SMOOTHING OF BUMP FUNCTIONS

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Abstract. Let X be a separable Banach space with a Schauder basis, admitting a continuous bump which depends locally on finitely many coordinates. Then X admits also a C^∞-smooth bump which depends locally on finitely many coordinates.

1. Introduction

In the present paper we investigate the properties of separable Banach spaces admitting bump functions depending locally on finitely many coordinates (LFC). The first use of the LFC notion for a function was the construction of C^∞-smooth and LFC renorming of c_0, due to Kuiper, which appeared in [BF]. The LFC notion was explicitly introduced and investigated in the paper [PWZ] of Pechanec, Whitfield and Zizler. In their work the authors have proved that every Banach space admitting a LFC bump is saturated with copies of c_0, providing in some sense a converse to Kuiper's result. Not surprisingly, it turns out that the LFC notion is closely related to the class of polyhedral spaces, introduced by Klee [K] and thoroughly investigated by many authors (see [JL, Chapter 15] for results and references). Indeed, prior to [PWZ], Fonf [F1] has proved that every polyhedral Banach space is saturated with copies of c_0. Later, it was independently proved in [F2] and [Haj1] that every separable polyhedral Banach space admits an equivalent LFC norm. Using the last result Fonf's result is a corollary of [PWZ]. The notion of LFC has been exploited (at least implicitly) in a number of papers, in order to obtain very smooth bump functions, norms and partitions of unity on non-separable Banach spaces, see e.g. [To], [Ta], [DGZ1], [GPWZ], [GTWZ], [FZ], [Hay1], [Hay2], [Hay3], [S1], [S2], [Haj1], [Haj2], [Haj3], and the book [DGZ]. In fact, it seems to be the only general approach to these problems. The reason is simple; it is relatively easy to check the (higher) differentiability properties of functions of several variables, while for functions defined on a Banach space it is very hard.

For separable spaces, one of the main known results is that a separable Banach space is isomorphic to a polyhedral space if and only if it admits a LFC renorming (resp. C^∞-smooth and LFC renorming) ([Haj1]). This smoothing up result is however obtained by using the boundary of a Banach space, rather than through some direct smoothing procedure. There is a variety of open questions, well known among the workers in the area, concerning the existence and possible smoothing of general non-convex LFC functions. In our note we are going to address the following one: Suppose a Banach space X admits a LFC bump. Does X admit a C^∞-smooth bump (norm)?

To this end, we develop some basic theory of LFC functions on separable Banach spaces.

The main result of this paper is that a separable Banach space with a Schauder basis has a C^∞-smooth and LFC bump function whenever it has a continuous LFC bump. This seems to be the first relatively general result in this direction. We establish some additional properties of such bumps, with an eye on the future developments.

We refer to [FHHMPZ], [LT] and [JL] for background material and results.

2. Preliminaries

We use a standard Banach space notation. If $\{e_i\}$ is a Schauder basis of a Banach space, we denote by $\{e_i^*\}$ its biorthogonal functionals. P_n are the canonical projections associated with the basis $\{e_i\}$, P_n^* are the operators adjoint to P_n, i.e. the canonical projections associated with the basis $\{e_i^*\}$. $U(x,\delta)$ denotes an open ball centered at x with radius δ. By $X^\#$ we denote an algebraic dual to a vector space X.

The notion of a function, defined on a Banach space with a Schauder basis, which is locally dependent on finitely many coordinates was introduced in [PWZ]. The following definition is a slight generalisation which was used by many authors.

Definition 1. *Let X be a topological vector space, $\Omega \subset X$ an open subset, E be an arbitrary set, $M \subset X^\#$ and $g\colon \Omega \to E$. We say that g* depends only on M *on a set $U \subset \Omega$ if $g(x) = g(y)$ whenever $x, y \in U$ are such that*

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$f(x) = f(y)$ for all $f \in M$. We say that g depends locally on finitely many coordinates from *M (LFC-M for short) if for each $x \in \Omega$ there are a neighbourhood $U \subset \Omega$ of x and a finite subset $F \subset M$ such that g depends only on F on U. We say that g* depends locally on finitely many coordinates *(LFC for short) if it is LFC-X^*.*

We may equivalently say that g depends only on $\{f_1, \dots, f_n\} \subset X^\#$ on $U \subset \Omega$ if there exist a mapping $G\colon \mathbb{R}^n \to E$ such that $g(x) = G(f_1(x), \dots, f_n(x))$ for all $x \in U$. Notice, that if $g\colon \Omega \to E$ is LFC and $h\colon E \to F$ is any mapping, then also $h \circ g$ is LFC.

The canonical example of a non-trivial LFC function is the sup norm on c_0, which is LFC-$\{e_i^*\}$ away from the origin. Indeed, take any $x = (x_i) \in c_0$, $x \neq 0$. Let $n \in \mathbb{N}$ be such that $|x_i| < \|x\|_\infty /2$ for $i > n$. Then $\|\cdot\|_\infty$ depends only on $\{e_1^*, \dots, e_n^*\}$ on $U(x, \|x\|_\infty /4)$.

A norm on a normed space is said to be LFC, if it is LFC away from the origin. Recall that a bump function (or bump) on a topological vector space X is a function $b\colon X \to \mathbb{R}$ with a bounded non-empty support.

The following theorem from [J] (see also [FZ]) shows that an existence of a LFC bump has a deep impact on the structure of the space.

Theorem 2. *Let X be a Banach space, $M \subset X^*$ and X admits an arbitrary LFC-M bump function. Then $\overline{\operatorname{span}}\, M = X^*$.*

Let X be a Banach lattice. We say that a function $f\colon X \to \mathbb{R}$ is a lattice function if it satisfies either $f(x) \leq f(y)$ whenever $|x| \leq |y|$, or $f(x) \geq f(y)$ whenever $|x| \leq |y|$. Recall that a Banach space X with an unconditional basis $\{e_i\}$ has a natural lattice structure defined by $\sum a_i e_i \geq 0$ if and only if $a_i \geq 0$ for all $i \in \mathbb{N}$. The same holds for ℓ_∞.

