometric space for every $\alpha \in [T]$,*
4. *if $\alpha_n \to_{[T]} \alpha$ and $\{\alpha_n\}_{n\in\mathbb{N}}$ is a $\mathbf{d}$-Cauchy sequence, then $\mathbf{d}(\alpha_n, \alpha) \to 0$.*

Moreover, we say that a Borel pseudoemtric is uniform *if*

- *for every $m \in \mathbb{N}$, $s, t \in T \cap \mathbb{N}^m$ and $\alpha, \beta \in [T_s] = [T_t]$ we have*

$$|\mathbf{d}(s^\frown\alpha, t^\frown\alpha) - \mathbf{d}(s^\frown\beta, t^\frown\beta)| < \frac{1}{2^m},$$

$$|\mathbf{d}(s^\frown\alpha, s^\frown\beta) - \mathbf{d}(t^\frown\alpha, t^\frown\beta)| < \frac{1}{2^m}$$

where we set $|+\infty - +\infty| = 0$.

First we show a canonical way how to find Borel pseudometrics. Recall that if G is a tsi Polish group, then d is a fixed two-sided invariant metric on G.

Proposition 4.2. *Let G be a tsi Polish group, X be a Polish G-space such that E_G^X is Borel, T be a finitely uniformly branching tree and $\varphi : [T] \to X$ be a continuous map. Then the function $\mathbf{d}_\varphi : [T] \times [T] \to [0, +\infty]$ defined as*

$$\mathbf{d}_\varphi(\alpha, \beta) = \inf\{d(g, 1_G) : g \in G \ \wedge \ g \cdot \varphi(\alpha) = \varphi(\beta)\}$$

is a Borel pseudometric.

Proof. The invariance of d guarantees that $d(g, 1_G) = d(g^{-1}, 1_G)$ for every $g \in G$ and consequently that $\mathbf{d}_\varphi$ is symmetric. Let $\alpha, \beta, \gamma \in [T]$. We may assume that $\mathbf{d}_\varphi(\alpha, \beta) + \mathbf{d}_\varphi(\beta, \gamma) < +\infty$. In that case for every $\epsilon > 0$ there is $g, h \in G$ such that $d(g, 1_G) < \mathbf{d}_\varphi(\alpha, \beta) + \epsilon$ and $d(h, 1_G) < \mathbf{d}_\varphi(\beta, \gamma) + \epsilon$. Then we have

$$\mathbf{d}_\varphi(\alpha, \gamma) - 2\epsilon \leq d(h \cdot g, 1_G) - 2\epsilon \leq d(h, 1_G) + d(g, 1_G) - 2\epsilon < \mathbf{d}_\varphi(\alpha, \beta) + \mathbf{d}_\varphi(\beta, \gamma)$$

because $d(h \cdot g, 1_G) \leq d(h \cdot g, g) + d(g, 1_G) = d(h, 1_G) + d(g, 1_G)$ by the invariance of d. That proves (1).

Recall that for $\epsilon > 0$ we defined $V_\epsilon = \{g \in G : d(g, 1_G) < \epsilon\}$. It follows, as in the proof of Proposition 2.1, that the relation R_{V_ϵ} defined as

$$(x, y) \in R_{V_\epsilon} \ \Leftrightarrow \ \exists g \in V_\epsilon \ g \cdot x = y$$

is Borel for every $\epsilon > 0$. Note that we have

$$\mathbf{d}_\varphi^{-1}([0, \epsilon)) = \{(\alpha, \beta) \in [T] \times [T] : \mathbf{d}_\varphi(\alpha, \beta) < \epsilon\} = \left(\varphi^{-1} \times \varphi^{-1}\right)(R_{V_\epsilon})$$

and that shows (2).

Let $\alpha \in [T]$ and $S_\alpha = \{\beta : \mathbf{d}(\alpha, \beta) < +\infty\}/\mathbf{d}_\varphi$ be the metric quotient. Then the space $G_\alpha = \{g \in G : \exists \beta \in [T] \ g \cdot \varphi(\alpha) = \varphi(\beta)\}$ endowed with d is a separable metric space and the assignment $g \mapsto \beta$ where $g \cdot \varphi(\alpha) = \varphi(\beta)$ is a contraction from (G_α, d) to $(S_\alpha, \mathbf{d})$. This shows (3).

Let $\{\alpha_n\}_{n \in \mathbb{N}}, \alpha \in [T]$ be such that the assumptions of (4) are satisfied. After possibly passing to a subsequence we may suppose that there is a sequence $\{g_n\}_{n \in \mathbb{N}} \subseteq G$ such that $g_n \cdot \varphi(\alpha_n) = \varphi(\alpha_{n+1})$ and $d(g_n, 1_G) < \frac{1}{2^n}$. Define $h_m^n = g_{n-1} \cdot \ldots \cdot g_m$ for every $m < n \in \mathbb{N}$. Then it follows that $\{h_m^n\}_{n \in \mathbb{N}}$ is d-Cauchy whenever $m \in \mathbb{N}$ is fixed and since d is complete there is $\{h_m\}_{m \in \mathbb{N}} \in G$ such that $h_m^n \to h_m$. Moreover we have $d(h_m, 1_G) < \frac{1}{2^{m-1}}$. Continuity of the action and of φ gives

$$h_m \cdot \varphi(\alpha_m) \leftarrow h_m^n \cdot \varphi(\alpha_m) = \varphi(\alpha_n) \to \varphi(\alpha).$$

This proves (4) and finishes the proof. □

It follows from (1) above that every Borel pseudoemtric $\mathbf{d}$ on $[T]$ defines a Borel equivalence relation $F_\mathbf{d}$ on $[T]$ as

$$(\alpha, \beta) \in F_\mathbf{d} \ \Leftrightarrow \ \mathbf{d}(\alpha, \beta) < +\infty.$$

Note that in the case of Proposition 4.2 we have that $F_{\mathbf{d}_\varphi} = (\varphi^{-1} \times \varphi^{-1})(E_G^X)$.

Theorem 4.3. *Let T be a finitely uniformly branching tree and $\mathbf{d}$ be a uniform Borel pseudometric such that $\mathbb{E}_0^T \subseteq F_\mathbf{d}$. Then the following are equivalent*

(a) *$F_\mathbf{d}$ is nonmeager,*

(b) $F_{\mathbf{d}} = [T] \times [T]$.

Proof. (b) $\Rightarrow$ (a) is trivial. We show that (a) $\Rightarrow$ (b). Suppose first, that for every $k \in \mathbb{N} \setminus \{0\}$ there is $m_k \in \mathbb{N}$ such that $\mathbf{d}(\alpha, \beta) < \frac{1}{k}$ for every $\alpha, \beta \in [T]$ such that $\{n \in \mathbb{N} : \alpha(n) \neq \beta(n)\} \cap m_k = \emptyset$ and $(\alpha, \beta) \in \mathbb{E}_0^T$. We may assume that $\{m_k\}_{k\in\mathbb{N}} \subseteq \mathbb{N}$ is strictly increasing and that $m_0 = 0$. Let $x, y \in [T]$ and define $y_k \in [T]$ such that $y_k \restriction m_k = y$ and $y_k(n) = x(n)$ for every $n \geq m_k$. Then clearly $y_0 = x$, $(y_r, y_s) \in \mathbb{E}_0^T \subseteq F_{\mathbf{d}}$ for every $r, s \in \mathbb{N}$ and $y_k \to_{[T]} y$. Let $k \in \mathbb{N} \setminus \{0\}$ and $r, s \geq k$. Then we have

$$|\{n \in \mathbb{N} : y_r(n) \neq y_s(n)\}| \cap m_k = \emptyset$$

and consequently $\mathbf{d}(y_r, y_s) < \frac{1}{k}$. This shows that $\{y_k\}_{k\in\mathbb{N}}$ is a $\mathbf{d}$-Cauchy sequence and by (4) from the definition of Borel pseudometric we have $\mathbf{d}(y_k, y) \to 0$. In particular, there is $k \in \mathbb{N}$ such that $\mathbf{d}(y_k, y) < +\infty$ and therefore $(y_k, y) \in F_{\mathbf{d}}$. Altogether we have $(x, y) \in F_{\mathbf{d}}$ and since $x, y \in [T]$ were arbitrary we have that $F_d = [T] \times [T]$.

The other case is when there is $\epsilon > 0$ such that for every $m \in \mathbb{N}$ there are $\alpha_m, \beta_m \in [T]$ such that $\mathbf{d}(\alpha, \beta) > \epsilon$, $\{n \in \mathbb{N} : \alpha(n) \neq \beta(n)\} \cap m = \emptyset$ and $(\alpha_m, \beta_m) \in \mathbb{E}_0^T$. We show that this contradicts $F_{\mathbf{d}}$ being non-meager.

