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MATHEMATICS OF THE HUBBLE LAW

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Abstract: A generalization of the Hubble law in the framework of the special theory of relativity is obtained. The relativistic Doppler effect allows us to find a nonlinear relationship between the redshift and distance. It is concluded that the Universe is expanding with an acceleration.

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1. About Hubble's law

We consider the law of Edwin Hubble (1929), independently discovered also by Georges Lemaître (1927), see [1, 2]. In standard terms, the law has the form:

$$v = H \cdot R, \quad 0 \le R \le R_* = \frac{c}{H},$$

where c is the speed of light in vacuum and H is the Hubble constant.

It is known that the Hubble law (or the Hubble–Lemaître law) is invariant under the Galilean transformations. Let us prove the opposite: the velocity distribution of galaxies, which is invariant with respect to the Galilean transformations or the Lorentz transformations, leads only to the Hubble law or to its generalization.

2. The Hubble law as a consequence of the invariance of the velocity distribution law with respect to the Galilean transformations

According to the cosmological principle, we choose a homogeneous and isotropic model of the Universe, and write the dependence of speed on distance by the formula $v = f(R)$. Let us consider three galaxies I, J, and K moving along one straight line. The distance between I and J is equal to x, the distance between J and K is equal to y, and the distance between I and K is equal to z. The galaxy J moves away from the galaxy I with speed U, the galaxy K moves away from the galaxy J with speed V, and the galaxy K moves away from the galaxy I with speed W. The following equalities hold:

$$U = f(x), \quad V = f(y), \quad W = f(z), \quad x + y = z, \quad U + V = W. \tag{1}$$

The last velocities-addition formula, given by Newton, is a consequence of the Galilean transformations. This formula is logically connected with the Corollary I from the Newton's laws of motion [3]. The five equalities (1) yield the functional equation:

$$f(x) + f(y) = f(x + y).$$

Augustin-Louis Cauchy found a solution to this equation in the class of continuous functions [4]:

$$f(x) = a \cdot x, \quad \text{where } a = \text{const}.$$

Equating this constant to the Hubble constant, we get the Hubble law, $v = H \cdot R$.

3. Relativistic generalization of the Hubble law as a consequence of the invariance of the velocity distribution law with respect to Lorentz transformations

We take the same three galaxies that move at high speeds and write five equalities for them:

$$U = f(x), \quad V = f(y), \quad W = f(z), \quad x + y = z, \quad \frac{U + V}{1 + \dfrac{U \cdot V}{c^2}} = W. \tag{2}$$

The last velocities-addition formula, given by Einstein in [5], is a consequence of the Lorentz transformations. The equalities (2) give a functional equation:

$$\frac{f(x) + f(y)}{1 + \dfrac{f(x) \cdot f(y)}{c^2}} = f(x + y).$$

The hyperbolic tangent gives the solution to this equation in the class of continuously differentiable functions:

$$f(x) = c \tanh \frac{a \cdot x}{c}, \quad \text{where } a = \text{const}.$$

Therefore, we get a generalization of the Hubble law:

$$v = c \tanh \frac{H \cdot R}{c}, \quad 0 \le R < \infty.$$

According to the correspondence principle, the generalization of the law goes over into the linear law for small values of R. Therefore, $a = H$. For large values of R, we obtain the asymptotic law $v = c$.

If $R = R_* = \dfrac{c}{H}$, then

$$v = \tanh(1) \cdot c \approx 0.76 \cdot c.$$

4. Redshift and distance to galaxies: Relativistic theory

In this section, z is the redshift, the remaining notation is standard. Einstein found the following formula for the relativistic Doppler effect, see [5]:

$$1 + z = 1 + \frac{\Delta\lambda}{\lambda} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}}.$$

The new formula for the Hubble law gives the equality:

$$1 + z = \sqrt{\frac{1 + \tanh \frac{H \cdot R}{c}}{1 - \tanh \frac{H \cdot R}{c}}}.$$

Taking into account the identity

$$\exp x = \sqrt{\frac{1 + \tanh x}{1 - \tanh x}},$$

we get a functional relationship between redshift and distance:

$$1 + z = \exp \frac{H \cdot R}{c} \quad \text{or} \quad R = \frac{c}{H} \ln(1 + z).$$

Therefore, we obtain the following correspondence of the quantities:

$$z = \mathrm{e} - 1 \approx 1.71828 \quad \Rightarrow \quad R = R_*,$$
$$z = 100 \quad \Rightarrow \quad R \approx 4.615 \cdot R_*,$$
$$z = 1\,000\,000 \quad \Rightarrow \quad R \approx 13.8 \cdot R_*.$$

For small values of R, we arrive at the well-known relation:

$$z = \frac{H \cdot R}{c} = \frac{v}{c}, \qquad 0 \le v \ll c.$$

Expanding the exponent in a power series, we obtain:

$$z = \frac{H \cdot R}{c} + \frac{1}{2}\left(\frac{H \cdot R}{c}\right)^2 + O\left(\left(\frac{R}{R_*}\right)\right)^3.$$

The last formula is close to the empirical formulas for the redshift.

5. Hubble's law and rapidity

Rapidity is determined by the formula:

$$\theta = c \cdot \operatorname{arctanh} \frac{v}{c}.$$

The Croatian physicist Varićak and the English mathematician and astronomer Whittaker proposed the variable θ, see [6, 7].

A generalization of the Hubble law for rapidity gives the formula:

$$\theta = H \cdot R.$$

In other words, the transition from the Hubble linear law to its generalization is associated with the replacement of speed by rapidity.

6. The universe expanding with acceleration

For the scale factor R, we consider the equation:

$$\dot{R} = c \cdot \tanh \frac{H \cdot R}{c}, \quad 0 \le R < \infty.$$

This allows us to find the deceleration parameter:

$$q = -\frac{\ddot{R}\,R}{(\dot{R})^2} = -\frac{\dfrac{2\,HR}{c}}{\sinh \dfrac{2\,HR}{c}}\,.$$

Therefore, we get the double inequality:

$$-1 < q < 0 \quad (0 < R < \infty),$$

which allows us to draw a conclusion about the accelerated expansion of the Universe. If $R = R_* = \dfrac{c}{H}$, then

$$q = -\frac{2}{\sinh(2)} \approx -0.55.$$

Next, we consider the function:

$$q = -\frac{2X}{\sinh(2X)}, \quad \text{where } X = \frac{R}{R_*}.$$

The inflection point of the function graph gives the following values:

$$X \approx 0.803, \quad q \approx -0.67.$$

Expanding the function in a power series, we obtain:

$$q = -1 + \frac{2}{3}\left(\frac{R}{R_*}\right)^2 + O\left(\left(\frac{R}{R_*}\right)^4\right).$$

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