The next (somewhat technical) lemmata (Lemma 3 and Lemma 5) will be useful later when dealing with lattice functions. The first one, the general formulation of which may seem out-of-place here, is taken from [J]. For the sake of completeness we include its proof.

If X is a topological vector space, let us recall, that a set-valued mapping $\psi\colon X \to 2^X$ is called a cusco mapping if for each $x \in X$, $\psi(x)$ is a non-empty compact convex subset of X and for each open set U in X, $\{x \in X; \psi(x) \subset U\}$ is open.

Lemma 3. *Let X be a locally convex space, E be an arbitrary set and $g\colon X \to E$ be a LFC-M mapping for some $M \subset X^\#$. Further, let $\psi\colon X \to 2^X$ be a cusco mapping with the following property: For any finite $F \subset M$, if $x, y \in X$ are such that $f(x) = f(y)$ for all $f \in F$, then for each $w \in \psi(x)$ there is $z \in \psi(y)$ such that $f(w) = f(z)$ for all $f \in F$. Then the mapping $G\colon X \to 2^E$, $G(x) = g(\psi(x))$, is LFC-M.*

For the proof we first need to know when it is possible to join together some of the neighbourhoods in the definition of LFC:

Lemma 4. *Let X be a topological vector space, E be an arbitrary set, $g\colon X \to E$ and $M \subset X^\#$. Let $U_\alpha \subset X$, $\alpha \in I$ be open sets such that $U = \bigcup_{\alpha \in I} U_\alpha$ is convex and g depends only on M on each U_α, $\alpha \in I$. Then g depends only on M on the whole of U.*

Proof. Pick any $x, y \in U$ such that $f(x) = f(y)$ for all $f \in M$. Since U is convex, the line segment $[x, y] \subset U$. Since $[x, y]$ is compact, there is a finite covering $U_1, \dots, U_n \in \{U_\alpha\}_{\alpha \in I}$ of $[x, y]$. Since $[x, y]$ is connected, without loss of generality we may assume that $x \in U_1$, $y \in U_n$ and there are $x_i \in U_i \cap U_{i+1} \cap [x, y]$ for $i = 1, \dots, n-1$. As $x_i \in [x, y]$, we have $f(x) = f(y) = f(x_i)$ for all $f \in M$ and $i = 1, \dots, n-1$. Therefore $g(x) = g(x_1) = \dots = g(x_{n-1}) = g(y)$. □

Proof of Lemma 3. Let $x_0 \in X$. We can find a finite covering of the compact $\psi(x_0)$ by open sets U_i, $i = 1, \dots, n$, so that g depends only on a finite set $F_i \subset M$ on U_i. Let W be a convex neighbourhood of zero such that $\psi(x_0) + W \subset \bigcup U_i$ and put $U = \psi(x_0) + W$ and $F = \bigcup F_i$. As U is convex and $U \subset \bigcup U_i$, by Lemma 4, g depends only on F on U.

Suppose $V \subset X$ is a neighbourhood of x_0 such that $\psi(V) \subset U$. Let $x, y \in V$ are such that $f(x) = f(y)$ for all $f \in F$. Choose $w' \in G(x)$ and find $w \in \psi(x)$ for which $g(w) = w'$. Then, by the assumption on ψ, there is $z \in \psi(y)$ such that $f(w) = f(z)$ for all $f \in F$. But we have also $w \in \psi(x) \subset U$ and $z \in \psi(y) \subset U$ and hence $g(w) = g(z)$. Therefore $w' \in G(y)$ and by the symmetry we can conclude that $G(x) = G(y)$. □

Lemma 5. *Let $f\colon \mathbb{R} \to \mathbb{R}$ be an even function that is non-decreasing on $[0, \infty)$ and let $\varphi\colon \mathbb{R} \to \mathbb{R}$ be an even function with bounded support that is non-increasing on $[0, \infty)$. Then $(f * \varphi)(x) = \int_\mathbb{R} f(x-t)\varphi(t)\, dt$ is an even function that is non-decreasing on $[0, \infty)$.*

Proof. Note that $f * \varphi$ is well defined as f and φ are bounded on bounded sets.

Obviously, $(f * \varphi)(-x) = \int_\mathbb{R} f(-x-t)\varphi(t)\, dt = \int_\mathbb{R} f(x+t)\varphi(t)\, dt = \int_\mathbb{R} f(x-t)\varphi(t)\, dt = (f * \varphi)(x)$, using first the fact that f is even, then the fact that φ is even.

Now pick any $0 \le x < y < \infty$. The function $\psi(t) = \varphi(\frac{y-x}{2} - t) - \varphi(\frac{x-y}{2} - t)$ is an odd function (this is obvious), such that $\psi(t) \ge 0$ for $t \ge 0$. Indeed, either we have $0 \ge \frac{y-x}{2} - t \ge \frac{x-y}{2} - t$, or $0 < \frac{y-x}{2} - t \le t - \frac{x-y}{2}$ and in both cases we use the properties of φ. Similarly we get that the function $t \mapsto f\left(\frac{x+y}{2} + t\right) - f\left(\frac{x+y}{2} - t\right)$ is non-negative for $t \ge 0$. Therefore,

$$\begin{aligned}(f * \varphi)(y) - (f * \varphi)(x) &= \int_{\mathbb{R}} f(t)(\varphi(y-t) - \varphi(x-t))\,dt = \int_{\mathbb{R}} f\left(\tfrac{x+y}{2} + t\right)\psi(t)\,dt \\ &= \int_{(-\infty,0)} f\left(\tfrac{x+y}{2} + t\right)\psi(t)\,dt + \int_{(0,\infty)} f\left(\tfrac{x+y}{2} + t\right)\psi(t)\,dt \\ &= -\int_{(0,\infty)} f\left(\tfrac{x+y}{2} - t\right)\psi(t)\,dt + \int_{(0,\infty)} f\left(\tfrac{x+y}{2} + t\right)\psi(t)\,dt \\ &= \int_{(0,\infty)} \left(f\left(\tfrac{x+y}{2} + t\right) - f\left(\tfrac{x+y}{2} - t\right)\right)\psi(t)\,dt \ge 0.\end{aligned}$$