Note that $F_{\mathbf{d}}$ is a Borel equivalence relation by (2) in the definition of Borel pseudometric and every $F_{\mathbf{d}}$-equivalence class is dense because $\mathbb{E}_0^T \subseteq F_{\mathbf{d}}$. This implies, by [8, Theorem 8.41], that there is $\alpha \in [T]$ such that $[\alpha]_{\mathbf{d}}$ is comeager in $[T]$. It follows from (3) in the definition of Borel pseudometric that there are Borel sets $\{U_l\}_{l\in\mathbb{N}}$ such that $\bigcup_{l\in\mathbb{N}} U_l = [\alpha]_{F_{\mathbf{d}}}$ and

$$\mathbf{d}(x, y) < \frac{\epsilon}{2}$$

for every $l \in \mathbb{N}$ and $x, y \in U_l$.

By [8, Proposition 8.26] we find $t' \in T$ and $l \in \mathbb{N}$ such that U_l is comeager in $t'^\frown[T_t]$. Pick $m \in \mathbb{N}$ such that $m \geq |t'|$ and $\frac{1}{m} < \frac{\epsilon}{4}$. We may suppose that $\alpha_m = s^\frown u_0{}^\frown x$ and $\beta_m = s^\frown u_1{}^\frown x$ where $|s| = m$, $|u_0| = |u_1|$ and $x \in [T_{s^\frown u_0}] = [T_{s^\frown u_1}]$.

Let $t \in T$ be such that $t' \sqsubseteq t$ and $|t| = |s| = m$. Then we have that U_l is comeager in $t^\frown[T_t]$ and therefore there is $y \in [T_{t^\frown u_0}] = [T_{t^\frown u_1}]$ such that

$$t^\frown u_0{}^\frown y, t^\frown u_1{}^\frown y \in U_l.$$

In particular we have $\mathbf{d}(t^\frown u_0{}^\frown y, t^\frown u_1{}^\frown y) < \frac{\epsilon}{2}$.

Last step is to use that $\mathbf{d}$ is uniform. We have

$$|\mathbf{d}(s^\frown(u_0{}^\frown x), s^\frown(u_1{}^\frown x)) - \mathbf{d}(t^\frown(u_0{}^\frown x), t^\frown(u_1{}^\frown x))| < \frac{1}{2^m} < \frac{1}{m} < \frac{\epsilon}{4}$$

and

$$|\mathbf{d}((t^\frown u_0)^\frown x, (t^\frown u_1)^\frown x) - ((t^\frown u_0)^\frown y, (t^\frown u_1)^\frown y)| < \frac{1}{2^{|t^\frown u_0|}} < \frac{1}{m} < \frac{\epsilon}{4}.$$

This implies

$$\mathbf{d}(t^\frown u_0{}^\frown y, t^\frown u_1{}^\frown y) \geq \mathbf{d}(s^\frown u_0{}^\frown x, s^\frown u_1{}^\frown x) - \frac{\epsilon}{2} > \frac{\epsilon}{2}$$

and that contradicts $\mathbf{d}(t^\frown u_0{}^\frown y, t^\frown u_1{}^\frown y) < \frac{\epsilon}{2}$. This finishes the proof. □

5. Base for Non-Classification

We describe the family that will serve as a base under $\leq_B$ for non-classification in the proof of Theorem 6.1. We denote the power set of $\mathbb{N}$ as $\mathcal{P}(\mathbb{N})$.

A map $\Theta : \mathcal{P}(\mathbb{N}) \to [0, +\infty]$ is a *lsc submeasure* if $\Theta(\emptyset) = 0$, $\Theta(M \cup N) \leq \Theta(M) + \Theta(N)$ whenever $M, N \in \mathcal{P}(\mathbb{N})$, $\Theta(\{m\}) < +\infty$ for every $m \in \mathbb{N}$ and

$$\Theta(M) = \lim_{m\to\infty} \Theta(M \cap m)$$

for every $M \in \mathcal{P}(\mathbb{N})$. We say that Θ is *tall* if $\lim_{m\to\infty} \Theta(\{m\}) = 0$.

Let Θ be a tall lsc submeausre. Then the equivalence relation E_Θ on $2^{\mathbb{N}}$ is defined as

$$(x, y) \in E_\Theta \ \Leftrightarrow \ \lim_{m\to\infty} \Theta(\{n \in \mathbb{N} \setminus m : x(n) \neq y(n)\}) = 0$$

for every $x, y \in 2^{\mathbb{N}}$. We remark that E_Θ is non-meager if and only if $E_\Theta = 2^{\mathbb{N}} \times 2^{\mathbb{N}}$, compare with Theorem 4.3.

A sequence of finite metric spaces $\{(Z_m, \mathfrak{d}_m)\}_{m\in\mathbb{N}}$ is called *non-trivial* if

$$\liminf_{m\to\infty} r(Z_m, \mathfrak{d}_m) > 0 \ \& \ \lim_{m\to\infty} j(Z_m, \mathfrak{d}_m) = 0$$

where $r(Z, \mathfrak{d}) = \max \mathfrak{d}$ and $j(Z, \mathfrak{d})$ is the minimal $\epsilon > 0$ such that there is $l \in \mathbb{N}$ and a sequence $(z_0, \dots z_l)$ that contains every element of Z and satisfies $\mathfrak{d}(z_i, z_{i+1}) < \epsilon$ for every $i < l$.

Let $\mathcal{Z} = \{(Z_m, \mathfrak{d}_m)\}_{m\in\mathbb{N}}$ be a non-trivial sequence of finite metric spaces and $\prod_{m\in\mathbb{N}} Z_m$ be endowed with the product topology. Then the equivalence relation $E_{\mathcal{Z}}$ on $\prod_{m\in\mathbb{N}} Z_m$ is defined as

$$(x, y) \in E_{\mathcal{Z}} \ \Leftrightarrow \ \lim_{m\to\infty} \mathfrak{d}_m(x(m), y(m)) = 0$$

for every $x, y \in \prod_{m\in\mathbb{N}} Z_m$.

Definition 5.1. *Denote as $\mathcal{B}$ the collection of all Borel meager equivalence relations that contain E_Θ for some tall lsc submeasure Θ or $E_{\mathcal{Z}}$ for some non-trivial sequence of finite metric spaces $\mathcal{Z}$, i.e., for every $E \in \mathcal{B}$ there is either tall lsc submeasure Θ such that $E_\Theta \subseteq E$ and E is a meager subset of $2^{\mathbb{N}} \times 2^{\mathbb{N}}$, or there is a non-trivial sequence of finite metric spaces $\mathcal{Z}$ such that $E_{\mathcal{Z}} \subseteq E$ and E is a meager subset of $\prod_{m\in\mathbb{N}} Z_m \times \prod_{m\in\mathbb{N}} Z_m$.*

Theorem 5.2. *Let $E \in \mathcal{B}$. Then E is not classifiable by countable structures.*

Proof. It is easy to see that if E_Θ is meager, then it is induced by a turbulent action of a Polish group on $2^{\mathbb{N}}$ whenever Θ is a tall lsc submeasure and $E_{\mathcal{Z}}$ is induced by a turbulent action of a Polish group on $\prod_{m\in\mathbb{N}} Z_m$ whenever $\mathcal{Z}$ is a non-trivial sequence of finite metric spaces, see [3, Appendix 3.7] and [6, Chapter 16].

Let $E \in \mathcal{B}$ be a Borel meager equivalence relation on Y. By the definition we find $F \subseteq E$ such that either $F = E_\Theta$ for some tall lsc submeasure Θ or $F = E_{\mathcal{Z}}$ for some non-trivial sequence of finite metric spaces $\mathcal{Z}$.

Let W be a Polish S_∞-space and $\psi : Y \to W$ be a Borel map that is a reduction from E to $E^W_{S_\infty}$. Then ψ is a Borel homomorphism from F to $E^W_{S_\infty}$ and it follows from [2, Theorem 10.4.3] that there is $y \in Y$ such that $\psi^{-1}([\psi(y)]_{E^W_{S_\infty}})$ is comeager in Y. Since ψ is a reduction we have

$$\psi^{-1}([\psi(y)]_{E^W_{S_\infty}}) \subseteq [y]_E.$$

An application of [8, Theorem 8.41] shows that E is comeager and that is a contradiction. □

6. Non-Classification by Countable Structures

The aim of this section is to show that **(B)** in Theorem 2.2 implies that E^X_G is complicated.

Theorem 6.1. *Let G be a tsi Polish group, X be a Polish G-space such that E^X_G is Borel and* **(B)** *in Theorem 2.2 holds for $A = X$. Then E^X_G is not classifiable by countable structures.*

Proof. Let $\mathbf{k} \in \mathbb{N}$, T', $\{s'_m\}_{m\in\mathbb{N}}$ and $\varphi : [T'] \to X$ be as in **(B)** Theorem 2.2. First we formulate the main technical result that uses crucially that G is tsi. See Section 8 for the proof.

Lemma 6.2 (Refinement)**.** *Suppose that $\mathbf{k} \in \mathbb{N}$, T', $\{s'_m\}_{m\in\mathbb{N}}$ and $\varphi : [T'] \to X$ are as in* **(B)** *Theorem 2.2. Then there are $\mathbf{k} \in \mathbb{N}$, T, $\{s_m\}_{m\in\mathbb{N}} \subseteq T$ and $\phi : [T] \to X$ as in* **(B)** *Theorem 2.2 such that $\mathbf{d}_\phi$ is a uniform Borel pseudometric and $\phi = \varphi \circ \zeta$ where $\zeta : [T] \to [T']$ is a continuous map.*

Let $\mathbf{k} \in \mathbb{N}$, T, $\{s_m\}_{m\in\mathbb{N}}$ and ϕ be as in Lemma 6.2. Observe that

$$\mathbb{E}^T_0 \subseteq F_{\mathbf{d}_\phi} = (\phi^{-1} \times \phi^{-1})(E^X_G)$$

because $s_m \in \mathbb{N}^m \cap T$ for every $m \in \mathbb{N}$. The rest of the proof consists of four steps.

(I). The Borel equivalence relation $F_{\mathbf{d}_\phi}$ is meager in $[T]\times[T]$. Otherwise there is $\alpha \in [T]$ such that $[\alpha]_{\mathbf{d}}$ is comeager in $[T]$ by [8, Theorem 8.41]. It follows from (3) in the definition of Borel pseudometric that there are Borel sets $\{U_l\}_{l\in\mathbb{N}}$ such that $\bigcup_{l\in\mathbb{N}} U_l = [\alpha]_{F_{\mathbf{d}}}$ and

$$\mathbf{d}_\phi(\alpha, \beta) < \frac{1}{2^{\mathbf{k}}}$$

for every $l \in \mathbb{N}$ and $\alpha, \beta \in U_l$. Using [8, Proposition 8.41] and the density of $\{s_m\}_{m\in\mathbb{N}}$ we find $m, l \in \mathbb{N}$ such that U_l is comeager in $s_m{}^\frown[T_{s_m}]$. This gives $x \in [T_{s_m{}^\frown(0)}] = [T_{s_m{}^\frown(l^T_m-1)}]$ such that

$$s_m{}^\frown(0)^\frown x, s_m{}^\frown(l^T_m - 1)^\frown x \in U_l.$$

Since ϕ is a homomorphism from $\mathbb{G}^T_{s_m}$ to $\mathcal{H}_{\mathbf{k},m}$ we have that

$$(\phi(s_m{}^\frown(i)^\frown x))_{i<l^T_m} \in \mathcal{H}_{\mathbf{k},m}$$

and consequently that

$$\phi(s_m{}^\frown(l^T_m - 1)^\frown x) \notin V_{\mathbf{k}} \cdot \phi(s_m{}^\frown(0)^\frown x).$$

This gives

$$\mathbf{d}_\phi(s_m{}^\frown(0)^\frown x, s_m{}^\frown(l^T_m - 1)^\frown x) > \frac{1}{2^{\mathbf{k}}}$$

and that contradicts the choice of $x \in [T_{s_m{}^\frown(0)}]$.

(II). Let $s, t \in T \cap \mathbb{N}^m$, $i, j < l^T_m$ and $x, y \in [T_{s^\frown(i)}] = [T_{s^\frown(j)}]$. Then

$$|\mathbf{d}_\phi(s^\frown(i)^\frown x, s^\frown(j)^\frown x) - \mathbf{d}_\phi(t^\frown(i)^\frown y, t^\frown(j)^\frown y| < \frac{1}{2^{m-1}}.$$

We use that $\mathbf{d}_\phi$ is uniform. Namely, we have

$$|\mathbf{d}_\phi(s^\frown((i)^\frown y), s^\frown((j)^\frown y)) - \mathbf{d}_\phi(t^\frown((i)^\frown y), t^\frown((j)^\frown y)| < \frac{1}{2^m}$$

$$|\mathbf{d}_\phi((s^\frown(i))^\frown x, (s^\frown(j))^\frown x) - \mathbf{d}_\phi((s^\frown(i))^\frown y, (t^\frown(j))^\frown y| < \frac{1}{2^{m+1}}$$

and that gives the estimate by the triangle inequality.

(III). Let $m \in \mathbb{N}$ and $\mathbf{0} = (0, 0, \dots)$. Since $(\{s_m{}^\frown(i)^\frown\mathbf{0}\}_{i<l^T_m}, \mathbf{d}_\phi)$ is a finite pseudometric space we find a metric space $(Z_m, \mathfrak{d}_m)$ where $Z_m = \{0, 1. \dots, l^T_m - 1\}$ and

$$|\mathbf{d}_\phi(s_m{}^\frown(i)^\frown\mathbf{0}, s_m{}^\frown(j)^\frown\mathbf{0}) - \mathfrak{d}_m(i, j)| < \frac{1}{2^{m-1}}$$

for every $i, j < l^T_m$. Then we have

$$\frac{1}{2^{\mathbf{k}}} - \frac{1}{2^{m-1}} \leq \mathfrak{d}_m(0, l^T_m - 1) \leq r(Z_m, \mathfrak{d}_m)$$

and $j(Z_m, \mathfrak{d}_m) < \frac{1}{2^{m-2}}$ because ϕ is a homomorphism from $\mathbb{G}_{s_m}$ to $\mathcal{H}_{\mathbf{k},m}$.

This implies immediately that $\mathcal{Z} = \{(Z_m, \mathfrak{d}_m)\}_{m\in\mathbb{N}}$ is a non-trivial sequence of finite metric spaces. Consider the bijective homeomorphism

$$\eta : \prod_{m\in\mathbb{N}} Z_m \to [T]$$

that is defined as

$$\eta(x)(m) = i \;\Leftrightarrow\; x(m) = i.$$

If $E_{\mathcal{Z}} \subseteq E = (\eta^{-1} \times \eta^{-1})(F_{\mathbf{d}_\phi})$, then we are done because $E \in \mathcal{B}$ by **(I)** and $\phi \circ \eta$ is a reduction from E to E^X_G. Hence, E^X_G is not classifiable by countable structures by Theorem 5.2.

(IV). Suppose that $E_{\mathcal{Z}} \not\subseteq E = (\eta^{-1} \times \eta^{-1})(F_{\mathbf{d}_\phi})$ in **(III)**. There is $x, y \in \prod_{m\in\mathbb{N}} Z_m$ such that

$$\mathfrak{d}_m(x(m), y(m)) \to 0$$

and $(\eta(x), \eta(y)) \notin F_{\mathbf{d}_\phi}$. Set $\alpha = \eta(x)$ and $\beta = \eta(y)$. Note that $|\{m \in \mathbb{N} : \alpha(m) \neq \beta(m)\}| = \aleph_0$ because $\mathbb{E}^T_0 \subseteq F_{\mathbf{d}_\phi}$.

Let

$$S = \{s \in T : \forall i < |s| \; (s(i) = \alpha(i) \vee s(i) = \beta(i))\}.$$

It follows that $S \subseteq T$ is isomorphic to a full binary tree. Moreover, the restriction of $\mathbf{d}_\phi$ to $[S]$ is a uniform Borel pseudometric, in the sense that the uniform condition holds for every $s, t \in \mathbb{N}^m \cap S$ and $x, y \in [S_s] = [S_t]$. Write F for the restriction of $F_{\mathbf{d}_\phi}$ to $[S] \times [S]$. Then it follows from Theorem 4.3 together with $(\alpha, \beta) \notin F$ that F is meager.

Let $\{m_l\}_{l\in\mathbb{N}}$ be an increasing enumeration of $\{m \in \mathbb{N} : \alpha(m) \neq \beta(m)\}$ and set $\mathbf{0}_l = \alpha(m_l)$, $\mathbf{1}_l = \beta(m_l)$ for every $l \in \mathbb{N}$. Then there is a sequence $\{t_l\}_{l\in\mathbb{N}} \subseteq \mathbb{N}^{<\mathbb{N}}$ such that

$$\alpha = t_0{}^\frown\mathbf{0}_0{}^\frown t_1{}^\frown\mathbf{0}_1{}^\frown \dots \ \& \ \beta = t_0{}^\frown\mathbf{1}_0{}^\frown t_1{}^\frown\mathbf{1}_1{}^\frown \dots$$

and consequently for every $s \in S$ there is $l \in \mathbb{N}$ such that

$$s \sqsubseteq t_0{}^\frown\mathbf{i}_0{}^\frown t_1{}^\frown\mathbf{i}_1{}^\frown \dots{}^\frown \mathbf{i}_{l-1}{}^\frown t_l$$

where $\mathbf{i}_j \in \{\mathbf{0}_j, \mathbf{1}_j\}$ for every $j < l$. Define $\Gamma : 2^{<\mathbb{N}} \to S$ as

$$\Gamma(s) = t_0{}^\frown\mathbf{s}(\mathbf{0})^\frown t_1{}^\frown\mathbf{s}(\mathbf{1})^\frown \dots{}^\frown \mathbf{s}(|\mathbf{s}|-\mathbf{1})^\frown t_{|s|} \in S$$

where $\mathbf{s}(\mathbf{j}) = \mathbf{0}_j$ if $s(j) = 0$ and $\mathbf{s}(\mathbf{j}) = \mathbf{1}_j$ if $s(j) = 1$. It is easy to see that the unique extension $\widetilde{\Gamma} : 2^{\mathbb{N}} \to [S]$ is a homeomorphism.