□

3. Spaces with Schauder Bases

The word "coordinate" in the term LFC originates of course from spaces with bases, where LFC was first defined using the coordinate functionals. In order to apply the LFC techniques to spaces without a Schauder basis, the notion had to be obviously generalised using arbitrary functionals from the dual. However, as we will show in this section, the generalisation does not substantially increase the supply of LFC functions on Banach spaces with a Schauder basis, and we can always in addition assume that the given LFC function in fact depends on the coordinate functionals. This fact is not only interesting in itself; it is the main tool for smoothing up LFC bumps on separable spaces with basis.

We begin with a simple related result for Markushevich bases:

Theorem 6. *Let E be a set, X be a separable Banach space and $g\colon X \to E$ be a LFC mapping. Then there is a Markushevich basis $\{x_i, x_i^*\} \subset X \times X^*$ such that g is LFC-$\{x_i^*\}$.*

Proof. By the Lindelöf property of X we can choose a countable $\{f_i\} \subset X^*$ such that g is LFC-$\{f_i\}$. Find a countable $\{g_i\} \subset X^*$ such that it separates points of X and $\{f_i\} \subset \{g_j\}$. Then we can use the Markushevich theorem (see e.g. [FHHMPZ]) to construct a Markushevich basis $\{x_i, x_i^*\}$ such that $\operatorname{span}\{x_i^*\} = \operatorname{span}\{g_i\} \supset \operatorname{span}\{f_i\}$.

Now let $x \in X$ and $U \subset X$ be a neighbourhood of x such that g depends only on $M = \{f_1, \dots, f_n\}$ on U. Let $M \subset \operatorname{span}\{x_1^*, \dots, x_m^*\}$. Then for any $y, z \in U$ such that $x_j^*(y) = x_j^*(z)$ for all $j = 1, \dots, m$ we have also $f_i(y) = f_i(z)$ for any $i = 1, \dots, n$ and hence $g(y) = g(z)$. Thus g depends only on $\{x_1^*, \dots, x_m^*\}$ on U.

□

We would like to establish a similar result for Schauder bases. In this context, shrinking Schauder bases emerge quite naturally, taking into account Theorem 2 (see also Theorem 12). We will use the following simple fact:

Fact 7. *Let X and Y be Banach spaces with equivalent Schauder bases $\{x_i\}$ and $\{y_i\}$ respectively. Then $\{x_i\}$ is shrinking if and only if $\{y_i\}$ is shrinking.*

Proof. Let $\{x_i\}$ be a shrinking basis and $T\colon Y \to X$ be an isomorphism of Y onto X such that $Ty_i = x_i$. Then $T^*\colon X^* \to Y^*$ is an isomorphism of X^* onto Y^* such that $T^* x_i^* = y_i^*$ and thus

$$Y^* = T^*(X^*) = T^*(\overline{\operatorname{span}}\{x_i^*\}) \subset \overline{T^*(\operatorname{span}\{x_i^*\})} = \overline{\operatorname{span}}\, T^*(\{x_i^*\}) = \overline{\operatorname{span}}\{y_i^*\}.$$

□

The next result is the main tool used in the sequel for the study of functions locally dependent on finitely many coordinates on spaces with shrinking Schauder bases.

Lemma 8. *Let X be a Banach space with a shrinking Schauder basis $\{e_i\}$. Let $f \in X^*$, $\varepsilon > 0$ and $n \in \mathbb{N}$. Then there is a (shrinking) Schauder basis $\{x_i\}$ of X and $N \in \mathbb{N}$, $N > n$, such that $x_i = e_i$ for $1 \le i < N$, $\{x_i\}$ is $(1+\varepsilon)$-equivalent to $\{e_i\}$, $\operatorname{span}\{x_i\}_{i=k}^m = \operatorname{span}\{e_i\}_{i=k}^m$ for all $1 \le k \le n$ and $m \ge k$, $x_i^* = e_i^*$ if $i < n$ or $i \ge N$, and $\overline{\operatorname{span}}\{x_i;\ i \ge N\} \subset \ker f$.*

Proof. Without loss of generality we may assume that there is a $z \in \operatorname{span}\{e_i;\ i \geq n\}$ for which $f(z) = 1$. Let us denote $f_k = f - P_{k-1}^* f$. As $\{e_i\}$ is shrinking, $\|f_k\| \to 0$ and hence we can find $N \in \mathbb{N}$ such that $N > \max\operatorname{supp} z \geq n$ and $\|f_N\| \leq \frac{\varepsilon}{(2+\varepsilon)\|z\|}$. Put $x_i = e_i$ for $1 \leq i < N$ and $x_i = e_i - f(e_i)z$ for $i \geq N$. For any m_1,$m_2 \in \mathbb{N}$ and any sequence $\{a_i\}$ of scalars we have

$$\left\|\sum_{i=m_1}^{m_2} a_i x_i\right\| = \left\|\sum_{i=m_1}^{m_2} a_i e_i - z \sum_{i=\max\{m_1,N\}}^{m_2} a_i f(e_i)\right\| \leq \left\|\sum_{i=m_1}^{m_2} a_i e_i\right\| + \left\|z f_N\left(\sum_{i=m_1}^{m_2} a_i e_i\right)\right\|$$
$$\leq \big(1 + \|z\|\,\|f_N\|\big)\left\|\sum_{i=m_1}^{m_2} a_i e_i\right\| \leq \left(1 + \frac{\varepsilon}{2+\varepsilon}\right)\left\|\sum_{i=m_1}^{m_2} a_i e_i\right\|$$

and

$$\left\|\sum_{i=m_1}^{m_2} a_i x_i\right\| \geq \left\|\sum_{i=m_1}^{m_2} a_i e_i\right\| - \left\|z f_N\left(\sum_{i=m_1}^{m_2} a_i e_i\right)\right\| \geq \left(1 - \frac{\varepsilon}{2+\varepsilon}\right)\left\|\sum_{i=m_1}^{m_2} a_i e_i\right\|.$$

This implies that $\{x_i\}$ is a basic sequence $(1+\varepsilon)$-equivalent to $\{e_i\}$. Since $z \in \operatorname{span}\{x_i;\ n \leq i < N\}$, we have $\operatorname{span}\{x_i\}_{i=k}^m = \operatorname{span}\{e_i\}_{i=k}^m$ for all $1 \leq k \leq n$ and $m \geq k$, and therefore $\operatorname{span}\{x_i\} = \operatorname{span}\{e_i\}$, which implies that $\{x_i\}$ is a basis of X. Moreover, $x_i^*(x) = \sum e_j^*(x) x_i^*(e_j) = \sum_{j<N} e_j^*(x) x_i^*(x_j) + \sum_{j\geq N} e_j^*(x) x_i^*(x_j + f(e_j)z) = \sum e_j^*(x) x_i^*(x_j) + x_i^*(z)\sum_{j\geq N} e_j^*(x) f(e_j) = e_i^*(x)$ if $i < n$ or $i \geq N$. Finally, $f(x_i) = 0$ for $i \geq N$.

□

It is perhaps worth noticing that the method used in the previous lemma (and the next theorem) does not rely on the classical argument of perturbation by the norm-summable sequence. In fact our new basis is "far" away from the original one.

Theorem 9. *Let X be a Banach space with a shrinking Schauder basis $\{e_i\}$, let $\{f_i\} \subset X^*$ be a countable subset and $\varepsilon > 0$. Then there is a (shrinking) Schauder basis $\{x_i\}$ of X such that it is $(1+\varepsilon)$-equivalent to $\{e_i\}$, $\operatorname{span}\{x_i\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$ and $\operatorname{span}\{f_i\} \subset \operatorname{span}\{x_i^*\}$.*

Proof. Choose a sequence of $\varepsilon_i > 0$ such that $\prod_i(1+\varepsilon_i) \leq (1+\varepsilon)$ and put $N_0 = 1$. We apply Lemma 8 to $\{e_i\}$, f_1, ε_1 and $n = 1$. We obtain a basis $\{x_i^1\}$ which is $(1+\varepsilon_1)$-equivalent to $\{e_i\}$ and $N_1 \in \mathbb{N}$ such that $\overline{\operatorname{span}}\{x_i^1;\ i \geq N_1\} \subset \ker f_1$. Moreover, $x_i^1 = e_i$ for $i < N_1$ and $\operatorname{span}\{x_i^1\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$.

We proceed by induction. Suppose the basis $\{x_i^k\}$ and $N_k \in \mathbb{N}$ have already been defined so that $\{x_i^k\}$ is $\prod_{j\leq k}(1+\varepsilon_j)$-equivalent to $\{e_i\}$, $x_i^k = x_i^{k-1}$ for $i < N_k$, $\operatorname{span}\{x_i^k\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$ and $\overline{\operatorname{span}}\{x_i^k;\ i \geq N_j\} \subset \ker f_j$ for $1 \leq j \leq k$. We apply Lemma 8 to $\{x_i^k\}$, f_{k+1}, ε_{k+1} and $n = N_k$ in order to obtain a basis $\{x_i^{k+1}\}$ which is $\prod_{j\leq k+1}(1+\varepsilon_j)$-equivalent to $\{e_i\}$ and $N_{k+1} \in \mathbb{N}$, $N_{k+1} > N_k$, such that $\overline{\operatorname{span}}\{x_i^{k+1};\ i \geq N_{k+1}\} \subset \ker f_{k+1}$. Moreover, $x_i^{k+1} = x_i^k$ for $i < N_{k+1}$ and $\operatorname{span}\{x_i^{k+1}\}_{i=1}^m = \operatorname{span}\{x_i^k\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$. Since also $\operatorname{span}\{x_i^{k+1}\}_{i=N_j}^m = \operatorname{span}\{x_i^k\}_{i=N_j}^m$ for all $1 \leq j \leq k$ and $m \geq N_j$, we have $\overline{\operatorname{span}}\{x_i^{k+1};\ i \geq N_j\} \subset \ker f_j$ for $1 \leq j \leq k+1$.

Clearly, there is a sequence $\{x_i\}$ such that $\lim_{j\to\infty} x_i^j = x_i$ for all $i \in \mathbb{N}$. (This is because the sequence N_k is increasing and thus x_i^j is eventually constant (in j).) It is straightforward to check that $\operatorname{span}\{x_i\}_{i=1}^m = \operatorname{span}\{e_i\}_{i=1}^m$ for all $m \in \mathbb{N}$, $\{x_i\}$ is a basis of X which is $(1+\varepsilon)$-equivalent to $\{e_i\}$ and $\overline{\operatorname{span}}\{x_i;\ i \geq N_j\} \subset \ker f_j$ (which means that $f_j \in \operatorname{span}\{x_i^*;\ i < N_j\}$) for any $j \in \mathbb{N}$.