Final step is to define a tall lsc submeasure Θ. Let $M \in \mathcal{P}(\mathbb{N})$ be a finite set. Define

$$\Theta(M) = \sup\left\{\mathbf{d}_\phi(\widetilde{\Gamma}(x), \widetilde{\Gamma}(y)) : x, y \in 2^{\mathbb{N}} \ \{l \in \mathbb{N} : x(l) \neq y(l)\} \subseteq M \ \right\} =$$

$$= \sup\left\{\mathbf{d}_\phi(x, y) : x, y \in [S] \ \{m \in \mathbb{N} : x(m) \neq y(m)\} \subseteq \{m_l\}_{l\in M} \ \right\}.$$

Let $M \in \mathcal{P}(\mathbb{N})$ be infinite. Then we define $\Theta(M) = \lim_{l\to\infty} \Theta(M \cap l)$.

To finish the proof we need to show that Θ is a tall lsc submeasure and $E_\Theta \subseteq E = \left(\widetilde{\Gamma}^{-1} \times \widetilde{\Gamma}^{-1}\right)(F) = \left(\widetilde{\Gamma}^{-1} \times \widetilde{\Gamma}^{-1}\right)(F_{\mathbf{d}_\phi})$. Indeed, then we have $E \in \mathcal{B}$ and $\phi \circ \widetilde{\Gamma}$ is a reduction from E to E^X_G.

(a). It is easy to see that Θ is monotone, $\Theta(\emptyset) = 0$ and $\Theta(M) = \lim_{l\to\infty} \Theta(M \cap l)$ for every $M \in \mathcal{P}(\mathbb{N})$. Let $M, N \in \mathcal{P}(\mathbb{N})$ be two finite sets and $x, y \in 2^{\mathbb{N}}$ such that $\{l \in \mathbb{N} : x(l) \neq y(l)\} \subseteq M \cup N$. Let $x'(l) = x(l)$ for every $l \in \mathbb{N} \setminus M$ and $x'(l) = y(l)$ for every $l \in M$. The fact that $\mathbf{d}_\phi$ is a pseudometric implies that

$$\mathbf{d}_\phi(\widetilde{\Gamma}(x), \widetilde{\Gamma}(y)) \le \mathbf{d}_\phi(\widetilde{\Gamma}(x), \widetilde{\Gamma}(x')) + \mathbf{d}_\phi(\widetilde{\Gamma}(x'), \widetilde{\Gamma}(y)) \le \Theta(M) + \Theta(N).$$

This shows that $\Theta(M \cup N) \le \Theta(M) + \Theta(N)$ for every finite $M, N \in \mathcal{P}(\mathbb{N})$ and one can easily check that it extends for any $M, N \in \mathcal{P}(\mathbb{N})$. Let $l \in \mathbb{N}$. It follows from **(II)**, definition of $\widetilde{\Gamma}$ and the definition of $\mathfrak{d}_m$ in **(III)** that

$$\Theta(\{l\}) \le \mathbf{d}_\phi(s_{m_l}{}^\frown\alpha(m_l)^\frown\mathbf{0}), s_{m_l}{}^\frown\beta(m_l)^\frown\mathbf{0})) + \frac{1}{2^{m_l-1}} \le \mathfrak{d}_{m_l}(\alpha(m_l), \beta(m_l)) + \frac{1}{2^{m_l-2}}.$$

This shows that $\Theta(\{l\}) < +\infty$ for every $l \in \mathbb{N}$ and the choice of $\alpha = \eta(x)$ and $\beta = \eta(y)$ in the beginning of **(IV)** guarantees that

$$\Theta(\{l\}) \le \mathfrak{d}_{m_l}(x(m_l), y(n_l)) + \frac{1}{2^{m_l-2}} \to 0.$$

Hence, Θ is a tall lsc submeasure.

(b) Let $x, y \in 2^{\mathbb{N}}$ such that $(x, y) \in E_\Theta$ and put $X = \{l \in \mathbb{N} : x(l) \neq y(n)\}$. Then we have that $\lim_{l\to\infty} \Theta(X \setminus l) = 0$ by the definition of E_Θ. Define $x_l(j) = y(j)$ for every $j < l$ and $x_l(j) = x(j)$ for every $j \ge l$ for every $l \in \mathbb{N}$. We have $(x_l, x) \in \mathbb{E}_0$ for every $l \in \mathbb{N}$ and $x_l \to y$. The definition of Γ easily implies that

$$\left(\widetilde{\Gamma}(x_l), \widetilde{\Gamma}(x)\right) \in \mathbb{E}_0^T$$

and $\widetilde{\Gamma}(x_l) \to \widetilde{\Gamma}(y)$. Let $l \leq r \leq s \in \mathbb{N}$. We have that $\{j \in \mathbb{N} : x_r(j) \neq x_s(j)\} = X \cap \{r, \ldots, s-1\} \subseteq X \setminus l$ and by the definition of Θ that

$$\mathbf{d}_\phi\left(\widetilde{\Gamma}(x_r), \widetilde{\Gamma}(x_s)\right) \leq \Theta(X \cap \{r, \ldots, s-1\}) \leq \Theta(X \setminus l).$$

This shows that $\{\widetilde{\Gamma}(x_l)\}_{l \in \mathbb{N}}$ is a $\mathbf{d}_\phi$-Cauchy sequence. By (4) in the definition of Borel pseudometric we find $l \in \mathbb{N}$ such that

$$\mathbf{d}_\phi\left(\widetilde{\Gamma}(x_l), \widetilde{\Gamma}(y)\right) < +\infty$$

and, in particular, $\left(\widetilde{\Gamma}(x_l), \widetilde{\Gamma}(y)\right) \in F_{\mathbf{d}_\phi}$. This gives $\left(\widetilde{\Gamma}(x), \widetilde{\Gamma}(y)\right) \in F_{\mathbf{d}_\phi}$ because $\left(\widetilde{\Gamma}(x), \widetilde{\Gamma}(x_l)\right) \in \mathbb{E}_0^T \subseteq F_{\mathbf{d}_\phi}$ and the proof is finished. $\square$

7. Remarks and Question

Our main result follows immediately from the following statement.

Theorem 7.1. *Let G be a tsi Polish group, X be a Polish G-space such that E_G^X is Borel and A be a $\mathbf{\Sigma}_1^1$ subset of X. Then exactly on of the following holds*

(1) *there is a Borel G-invariant set $B \subseteq X$ such that $A \subseteq B$ and $E_G^X \restriction B \times B$ is classifiable by countable structures,*
(2) *there is $E \in \mathcal{B}$ on a Polish space Y and a continuous map $\zeta : Y \to A$ that is a reduction from E to E_G^X.*

Moreover, (1) *is equivalent to*

(1)' *there is a Borel G-invariant set $B \subseteq X$ such that $A \subseteq B$ and B satisfies property (IC).*

Proof. Apply Theorem 2.2. Note that **(A)** implies (1)′ by Corollary 3.4 and (1)′ implies (1) by Theorem 3.5.

On the other hand **(B)** implies by the proof of Theorem 6.1 that there is $E \in \mathcal{B}$ on Y and a continuous map $\zeta : Y \to X$ that is a reduction from E to E_G^X. Note that ζ is of the form $\phi \circ \widetilde{\Gamma}$ or $\phi \circ \eta$ where ϕ is given by Lemma 6.2 and satisfies $\mathrm{rng}(\phi) \subseteq \mathrm{rng}(\varphi) \subseteq A$. This shows that $\zeta : Y \to A$ and (2) follows.

Finally observe that (1) implies $\neg$**(B)** by Theorem 5.2 and consequently (1) implies (1)′. That completes the proof. $\square$

It is a very interesting question if the base in (2) can be smaller.

Question 7.2. *Let $\mathcal{C}$ be the collection of meager equivalence relations E_Θ and $E_\mathcal{Z}$ where Θ runs over all tall lsc submeasures and $\mathcal{Z}$ over non-trivial sequences of finite metric spaces. Is it enough to take $\mathcal{C}$ instead of $\mathcal{B}$ in Theorem 7.1* (2)*?*

Maybe mention Hjorth's summable ideal dichotomy.
Next, we sketch another application of our approach.

Theorem 7.3. *Let G be a tsi Polish group, X a Polish G-space such that E_G^X is Borel and F an equivalence relation on a Polish space Y that is classifiable by countable structures. Suppose that $\varphi : Y \to X$ is a Borel map that is a reduction from F to E_G^X. Then there is a Borel G-invariant set $B \subseteq X$ such that $\varphi(Y) \subseteq B$ and $E_G^X \restriction B \times B$ is classifiable by countable structures.*

Proof Sketch. Put $A = \varphi(Y)$ an apply Theorem 7.1. We show that we get (1). Define $\mathbf{d}_\varphi$ on Y as in Proposition 4.2. Then $\mathbf{d}_\varphi$ is a Borel pseudometric and $F_{\mathbf{d}_\varphi} = F$ since φ is a reduction. In another words, we pull back the metric structure from G on any F-orbit via the reduction φ, see Proposition 4.2.