□

If a Banach space X has a shrinking Schauder basis, using the Lindelöf property of X (as in the proof of Theorem 6) and Theorem 9 we obtain the following corollary, which allows us to work only with LFC-$\{e_i^*\}$ functions.

Corollary 10. *Let E be a set, X be a Banach space with a shrinking Schauder basis $\{e_i\}$, $g\colon X \to E$ be a LFC mapping and $\varepsilon > 0$. Then there is a (shrinking) Schauder basis $\{x_i\}$ of X, $(1+\varepsilon)$-equivalent to $\{e_i\}$, such that g is LFC-$\{x_i^*\}$.*

The following lemma seems to be the crucial reason why we need to work with Schauder bases.

Lemma 11. *Let X be a Banach space with a Schauder basis $\{e_i\}$ and E be an arbitrary set. Then $f\colon X \to E$ is LFC-$\{e_i^*\}$ if and only if for each $x \in X$ there is $\delta > 0$ and $n_0 \in \mathbb{N}$ such that $f(y) = f(P_n y)$ whenever $\|x - y\| < \delta$ and $n \geq n_0$.*

Proof. The "if" part is trivial: $P_{n_0} y = P_{n_0} z$ whenever $e_i^*(y) = e_i^*(z)$ for $1 \leq i \leq n_0$. Thus $f(y) = f(P_{n_0} y) = f(P_{n_0} z) = f(z)$ if moreover $y, z \in U(x,\delta)$, which means that f depends only on $\{e_1^*, \ldots, e_{n_0}^*\}$ on $U(x,\delta)$.

The "only if" part is also simple. Let K be a basis constant of $\{e_i\}$ and $x \in X$. There is $m \in \mathbb{N}$ and $\delta > 0$ such that $f(y) = f(z)$ if $y, z \in U(x, \delta(1+K))$ and $e_i^*(y) = e_i^*(z)$ for $1 \le i \le m$. Choose $n_0 \ge m$ such that $\|x - P_n x\| < \delta$ for all $n \ge n_0$. Then for any $n \ge n_0$ and $y \in X$ such that $\|x - y\| < \delta$ we have $\|P_n y - x\| \le \|P_n y - P_n x\| + \|P_n x - x\| < \delta(1+K)$ and therefore $f(y) = f(P_n y)$.

□

4. Main results

Theorem 12. *Let X be a Banach space with a Schauder basis $\{e_i\}$. The following statements are equivalent:*

(i) $\{e_i\}$ is shrinking and X admits a continuous LFC bump.
(ii) X admits a continuous LFC-$\{e_i^*\}$ bump.
(iii) X admits a C^∞-smooth LFC-$\{e_i^*\}$ bump.

For the proof of Theorem 12 we will need the following lemma, the basic idea of which is implicitly contained in [Haj1]. Let $\Delta = \{\delta_n\}_{n=1}^\infty$ be a sequence of positive real numbers. We denote by A_Δ an open subset of ℓ_∞ such that $x \in A_\Delta$ if and only if there is $n_x \in \mathbb{N}$ satisfying $|x(n_x)| - \delta_{n_x} > \sup_{n>n_x} |x(n)| + \delta_{n_x}$. For any $x \in A_\Delta$, the set $V_{n_x}^\Delta = \{y \in \ell_\infty : |y(n_x)| - \delta_{n_x} > \sup_{n>n_x} |y(n)| + \delta_{n_x}\} \subset A_\Delta$ is an open neighbourhood of x in ℓ_∞.

Lemma 13. *Let $\varepsilon > 0$ and a sequence $\Delta = \{\delta_n\}_{n=1}^\infty$, $\delta_n > 0$ be given. There is a convex lattice 1-Lipschitz function $F\colon \ell_\infty \to \mathbb{R}$ such that $\|x\|_\infty \le F(x) \le \|x\|_\infty + \varepsilon$ for any $x \in \ell_\infty$ and F is LFC-$\{e_i^*\}$ and C^∞ on A_Δ. Moreover, for any $x \in A_\Delta$, F depends only on $\{e_i^*\}_{i=1}^{n_x}$ on $V_{n_x}^\Delta$, where e_i^* are the coordinate functionals on ℓ_∞.*

Proof. Let $\varepsilon_1 = \min\{\delta_1, \varepsilon\}$ and $\varepsilon_n = \min\{\delta_n, \varepsilon_{n-1}\}$ for $n > 1$. Choose a sequence $\{\varphi_n\}_{n=1}^\infty$ of C^∞-smooth even functions $\varphi_n\colon \mathbb{R} \to [0,\infty)$ such that $\operatorname{supp}\varphi_n \subset [-\varepsilon_n, \varepsilon_n]$, φ_n is non-increasing on $[0,\infty)$ and $\int_\mathbb{R} \varphi_n(t)\,dt = 1$. Define a sequence $\{F_n\}_{n=0}^\infty$ of functions $F_n\colon \ell_\infty \to \mathbb{R}$ by the inductive formula

$$F_0(x) = \|x\|_\infty,$$
$$F_n(x) = \int_\mathbb{R} F_{n-1}(x + te_n)\varphi_n(t)\,dt.$$