Define $\mathcal{H}^{\mathbf{d}_\varphi}_{k,m}$ on Y as

$$\overline{y} \in \mathcal{H}^{\mathbf{d}_\varphi}_{k,m} \ \Leftrightarrow \ \forall i < (l(\overline{y}) - 1) \ \mathbf{d}_\varphi \left(\overline{y}_i, \overline{y}_{i+1}\right) < \frac{1}{2^m} \ \wedge \mathbf{d}_\varphi \left(\mathfrak{s}(\overline{y}), \mathfrak{t}(\overline{y})\right) > \frac{1}{2^k}.$$

Then one can verify that $\{\mathcal{H}^{\mathbf{d}_\varphi}_{k,l}\}_{k,l\in\mathbb{N}}$ is a Borel sequence of dihypergraphs on Y and a version of Theorem 2.2 applies.

If we get a version of **(A)** we compose the $\mathcal{H}^{\mathbf{d}_\varphi}_{k,m}$-independent sets with φ and obtain $\mathcal{H}_{k,m}$-independent subsets of X that cover A, hence Theorem 3.5 applies.

In the case of a version of **(B)** we get a map $\zeta : [T] \to Y$ that satisfies all the properties of a version of **(B)**. Note that $\varphi \circ \zeta$ is as in **B** of Theorem 2.2. Applying Theorem 6.1 we obtain a refinement of $\varphi \circ \zeta \circ \eta$ that is a reduction from E to E_G^X for some $E \in \mathcal{B}$. However, $\zeta \circ \eta$ is a reduction from E to F and that is a contradiction. □

8. Proof of Lemma 6.2

Before we prove Lemma 6.2 we introduce some auxiliary notion and technical results. Let T be a finitely uniformly branching tree. Let $(A, \alpha) \in [\mathbb{N}]^{\mathbb{N}} \times [T]$ where $[\mathbb{N}]^{\mathbb{N}}$ denotes the set of all infinite subsets of $\mathbb{N}$. Then we define $T_{(A,\alpha)} \subseteq T$ as

$$s \in T_{(A,\alpha)} \ \Leftrightarrow \ \forall n \notin A \ s(n) = \alpha(n)$$

and denote as $[T_{(A,\alpha)}]$ the branches of $T_{(A,\alpha)}$. Note that $[T_{(A,\alpha)}]$ is closed in $[T]$.

Write $\{n_l\}_{l\in\mathbb{N}} = A$ for the increasing enumeration of A. Then there is a unique finitely uniformly branching tree $S = S_{(A,\alpha)}$ and a unique map $e_{(A,\alpha)} : S \to T_{(A,\alpha)}$ that satisfy

- $l^S_l = l^T_{n_l}$ for every $l \in \mathbb{N}$,
- $|e_{(A,\alpha)}(s)| = n_{|s|}$
- $e_{(A,\alpha)}(s)(n_l) = s(l)$ for every $l < |s|$,
- $e_{(A,\alpha)}(s)(j) = \alpha(j)$ for every $j < n_{|s|}$ such that $j \notin A$.

It is easy to verify that $e_{(A,\alpha)}$ extends to a unique continuous homeomorphism

$$\widetilde{e}_{(A,\alpha)} : [S] \to [T_{(A,\alpha)}]$$

that is a reduction from $\mathbb{G}^S_s$ to $\mathbb{G}^T_{e_{(A,\alpha)}(s)}$ for every $s \in S$. This is because if $s(l) = t(l)$, then we have $e_{(A,\alpha)}(s)(j) = e_{(A,\alpha)}(t)(j)$ for every $n_l \leq j < n_{l+1}$.

Lemma 8.1. *Let $\{T_r\}_{r\in\mathbb{N}}$ be a sequence of finitely uniformly branching trees, $(A_r,\alpha_r)\in[\mathbb{N}]^{\mathbb{N}}\times[T_r]$ be such that $A_r\cap r+1=r+1$ for every $r\in\mathbb{N}$ and $S_{(A_r,\alpha_r)}=T_{r+1}$ for every $r\in\mathbb{N}$. Then there is a finitely uniformly branching tree S and a sequence of continuous maps $\{\widetilde{\psi}_{r,\infty}:[S]\to[T_r]\}_{r\in\mathbb{N}}$ such that*

1. $l^S_r=l^{T_{r'}}_r$ *for every $r\leq r'\in\mathbb{N}$,*
2. *for every $s\in S\cap\mathbb{N}^r$ and $x\in[S_s]$ there is $y\in[(T_r)_s]$ such that $\widetilde{\psi}_{r,\infty}(t^\frown x)=t^\frown y$ whenever $t\in S\cap\mathbb{N}^r$ for every $r\in\mathbb{N}$,*
3. $\widetilde{\psi}_{r,\infty}=\widetilde{e}_{(A_r,\alpha_r)}\circ\widetilde{\psi}_{r+1,\infty}$,
4. $\widetilde{\psi}_{r,\infty}$ *is a reduction from $\mathbb{G}^S_s$ to $\mathbb{G}^{T_r}_s$ for every $s\in T_r\cap\mathbb{N}^r$.*

Proof. Observe that if $r\leq r'\in\mathbb{N}$, then $l^{T_{r'}}_r=l^{T_r}_r$ and define $l^S_r=l^{T_r}_r$. This defines S and (1) is satisfied.

For $s\in S\cap\mathbb{N}^r$ we define $\psi_{r',\infty}(s)=s$ for every $r\leq r'\in\mathbb{N}$ and inductively $\psi_{r',\infty}(s)=e_{(A_r,\alpha_r)}\circ\psi_{r'+1,\infty}$ for every $0\leq r'<r$. Then we have $\psi_{r,\infty}=e_{(A_r,\alpha_r)}\circ\psi_{r+1,\infty}$ for every $r\in\mathbb{N}$ and if $s\sqsubseteq t\in S$, then $\psi_{r,\infty}(s)\sqsubseteq\psi_{r,\infty}(t)$ for every $r\in\mathbb{N}$.

Define

$$\widetilde{\psi}_{r,\infty}(x)=\bigcup_{l\in\mathbb{N}}\psi_{r,\infty}(x\restriction l)$$

for every $x\in[S]$ and $r\in\mathbb{N}$. We have

$$\widetilde{\psi}_{r,\infty}(x)=\bigcup_{l\in\mathbb{N}}\psi_{r,\infty}(x\restriction l)=\bigcup_{l\in\mathbb{N}}e_{(A_r,\alpha_r)}\circ\psi_{r+1,\infty}(x\restriction l)=$$

$$=\widetilde{e}_{(A_r,\alpha_r)}\left(\bigcup_{l\in\mathbb{N}}\psi_{r+1,\infty}(x\restriction l)\right)=\widetilde{\psi}_{r+1,\infty}(x)$$

for every $x\in[S]$ and that shows (3).

Note that (1) and (2) imply (4) and therefore it remains to show (2). Let $s\in S\cap\mathbb{N}^r$ and $x\in[S_s]$. Put $y\in[(T_r)_s]$ such that

$$\widetilde{\psi}_{r,\infty}(s^\frown x)=s^\frown y.$$

Let $t\in S\cap\mathbb{N}^r$ and $r<l\in\mathbb{N}$. It is clearly enough to show that $\psi_{r,\infty}(s^\frown x\restriction l)(j)=\psi_{r,\infty}(t^\frown x\restriction l)(j)$ for every $r\leq j<l$.

We show inductively that $\psi_{r',\infty}(s^\frown x\restriction l)(j)=\psi_{r',\infty}(t^\frown x\restriction l)(j)$ for every $r\leq j<l$ where $r\leq r'\leq l$. By the definition we have

$$\psi_{l,\infty}(s^\frown x\restriction l)(j)=(s^\frown x\restriction l)(j)=(t^\frown x\restriction l)(j)=\psi_{l,\infty}(t^\frown x\restriction l)(j)$$

for every $r\leq j<l$. Suppose that it holds for $r'+1$ where $r\leq r'<l$. Fix an enumeration $\{m_p\}_{p\in\mathbb{N}}$ of $A_{r'}$. Then for every $r\leq j<l$ there is $p\in\mathbb{N}$ such that $r\leq p<l$ and $m_p\leq j<m_{p+1}$. This is because $A_{r'}\cap r+1=r+1$. If $m_p=j$, then we have

$$\psi_{r',\infty}(s^\frown x\restriction l)(j)=\left(e_{(A_{r'},\alpha_{r'})}\circ\psi_{r'+1,\infty}(s^\frown x\restriction l)\right)(m_p)=\psi_{r'+1,\infty}(s^\frown x\restriction l)(p)=$$

$$=\psi_{r'+1,\infty}(t^\frown x\restriction l)(p)=\left(e_{(A_{r'},\alpha_{r'})}\circ\psi_{r'+1,\infty}(t^\frown x\restriction l)\right)(m_p)=\psi_{r',\infty}(t^\frown x\restriction l)(j)$$

from the inductive assumption. If $m_p < j$, then

$$\psi_{r',\infty}(s^\frown x \restriction l)(j) = \left(e_{(A_{r'},\alpha_{r'})} \circ \psi_{r'+1,\infty}(s^\frown x \restriction l)\right)(j) = \alpha_{r'}(j) =$$

$$= \left(e_{(A_{r'},\alpha_{r'})} \circ \psi_{r'+1,\infty}(t^\frown x \restriction l)\right)(j) = \psi_{r',\infty}(t^\frown x \restriction l)(j)$$

and the proof is finished. $\square$

Lemma 8.2. *Let T be a finitely uniformly branching tree, $\mathcal{A} \in [\mathbb{N}]^{\mathbb{N}}$, $\mathbf{m} \in \mathbb{N}$, $\mathbf{p} \in T \cap \mathbb{N}^{\mathbf{m}}$, $\{X_r\}_{r\in\mathbb{N}}$ be a sequence of subsets of $[T]$ with the Baire property such that $\bigcup_{r\in\mathbb{N}} X_r = [T]$ and $\{s_n\}_{n\in\mathcal{A}} \subseteq T$ be dense in T and $|s_n| = n$. Then there is $(A,\alpha) \in [\mathbb{N}]^{\mathbb{N}} \times [T]$ such that, if we put $S = S_{(A,\alpha)}$, we have*

1. *$A \cap \mathbf{m} = \mathbf{m}$,*
2. *for every $s \in S \cap \mathbb{N}^{\mathbf{m}}$ there is $r \in \mathbb{N}$ such that $s^\frown[S_s] \subseteq (\widetilde{e}_{(A,\alpha)})^{-1}(X_r)$,*
3. *$\{v \in S : \exists n \in \mathcal{A}\ e_{(A,\alpha)}(v) = s_n\}$ is dense in S,*
4. *there is $n \in \mathcal{A}$ such that $\mathbf{p} \sqsubseteq e_{(A,\alpha)}(\mathbf{p}) = s_n$.*

Proof. Let $\{p_l\}_{l\in\mathbb{N}}$ be an enumeration of T such that $|\{l \in \mathbb{N} : s = p_l\}| = \aleph_0$ for every $s \in T$. The construction proceeds by induction on $l \in \mathbb{N}$. Namely, in every step we construct $t_l \in \mathbb{N}^{<\mathbb{N}}$, $n_l \in \mathbb{N}$, $\alpha_l \in T$ and $S_l \subseteq T$ such that $n_l = |\alpha_l|$,

$$\alpha_l = \mathbf{p}^\frown t_0{}^\frown(0)^\frown t_1{}^\frown(0)^\frown \dots {}^\frown (0)^\frown t_l$$

and

$$S_l = \{s \in T : |s| = n_l + 1\ \wedge\ \forall \mathbf{m} \le j < n_l\ (\forall l' \le l\ j \neq n_{l'} \to s(j) = \alpha_l(j))\}.$$

In the end we put $\alpha = \bigcup_{l\in\mathbb{N}} \alpha_l$ and $A = \mathbf{m} \cup \{n_l\}_{l\in\mathbb{N}}$.

(I) $l = 0$. Let $\{u_i\}_{i<N_0}$ be an enumeration of $\{s \in T : |s| = \mathbf{m}\}$. Define inductively $v_i \in \mathbb{N}^{<\mathbb{N}}$ such that

- $u_i \frown v_i \in T$ for every $i < N_0$,
- $v_i \sqsubseteq v_{i+1}$ for every $i < N_0 - 1$,
- for every $i < N_0$ there is $r(i) \in \mathbb{N}$ such that $X_{r(i)}$ is comeager in $u_i{}^\frown v_i{}^\frown[T_{u_i{}^\frown v_i}]$.

This can be achieved by [8, Proposition 8.26]. Write $v = v_{N_0-1}$ and use the density of $\{s_n\}_{n\in\mathcal{A}}$ to find $n \in \mathbb{N}$ such that $\mathbf{p}^\frown v \sqsubseteq s_n$. Let $t_0 \in \mathbb{N}^{<\mathbb{N}}$ be such that $\alpha_0 = \mathbf{p}^\frown t_0 = s_n$ and $n_0 = |\mathbf{p}^\frown t_0|$.

Define

$$X = \bigcup_{i<N_0} u_i{}^\frown t_0{}^\frown[T_{u_i{}^\frown t_0}] \cap X_{r(i)}.$$

Note that X is comeager in $u_i{}^\frown t_0{}^\frown[T_{u_i{}^\frown t_0}]$ for every $i < N_0$. Let $\{\mathcal{O}_l\}_{l\in\mathbb{N}}$ be a decreasing collection of open subsets of $[T]$ such that $\mathcal{O}_0 = [T]$, $\bigcap_l \mathcal{O}_l \subseteq X$ and $\mathcal{O}_l$ is dense in $u_i{}^\frown t_0{}^\frown[T_{u_i{}^\frown t_0}]$ for every $i \in N_0$.

(II) $l \mapsto l + 1$. Suppose that we have $\{n_m\}_{m\le l}$, $\{\alpha_m\}_{m\in l}$, $\{S_m\}_{m\le l}$ and $\{t_m\}_{m\le l}$ that satisfies

(a) $|\alpha_m| = n_m$ and $\alpha_m = \mathbf{p}^\frown t_0{}^\frown(0)^\frown \dots {}^\frown (0)^\frown t_m$ for every $m \le l$,
(b) $u^\frown[T_u] \subseteq \mathcal{O}_l$ for every $u \in S_l$,

(c) if $m < l$ and $p_m \sqsubseteq u$ for some $u \in S_m$, then there is $n \in \mathcal{A}$ such that $|s_n| = n_{m+1} = n$, $p_m \sqsubseteq u \sqsubseteq s_n$ and $s_n(j) = \alpha_{m+1}(j)$ for every $j < n_{m+1}$ such that $j \notin \mathbf{m} \cup \{n_r\}_{r<m+1}$.

Note that if $l = 0$, then (a)–(c) are satisfied. Next we show how to find $t_{l+1} \in 2^{<\mathbb{N}}$, α_{l+1} and $n_{l+1} \in \mathbb{N}$ such that (a)–(c) holds.

Let $\{u_i\}_{i<N_l}$ be an enumeration of S_l. Construct inductively $\{v_i\}_{i<N_l}$ such that

- $u_i{}^\frown v_i \in T$ for every $i < N_l$,
- $v_i \sqsubseteq v_{i+1}$ for every $i < N_l - 1$,
- $u_i{}^\frown v_i{}^\frown [T_{u_i{}^\frown v_i}] \subseteq \mathcal{O}_{l+1}$ for every $i < N_l$.

This can be done because for every $i < N_l$ there is $u \in T$ such that $u^\frown t_0 \sqsubseteq u_i$ by the definition of S_l and we have $\mathcal{O}_{l+1}$ is dense in $u^\frown t_0{}^\frown [T_{u^\frown t_0}]$. Put $v = v_{N_l - 1}$. If p_l satisfies the assumption of (c), then pick $i < N_l$ such that $p_l \sqsubseteq u_i$. Otherwise pick any $i < N_l$. It follows from the density of $\{s_n\}_{n\in\mathcal{A}}$ that there is $n \in \mathbb{N}$ such that $u_i{}^\frown v \sqsubseteq s_n$. Define $t_{l+1} \in \mathbb{N}^{<\mathbb{N}}$ such that $u_i{}^\frown t_{l+1} = s_n$, $\alpha_{l+1} = \alpha_l{}^\frown (0)^\frown t_{l+1}$ and $n_{l+1} = |u_i{}^\frown t_{l+1}|$.

It is easy to see that we have (a). Let $u \in S_{l+1}$, then there is $i < N_l$ such that $u_i \sqsubseteq u$. Moreover, we have $u_i{}^\frown v_i \sqsubseteq u$ by the definition of t_{l+1} and S_{l+1}. We have

$$u^\frown [T_u] \sqsubseteq u_i{}^\frown v_i{}^\frown [T_{u_i{}^\frown v_i}] \subseteq \mathcal{O}_{l+1}$$

and that shows (b). Item (c) follows directly from the construction.

(III). Let $A = \mathbf{m} \cup \{n_l\}_{l\in\mathbb{N}}$ and $\alpha = \bigcup_{l\in\mathbb{N}} \alpha_l$. Property (1) is trivial. Let $s \in S \cap \mathbb{N}^{\mathbf{m}}$. It is easy to see that $e_{(A,\alpha)}(s) = s^\frown t_0$ and that gives

$$\widetilde{e}_{(A,\alpha)}(s^\frown [S_s]) \subseteq s^\frown t_0{}^\frown [T_{s^\frown t_0}].$$

By the definition in **(I)** there is $r \in \mathbb{N}$ such that

$$X \cap s^\frown t_0{}^\frown [T_{s^\frown t_0}] \subseteq X_r.$$

Let $c \in [T_s]$ and $l \in \mathbb{N}$ Then we have

$$e_{(A,\alpha)}(s^\frown (c \restriction l)) \sqsubseteq s^\frown t_0{}^\frown c(0)^\frown t_1{}^\frown \ldots {}^\frown c(l-1)^\frown t_l{}^\frown c(l) \in S_l$$
$$s^\frown t_0{}^\frown c(0)^\frown t_1{}^\frown \ldots {}^\frown c(l-1)^\frown t_l{}^\frown c(l) \sqsubseteq e_{(A,\alpha)}(s^\frown (c \restriction l+1))$$

and using (b) from the inductive assumption

$$\widetilde{e}_{(A,\alpha)}(s^\frown c) \in e_{(A,\alpha)}(s^\frown (c \restriction l+1))^\frown [T_{e_{(A,\alpha)}(s^\frown (c\restriction l+1))}] \subseteq \mathcal{O}_l.$$

Therefore

$$\widetilde{e}_{(A,\alpha)}(s^\frown c) \in s^\frown t_0{}^\frown [T_{s^\frown t_0}] \cap \bigcap_{l\in\mathbb{N}} \mathcal{O}_l \subseteq X_r$$

and that shows (2).