It is easily checked that each F_n is convex, 1-Lipschitz and $F_n(x) - \|x\|_\infty \le \varepsilon$ for any $x \in \ell_\infty$. To see that F_n is lattice, pick $x, y \in \ell_\infty$, $x = (x_i)$, $y = (y_i)$, satisfying $|y| \le |x|$. Define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(u) = F_{n-1}(y + (u - y_n)e_n)$. Then

$$\begin{aligned}
F_n(x) - F_n(y) &= \int_\mathbb{R} \big(F_{n-1}(x+te_n) - F_{n-1}(y+te_n)\big)\varphi_n(t)\,dt \\
&= \int_\mathbb{R} \Big(F_{n-1}(x+te_n) - F_{n-1}\big(y + (x_n - y_n + t)e_n\big)\Big)\varphi_n(t)\,dt \\
&\quad + \int_\mathbb{R} \Big(F_{n-1}\big(y + (x_n - y_n + t)e_n\big) - F_{n-1}(y+te_n)\Big)\varphi_n(t)\,dt \\
&= \int_\mathbb{R} \Big(F_{n-1}(x+te_n) - F_{n-1}\big(y + (x_n - y_n + t)e_n\big)\Big)\varphi_n(t)\,dt + g*\varphi_n(x_n) - g*\varphi_n(y_n) \ge 0,
\end{aligned}$$

because $F_{n-1}(x+te_n) \ge F_{n-1}(y+(x_n-y_n+t)e_n)$ by the induction hypothesis (notice that $x+te_n = (x_1, \dots, x_{n-1}, x_n + t, x_{n+1}, \dots)$ and $y + (x_n - y_n + t)e_n = (y_1, \dots, y_{n-1}, x_n + t, y_{n+1}, \dots)$, thereby $|x + te_n| \ge |y + (x_n - y_n + t)e_n|$ in the lattice sense), g is an even function non-decreasing on $[0,\infty)$ also by the induction hypothesis and we may use Lemma 5.

Further, by Jensen's inequality,

$$F_n(x) = \int_\mathbb{R} F_{n-1}(x+te_n)\varphi_n(t)\,dt \ge F_{n-1}\left(x + e_n \int_\mathbb{R} t\varphi_n(t)\,dt\right) = F_{n-1}(x),$$

which means that the sequence $\{F_n\}$ is non-decreasing. Consequently the function $F = \lim_n F_n = \sup_n F_n$ is convex, lattice, 1-Lipschitz and $\|x\|_\infty \le F(x) \le \|x\|_\infty + \varepsilon$ for any $x \in \ell_\infty$.

For any $y \in \ell_\infty$ and $k \in \mathbb{N}$ we have

$$F_k(y) = \int_{-\varepsilon_k}^{\varepsilon_k} \dots \int_{-\varepsilon_1}^{\varepsilon_1} \|y + t_1 e_1 + \dots + t_k e_k\|_\infty \,\varphi_1(t_1)\cdots\varphi_k(t_k)\,dt_1 \dots dt_k.$$

Fix an arbitrary $x \in A_\Delta$ and pick any $y \in V_{n_x}^\Delta$ and $k > n_x$. Then

$$\|y + t_1e_1 + \cdots + t_ke_k\|_\infty = \|y + t_1e_1 + \cdots + t_{n_x}e_{n_x}\|_\infty = \|P_{n_x}y + t_1e_1 + \cdots + t_{n_x}e_{n_x}\|_\infty,$$

as long as $|t_i| \leq \delta_{n_x}$ for $n_x \leq i \leq k$. Since $\varepsilon_n \leq \delta_{n_x}$ for $n \geq n_x$ and $\int_{\mathbb{R}} \varphi_n = 1$, it follows that $F_k(y) = F_{n_x}(y) = F_{n_x}(P_{n_x}y)$. This means that $F(y) = F_{n_x}(P_{n_x}y)$ and therefore F is C^∞-smooth and depends only on $\{e_i^*\}_{i=1}^{n_x}$ on $V_{n_x}^\Delta$. □

Proof of Theorem 12. (iii)⇒(i) follows from Theorem 2.

(i)⇒(ii) follows from Corollary 10: If g is a continuous LFC bump on X, let $\{x_i\}$ be a basis obtained from Corollary 10 and T be an isomorphism $e_i \mapsto x_i$. Then the function $g \circ T$ is a continuous LFC-$\{e_i^*\}$ bump.

It remains to prove (ii)⇒(iii). Since X admits a continuous LFC-$\{e_i^*\}$ bump, using an affine transformation and a composition with a suitable function we can produce a continuous LFC-$\{e_i^*\}$ function $b\colon X \to [1,2]$ such that $b(0) = 1$ and $b(x) = 2$ whenever $\|x\| \geq 1$. Choose a sequence of real numbers $\{\eta_n\}$ decreasing to 1 such that $\eta_1 < 1 + \frac{1}{4}$ and a decreasing sequence $\Delta = \{\delta_n\}$ such that $0 < \delta_n < \frac{1}{4}(\eta_n - \eta_{n+1})$ and $\delta_1 < \frac{1}{8}$.

For a fixed $n \in \mathbb{N}$, let $T_n\colon \mathbb{R}^n \to P_nX$ be a canonical isomorphism, i.e. $T_n(t_1, \ldots, t_n) = t_1e_1 + \cdots + t_ne_n$. Since $b \circ T_n \in C(\mathbb{R}^n)$ and it is constant outside a sufficiently large ball in $\mathbb{R}^n$, using standard finite-dimensional smooth approximations we can find $\widetilde{b}_n \in C^\infty(\mathbb{R}^n)$ such that $\sup_{\mathbb{R}^n} |\widetilde{b}_n(y) - \eta_n b(T_ny)| < \delta_n$. We define $b_n(x) = \widetilde{b}_n(T_n^{-1}P_nx)$ and thus $b_n \in C^\infty(X)$ and $\sup_X |b_n(x) - \eta_n b(P_nx)| < \delta_n$.

Further, let us define $\Phi\colon X \to \ell_\infty$ by $\Phi(x)(n) = b_n(x)$. Pick any $x \in X$. By Lemma 11 there is $\delta > 0$ and $n_x \in \mathbb{N}$ such that $b(y) = b(P_ny)$ whenever $\|x - y\| < \delta$ and $n \geq n_x$. Thus for $n > m \geq n_x$ and $\|x - y\| < \delta$ we have

$$\begin{aligned} |\Phi(y)(m)| - \delta_m &= b_m(y) - \delta_m > \eta_m b(P_my) - 2\delta_m = \eta_m b(y) - 2\delta_m > \eta_{m+1}b(y) + 2\delta_m \\ &> \eta_n b(y) + \delta_n + \delta_m = \eta_n b(P_ny) + \delta_n + \delta_m > b_n(y) + \delta_m = |\Phi(y)(n)| + \delta_m. \end{aligned}$$

(The second inequality follows from the definition of δ_m.) It means that $|\Phi(y)(n_x)| - \delta_{n_x} > |\Phi(y)(n_x + 1)| + \delta_{n_x} = \sup_{n>n_x} |\Phi(y)(n)| + \delta_{n_x}$. As $x \in X$ is arbitrary, these inequalities show that $\Phi(X) \subset A_\Delta$ and moreover

$$\Phi(y) \in V_{n_x}^\Delta \quad \text{whenever } \|x - y\| < \delta. \tag{1}$$

We now apply Lemma 13 to the sequence Δ and $\varepsilon < \frac{1}{8}$ in order to obtain the corresponding function F, and we set $f = F \circ \Phi$. The properties of F together with (1) and the fact that b_n depends only on $\{e_i^*\}_{i=1}^n$ imply that f is a C^∞-smooth LFC-$\{e_i^*\}$ function.

Further,

$$f(0) = F(\Phi(0)) \leq \|\Phi(0)\|_\infty + \varepsilon = \sup_n b_n(0) + \varepsilon \leq \sup_n \big(\eta_n b(0) + \delta_n\big) + \varepsilon = \eta_1 + \delta_1 + \varepsilon < 1 + \frac{1}{2}.$$

On the other hand, if $\|x\| \geq 1$ we get

$$f(x) \geq \|\Phi(x)\|_\infty = \sup_n b_n(x) \geq b_{n_x}(x) > \eta_{n_x} b(P_{n_x}x) - \delta_{n_x} = \eta_{n_x} b(x) - \delta_{n_x} > 2 - \delta_1 > 2 - \frac{1}{8}.$$

Therefore f is a separating function on X and we obtain the desired bump by composing f with a suitable smooth real function. □

Theorem 14. *Let X be a Banach space with an unconditional Schauder basis $\{e_i\}$, which admits a continuous LFC bump. Then X admits a C^∞-smooth LFC-$\{e_i^*\}$ lattice bump.*

Proof. Since X is c_0-saturated ([PWZ]), it does not contain ℓ_1 and so by James's theorem $\{e_i\}$ is shrinking. By Theorem 12 there is a continuous LFC-$\{e_i^*\}$ bump b on X and without loss of generality we may assume $b\colon X \to [0,1]$ and $b(0) > 0$. We may further assume that the norm $\|\cdot\|$ on X is lattice.

First we show that there is a continuous lattice LFC-$\{e_i^*\}$ bump on X. Put $g(x) = \inf_{|y| \leq |x|} b(y)$. If $b(x) = 0$ then also $g(x) = 0$ and $g(0) = b(0) > 0$, hence g is a bump function. Further, for any $x, y \in X$ such that $|y| \leq |x|$ we have $g(y) = \inf_{|z| \leq |y|} b(z) \geq \inf_{|z| \leq |x|} b(z) = g(x)$, thus g is lattice.

For any $y \in X$ we denote $y(i) = e_i^*(y)$. Define a mapping $\psi\colon X \to 2^X$ by $\psi(y) = \{z \in X;\ |z| \leq |y|\}$. Clearly, $\psi(y)$ is a convex set for any $y \in X$. Furthermore, as $\{e_i\}$ is unconditional, $\psi(y)$ is a compact set for any $y \in X$ (consider the mapping from a compact space $\prod_i [-|y(i)|, |y(i)|]$ into X defined by $(t_1, t_2, \ldots) \mapsto \sum t_ie_i$).

Now fix an arbitrary $x \in X$. Let us define a projection $y \mapsto \tilde{y}$ from X onto $\psi(x)$: For any $y \in X$ we put $\tilde{y}(i) = y(i)$ if $|y(i)| \leq |x(i)|$, $\tilde{y}(i) = \operatorname{sgn} y(i)\,|x(i)|$ otherwise. Notice that $|\tilde{y}| \leq |x|$ and so indeed $\tilde{y} \in \psi(x)$. Let $z \in X$. Then $\|y - \tilde{y}\| \leq \|x - z\|$ for any $y \in \psi(z)$. Indeed, $|\tilde{y}(i) - y(i)| = \big|\operatorname{sgn} y(i)\,|x(i)| - y(i)\big| = \big||x(i)| - |y(i)|\big| = |y(i)| - |x(i)| \leq |z(i)| - |x(i)| \leq |z(i) - x(i)|$ whenever $|y(i)| > |x(i)|$. Thus $|\tilde{y} - y| \leq |x - z|$ and we use the fact that $\|\cdot\|$ is lattice.

Let U be a neighbourhood of $\psi(x)$ and $\delta = \operatorname{dist}(\psi(x), X \setminus U)$. Suppose $z \in X$, $\|x - z\| < \delta$. Then $\|y - \tilde{y}\| \leq \|x - z\| < \delta$ for any $y \in \psi(z)$ and hence $\psi(z) \subset U$. This implies that ψ is a cusco mapping.