Let $s \in T \cap \mathbb{N}^{\mathbf{m}}$ and $u \in \mathbb{N}^{<\mathbb{N}}$ such that $s^\frown u \in S$. Find $l \in \mathbb{N}$ such that $|p_l| \leq n_l$ and

$$p_l = e_{(A,\alpha)}(s^\frown u) = s^\frown t_0{}^\frown \ldots {}^\frown u(|u|-1)^\frown t_{|u|}.$$

It follows that there is $w \in S_l$ such that $p_l \sqsubseteq w$ and by (c) in **(II)** we have $n \in \mathcal{A}$ such that $|s_n| = n_{l+1} = n$, $p_l \sqsubseteq s_n$. It is easy to see from the construction that

$$s_n = s^\frown t_0{}^\frown \ldots {}^\frown s_n(n_l)^\frown t_{l+1} = w^\frown t_{l+1}.$$

Put

$$v = s^\frown s_n(n_0)^\frown \dots ^\frown s_n(n_l).$$

Then we have $v \in S$, $e_{(A,\alpha)}(v) = s_n$ and $s^\frown u \sqsubseteq v$ because $e_{(A,\alpha)}(s^\frown u) = p_l \sqsubseteq s_n = e_{(A,\alpha)}(v)$. This shows (3).

Finally, we have $\mathbf{p} \sqsubseteq e_{(A,\alpha)}(\mathbf{p}) = \mathbf{p}^\frown t_0 = s_n$ where $n \in \mathcal{A}$ by the construction in **(I)**. □

Proof of Lemma 6.2. Let $\{g_a\}_{a\in\mathbb{N}}$ be a dense subset of G. The construction proceeds by induction on $r \in \mathbb{N}$. Let $\{p_r\}_{r\in\mathbb{N}}$ be an enumeration of $\mathbb{N}^{<\mathbb{N}}$ such that $|\{r \in \mathbb{N} : p_r = s\}| = \aleph_0$ for every $s \in \mathbb{N}^{<\mathbb{N}}$. We construct a sequence of finitely uniformly branching trees $\{T_r\}_{r\in\mathbb{N}}$ together with $(A_r, \alpha_r) \in [\mathbb{N}]^{\mathbb{N}} \times [T_r]$ such that $S_{(A_r,\alpha_r)} = T_{r+1}$ for every $r \in \mathbb{N}$, $\{\mathcal{A}^r\}_{r\in\mathbb{N}} \subseteq [\mathbb{N}]^{\mathbb{N}}$, $\{s^r_n\}_{n\in\mathcal{A}^r} \subseteq T_r$ for every $r \in \mathbb{N}$ and $\{\varphi_r : [T_r] \to X\}_{r\in\mathbb{N}}$ such that the following holds

1. (1) $A_r \cap r+1 = r+1$ for every $r \in \mathbb{N}$,
2. (2) $\varphi_r = \varphi \circ \widetilde{e}_{(A_0,\alpha_0)} \circ \dots \widetilde{e}_{(A_{r-1},\alpha_{r-1})}$ is a homomorphism from $\mathbb{E}_0^{T_r}$ to E^X_G for every $r \in \mathbb{N}$ (where in the case $r = 0$ we put $\varphi_0 = \varphi$),
3. (3) $r \in \mathcal{A}^r$ for every $r \in \mathbb{N}$,
4. (4) $\{s^r_n\}_{n\in\mathcal{A}^r}$ is a dense subset of T_r such that $|s^r_n| = n$ and φ_r is a homomorphism from $\mathbb{G}^{T_r}_{s^r_n}$ to $\mathcal{H}_{\mathbf{k},n}$ for every $r, n \in \mathbb{N}$,
5. (5) if $p_r \in T_r$ is such that $|p_r| \leq r$, then $p_r \sqsubseteq s^{r+1}_{r+1}$ (where $p_r \in T_{r+1}$ by (1)),
6. (6) for every $s \in T_r$ such that $|s| = r$ there is $g^{s,r} \in G$ such that for every $c \in s^\frown[(T_r)_s]$ there is $g^{s,r}_c \in G$ such that we have

$$|d(g^{s,r}, 1_G) - \mathbf{d}_{\varphi_r}(s^{r\frown}_r c, s^\frown c)| < \frac{1}{2^{r+2}},$$

$$g^{s,r}_c \cdot \varphi_r(s^{r\frown}_r c) = \varphi_r(s^\frown c)$$

$$d(g^{s,r}, g^{s,r}_c) < \frac{1}{2^{r+2}}$$

for every $r \in \mathbb{N}$.

If $r = 0$, then we put $T_0 = T'$, $\mathcal{A} = \mathbb{N}$, $s^0_m = s'_m$ for every $m \in \mathbb{N}$ and $\varphi_0 = \varphi'$. Conditions (1) and (5) are empty, (2)–(4) are satisfied by **(B)** Theorem 2.2 and for (6) it is enough to take $g^{\emptyset,0} = g^{\emptyset,0}_c = 1_G$ for every $c \in [T_0]$.

In the inductive step $r \mapsto r+1$ we construct (A_r, α_r), $\mathcal{A}^{r+1}$, $\{s^{r+1}_n\}_{n\in\mathbb{N}}$ and φ_{r+1} such that (1)–(6) holds.

$\mathbf{r \mapsto r+1}$. We use Lemma 8.2 with $T = T_r$, $\mathcal{A} = \mathcal{A}^r$, $\mathbf{m} = r+1$, $\{s^r_n\}_{n\in\mathcal{A}}$, $p_r \sqsubseteq \mathbf{p} \in T_r \cap \mathbb{N}^{\mathbf{m}}$ if $p_r \in T_r \cap \mathbb{N}^{<\mathbf{m}}$ otherwise we put $\mathbf{p} = (0, \dots, 0) \in \mathbb{N}^{\mathbf{m}}$ and $\{X_q\}_{q\in\mathbb{N}^{N_r}}$ where $N_r = \{s \in T_r : |s| = r+1\}$ and

- if $s \in N_r$ and $s \neq \mathbf{p}$, then $s^\frown x \in X_q$ for every $q \in \mathbb{N}^{N_r}$ and $x \in [(T_r)_s]$,
- if $x \in [(T_r)_{\mathbf{p}}]$, then $\mathbf{p}^\frown x \in X_q$ if and only if

$$\forall s \in N_r \ \left(\exists g^s_x \in G\ d(g^s_x, g_{q(s)}) < \frac{1}{2^{r+2}} \ \wedge \ g^s_x \cdot \varphi_r(\mathbf{p}^\frown x) = \varphi_r(s^\frown x)\right) \ \wedge$$

$$\wedge\ |d(g_{q(s)}, 1_G) - \mathbf{d}_{\varphi_r}(s^\frown x, \mathbf{p}^\frown x)| < \frac{1}{2^{r+2}}.$$

It is easy to see that the first line in the second item defines $\mathbf{\Sigma}^1_1$ set and it follows from Proposition 4.2 that the second line defines Borel set. Altogether, X_q is $\mathbf{\Sigma}^1_1$ subset of $[T_r]$ for every $q \in \mathbb{N}^{N_r}$ and $[T_r] = \bigcup_{q \in \mathbb{N}^{N_r}} X_q$.

Lemma 8.2 produces $(A_r, \alpha_r) \in [\mathbb{N}]^{\mathbb{N}} \times [T_r]$. Define $T_{r+1} = S_{(A_r,\alpha_r)}$, $\varphi_{r+1} = \varphi_r \circ \widetilde{e}_{(A_r,\alpha_r)}$,

$$\mathcal{A}^{r+1} = \{|v| \in T_{r+1} : \exists n \in \mathcal{A}^r \ s^r_n = e_{(A_r,\alpha_r)}(v)\}$$

and $\{s^{r+1}_n\}_{n \in \mathcal{A}^{r+1}}$ be any enumeration of $e^{-1}_{(A_r,\alpha_r)}(\{s^r_n\}_{n\in\mathcal{A}^r})$ that satisfies $|s^{r+1}_n| = n$ for every $n \in \mathbb{N}$.