Given any $\varepsilon > 0$ we can find a neighbourhood U of $\psi(x)$ and $0 < \delta < \operatorname{dist}(\psi(x), X \setminus U)$ such that $|b(y) - b(z)| < \varepsilon$ whenever $y, z \in U$, $\|y - z\| < \delta$. Suppose $z \in X$, $\|x - z\| < \delta$. Then, by the previous paragraph, $|b(\tilde{y}) - b(y)| < \varepsilon$. Therefore, $g(z) = \inf_{y\in\psi(z)} b(y) \geq \inf_{y\in\psi(z)} b(\tilde{y}) - \varepsilon \geq \inf_{y\in\psi(x)} b(y) - \varepsilon = g(x) - \varepsilon$. Similarly, considering a projection onto $\psi(z)$, we obtain $g(x) \geq g(z) - \varepsilon$. This shows that g is continuous.

Suppose that for some $F \subset \mathbb{N}$ we have $x(i) = y(i)$ for all $i \in F$ and let $w \in \psi(x)$. Define $z \in X$ such that $z(i) = w(i)$ for $i \in F$ and $z(i) = y(i)$ otherwise. Then $z \in \psi(y)$ and the assumption of Lemma 3 is satisfied. Hence g is LFC-$\{e_i^*\}$.

We note that the process described above does not preserve smoothness as can be easily seen on a one-dimensional example.

Finally, we smoothen up the bump g by repeating the proof of Theorem 12. Notice only that the finite-dimensional smooth approximations can be made lattice similarly as in the proof of Lemma 13, consequently $\Phi(\cdot)(n)$ is lattice for each $n \in \mathbb{N}$ and since F from Lemma 13 is lattice too, we can conclude that the resulting function $f = F \circ \Phi$ is lattice. □

References

[BF] R. Bonic and J. Frampton, *Smooth Functions on Banach Manifolds*, J. Math. Mech. **15** (1966), 877–898.

[DGZ] R. Deville, G. Godefroy and V. Zizler, *Smoothness and Renormings in Banach Spaces*, Monographs and Surveys in Pure and Applied Mathematics **64**, Pitman, 1993.

[DGZ1] R. Deville, G. Godefroy and V. Zizler, *The Three Space Problem for Smooth Partitions of Unity and $C(K)$ Spaces*, Math. Ann. **288** (1990), no. 4, 613–625.

[F1] V.P. Fonf, *Polyhedral Banach Spaces*, Math. Notes Acad. Sci. USSR **30** (1981), 809–813.

[F2] V.P. Fonf, *Three Characterizations of Polyhedral Banach Spaces*, Ukrainian Math. J. **42** (1990), no. 9, 1145–1148.

[FHHMPZ] M. Fabian, P. Habala, P. Hájek, V. Montesinos, J. Pelant and V. Zizler, *Functional Analysis and Infinite Dimensional Geometry*, CMS Books in Mathematics **8**, Springer-Verlag, 2001.

[FZ] M. Fabian and V. Zizler, *A Note on Bump Functions That Locally Depend on Finitely Many Coordinates*, Bull. Austral. Math. Soc. **56** (1997), no. 3, 447–451.

[GPWZ] G. Godefroy, J. Pelant, J.H.M. Whitfield and V. Zizler, *Banach Space Properties of Ciesielski-Pol's $C(K)$ Space*, Proc. Amer. Math. Soc. **103** (1988), no. 4, 1087–1093.

[GTWZ] G. Godefroy, S. Troyanski, J.H.M. Whitfield and V. Zizler, *Smoothness in Weakly Compactly Generated Banach Spaces*, J. Funct. Anal. **52** (1983), no. 3, 344–352.

[Haj1] P. Hájek, *Smooth Norms That Depend Locally on Finitely Many Coordinates*, Proc. Amer. Math. Soc. **123** (1995), no. 12, 3817–3821.

[Haj2] P. Hájek, *Smooth Norms on Certain $C(K)$ Spaces*, Proc. Amer. Math. Soc. **131** (2003), no. 7, 2049–2051.

[Haj3] P. Hájek, *Smooth Partitions of Unity on Certain $C(K)$ Spaces*, Mathematika **52** (2005), 131–138.

[Hay1] R.G. Haydon, *Normes infiniment différentiables sur certains espaces de Banach*, C. R. Acad. Sci. Paris Sér. I Math. **315** (1992), no. 11, 1175–1178.

[Hay2] R.G. Haydon, *Smooth Functions and Partitions of Unity on Certain Banach Spaces*, Quart. J. Math. Oxford Ser. **47** (1996), no. 188, 455–468.

[Hay3] R.G. Haydon, *Trees in Renorming Theory*, Proc. London Math. Soc. **78** (1999), no. 3, 541–584.

[J] M. Johanis, *Locally Flat Banach Spaces*, to appear in Czechoslovak Math. J.

[JL] W.B. Johnson and J. Lindenstrauss (Eds.), *Handbook of the Geometry of Banach Spaces, vol. 1*, Elsevier, 2001.

[K] V.L. Klee, *Polyhedral Sections of Convex Bodies*, Acta Math. **103** (1960), 243–267.

[LT] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I. Sequence Spaces*, Springer-Verlag, 1977.

[PWZ] J. Pechanec, J.H.M. Whitfield and V. Zizler, *Norms Locally Dependent on Finitely Many Coordinates*, An. Acad. Brasil. Ciênc. **53** (1981), no. 3, 415–417.

[S1] R.J. Smith, *Bounded Linear Talagrand Operators on Ordinal Spaces*, Q. J. Math. **56** (2005), 383–395.

[S2] R.J. Smith, *Trees, Gâteaux Norms and a Problem of Haydon*, Bull. Lond. Math. Soc. **39** (2007), 112–120.

[Ta] M. Talagrand, *Renormages de quelques $\mathcal{C}(K)$*, Israel J. Math. **54** (1986), no. 3, 327-334.

[To] H. Toruńczyk, *Smooth Partitions of Unity on Some Non-separable Banach Spaces*, Studia Math. **46** (1973), 43–51.

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