It is easy to see that (1) and (2) hold. Note that $\mathbf{p} = s^{r+1}_{r+1} \in T_{r+1}$ because by Lemma 8.2 (4) we have $\mathbf{p} \sqsubseteq e_{(A_r,\alpha_r)}(\mathbf{p}) = s^r_n$ for some $n \in \mathcal{A}^r$. This shows (3) and (5) follows from $p_r \sqsubseteq \mathbf{p}$. First part of item (4) follows from Lemma 8.2 (3). Second part follows from the inductive hypothesis and definition of $\{s^{r+1}_n\}_{n\in\mathbb{N}}$. Namely, for every $n \in \mathcal{A}^{r+1}$ there is $n' \in \mathcal{A}^r$ such that $e_{(A_r,\alpha_r)}(s^{r+1}_n) = s^r_{n'}$. Note that $n \leq n'$. Then we have that φ_r is a homomorphism from $\mathbb{G}^{T_r}_{s^r_{n'}}$ to $\mathcal{H}_{\mathbf{k},n'}$ and $\widetilde{e}_{(A_r,\alpha_r)}$ is a reduction from $\mathbb{G}^{T_{r+1}}_{s^{r+1}_n}$ to $\mathbb{G}^{T_r}_{s^r_{n'}}$. This shows that φ_{r+1} is a homomorphism from $\mathbb{G}^{T_{r+1}}_{s^{r+1}_n}$ to $\mathcal{H}_{\mathbf{k},n'} \subseteq \mathcal{H}_{\mathbf{k},n}$ because $n \leq n'$.

It remains to show (6). Recall that $\mathbf{p} = s^{r+1}_{r+1}$. It follows from Lemma 8.2 (2) that there is $q \in \mathbb{N}^{N_r}$ such that $\mathbf{p}^\frown[(T_{r+1})_{\mathbf{p}}] \subseteq \widetilde{e}^{-1}_{(A_r,\alpha_r)}(X_q)$. Let $s \in T_{r+1}$ and define $g^{s,r+1} = g_{q(s)} \in G$. Take any $c \in [(T_{r+1})_s]$. By the definition of $\widetilde{e}_{(A_r,\alpha_r)}$ we find $d \in [(T_r)_s] = [(T_r)_{\mathbf{p}}]$ such that

$$\widetilde{e}_{(A_r,\alpha_r)}(s^\frown c) = s^\frown d \ \& \ \widetilde{e}_{(A_r,\alpha_r)}(\mathbf{p}^\frown c) = \mathbf{p}^\frown d.$$

Since $\mathbf{p}^\frown d \in X_q$ we find $g^{s,r+1}_c = g^s_d \in G$ such that

$$d(g^{s,r+1}_c, g^{s,r+1}) = d(g^s_d, g_{q(s)}) < \frac{1}{2^{r+2}}$$

$$|d(g^{s,r+1}, 1_G) - \mathbf{d}_{\varphi_{r+1}}(s^\frown c, \mathbf{p}^\frown c)| = |d(g_{q(s)}, 1_G) - \mathbf{d}_{\varphi_r}(s^\frown d, \mathbf{p}^\frown d)| < \frac{1}{2^{r+2}}$$

$$g^{s,r+1}_c \cdot \varphi_{r+1}(\mathbf{p}^\frown c) = g^s_d \cdot \varphi_r \circ \widetilde{e}_{(A_r,\alpha_r)}(\mathbf{p}^\frown c) = \varphi_r \circ \widetilde{e}_{(A_r,\alpha_r)}(s^\frown c) = \varphi_{r+1}(s^\frown c)$$

by the definition of X_q. That shows (6) an the proof is finished.

Constructing ϕ. Lemma 8.1 gives a finitely uniformly branching tree T and a sequence of continuous maps $\left\{\widetilde{\psi}_{r,\infty} : [T] \to [T_r]\right\}_{r\in\mathbb{N}}$. Define $\phi = \varphi_r \circ \widetilde{\psi}_{r,\infty}$ for some, or equivalently (by Lemma 8.2 (3)) any, $r \in \mathbb{N}$. Note that ϕ is a continuous map and $\phi = \varphi \circ \zeta$ where $\zeta = \widetilde{\psi}_{0,\infty}$.

Define $\{s_r\}_{r\in\mathbb{N}} = \{s^r_r\}_{r\in\mathbb{N}}$. It follows from (1) and Lemma 8.1 (1) that $s^r_r = s_r \in T$ for every $r \in \mathbb{N}$ and $|s_r| = r$. By (4) and Lemma 8.1 (4) we have that φ is a homomorphism from $\mathbb{G}^T_{s_r}$ to $\mathcal{H}_{\mathbf{k},r}$ for every $r \in \mathbb{N}$. Let $s \in T$. Then there is $r \geq |s|$ such that $p_r = s$. It follows by (5) that $s = p_r \sqsubseteq s_{r+1} = s^{r+1}_{r+1}$ and consequently that $\{s_r\}_{r\in\mathbb{N}}$ is dense in T.

It remains to show that $\mathbf{d}_\varphi$ is uniform. Let $s \in T \cap \mathbb{N}_r$ and $x, y \in [T_s]$. It follows from Lemma 8.1 (2) that there is $c, d \in [(T_r)_s]$ such that

$$\widetilde{\psi}_{r,\infty}(t^\frown x) = t^\frown c \ \wedge \widetilde{\psi}_{r,\infty}(t^\frown y) = t^\frown d$$

whenever $t \in T \cap \mathbb{N}^r$. Let $t = s_r$ and $g^{s,r}, g_c^{s,r}, g_d^{s,r} \in G$ be as in (6). Then we have

$$|\mathbf{d}_\varphi(s_r{}^\frown x, s^\frown x) - \mathbf{d}_\varphi(s_r{}^\frown y, s^\frown y)| = |\mathbf{d}_{\varphi_r}(s_r^r{}^\frown c, s^\frown c) - \mathbf{d}_{\varphi_r}(s_r^r{}^\frown d, s^\frown d)| \leq$$

$$\leq |\mathbf{d}_{\varphi_r}(s_r^r{}^\frown c, s^\frown c) - d(g^{s,r}, 1_G)| + |d(g^{s,r}, 1_G) - \mathbf{d}_{\varphi_r}(s_r^r{}^\frown d, s^\frown d)| \leq \frac{1}{2^{r+1}}$$

and consequently

$$|\mathbf{d}_\varphi(t^\frown x, s^\frown x) - \mathbf{d}_\varphi(t^\frown y, s^\frown y)| \leq \frac{1}{2^r}$$

for any $t \in T \cap \mathbb{N}^r$.

Pick any $g, h \in G$ such that $g \cdot \varphi(s^\frown x) = \varphi(s^\frown y)$ and $h \cdot \varphi(s_r{}^\frown x) = \varphi(s_r{}^\frown y)$ if they exist. Then we have

$$(g_d^{s,r})^{-1} \cdot g \cdot g_c^{s,r} \cdot \varphi(s_r{}^\frown x) = (g_d^{s,r})^{-1} \cdot g \cdot g_c^{s,r} \cdot \varphi_r(s_r{}^\frown c) = \varphi_r(s_r{}^\frown d) = \varphi(s_r{}^\frown y)$$

$$g_d^{s,r} \cdot h \cdot (g_c^{s,r})^{-1} \cdot \varphi(s^\frown x) = g_d^{s,r} \cdot h \cdot (g_c^{s,r})^{-1} \cdot \varphi_r(s^\frown c) = \varphi_r(s^\frown d) = \varphi(s^\frown y)$$

by (6). The invariance of d gives

$$d((g_d^{s,r})^{-1} \cdot g \cdot g_c^{s,r}, 1_G) = d(g, g_d^{s,r} \cdot (g_c^{s,r})^{-1}) \leq d(g, 1_G) + d(g_d^{s,r}, g_c^{s,r}) \leq d(g, 1_G) + \frac{1}{2^{r+1}}$$

where the last inequality follows from

$$d(g_d^{s,r}, g_c^{s,r}) \leq d(g_d^{s,r}, g^{s,r}) + d(g^{s,r}, g_c^{s,r}).$$

Similarly

$$d(g_d^{s,r} \cdot h \cdot (g_c^{s,r})^{-1}, 1_G) \leq d(h, 1_G) + \frac{1}{2^{r+1}}.$$

This implies

$$|\mathbf{d}_\varphi(s^\frown x, s^\frown y) - \mathbf{d}_\varphi(s_r{}^\frown x, s_r{}^\frown y)| \leq \frac{1}{2^{r+1}}$$

and consequently

$$|\mathbf{d}_\varphi(s^\frown x, s^\frown y) - \mathbf{d}_\varphi(t^\frown x, t^\frown y)| \leq \frac{1}{2^r}$$

for any $t \in T \cap \mathbb{N}^r$. If such $g, h \in G$ do not exist, then we have

$$\mathbf{d}_\varphi(s^\frown x, s^\frown y) = \mathbf{d}_\varphi(t^\frown x, t^\frown y) = +\infty$$

and trivially

$$|\mathbf{d}_\varphi(s^\frown x, s^\frown y) - \mathbf{d}_\varphi(t^\frown x, t^\frown y)| \leq \frac{1}{2^r}.$$

This finishes the proof. $\square$

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