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ON THE PROBLEM OF FLATNESS IN VARIOUS COSMOLOGICAL MODELS

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Abstract: The problem of flatness of the Universe (i.e., an almost zero curvature of its three-dimensional space) is one of the major methodological issues, determining the viability of various cosmological models. In fact, just the inability to resolve this problem was one of the reasons to reject the old, "pre-inflationary" cosmological model in the early 1980's and to replace it by the inflationary scenario. Here, we outline the corresponding mathematical formalism and pose a question if flatness of the Universe can be satisfactorily explained in the "uncertainty-mediated" inflationary model, which was suggested in our recent works [Y. V. Dumin. In: *Cosmology on Small Scales 2018*, p. 136; Grav. Cosmol. **25** (2019), 169]? As follows from the respective calculations, the curvature term in the Friedmann equation becomes insignificant in the course of time and, therefore, the Universe tends to be flat. From this point of view, the uncertainty-mediated cosmological model can serve as a reasonable alternative to the standard (exponential) inflationary scenario.

Keywords: flatness of the Universe, inflationary model, uncertainty relation

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1. Introduction

Since the observed three-dimensional space is almost perfectly flat, it is one of key problems of theoretical cosmology to explain how this flatness developed in the course of evolution of the Universe. In other words, does a particular cosmological model predict the flat spatial asymptotics for a sufficiently generic class of initial conditions?

A change in the cosmological paradigm in the early 1980's from the old model with power-like expansion, governed by the ordinary matter and radiation, to the inflationary scenario, governed by the vacuum energy or Λ-term [1, 2], was caused just by the inability of the first-mentioned model to resolve a few conceptual puzzles of the observed Universe. They were the causal connectivity between the remote subregions of space–time, a presence of the initial singularity and, in particular, a surprising flatness of the three-dimensional space. In the next sections, we shall discuss how the mathematical formalism can or cannot predict such flatness for the different cosmological models; and special attention will be paid to the uncertainty-mediated scenario, which was put forward in our recent publications [3, 4].

2. Dynamics of the scale factor in various cosmological models

The commonly-accepted paradigm of the modern cosmology both for the early and late Universe is based on the Robertson–Walker space–time metric:

$$\mathrm{d}s^2 = c^2 \mathrm{d}t^2 - R^2(t) \Big[\frac{\mathrm{d}r^2}{1 - kr^2} + r^2 (\mathrm{d}\theta^2 + \sin^2\theta \, \mathrm{d}\varphi^2) \Big] \, , \tag{1}$$

whose temporal dynamics is described by the Friedmann equation [5]:

$$H^2 \equiv \left(\frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3c^2} \, \rho_0 \left(\frac{R}{R_0} \right)^{-3(1+w)} + \frac{c^2}{3} \, \Lambda \, - \, kc^2 \frac{1}{R^2} \, . \tag{2}$$

Here, r, θ, and φ are the dimensionless spherical coordinates, the coefficient k equals 1, 0, and -1 for the closed, flat, and open 3D space, respectively; R is the scale factor of the Universe, H is the Hubble parameter, G is the gravitational constant, c is the speed of light in vacuum, and dot denotes a differentiation with respect to the time t.

The first term in the right-hand side of formula (2) describes a contribution of the ordinary matter, whose density is ρ_0 at the instant $t = 0$, when the scale factor equals R_0. The equation-of-state parameter w is equal to 0 for the dust-like (i.e., non-relativistic) substance and 1/3 for the radiation.

The second term represents a contribution of the vacuum energy or Λ-term, which is usually assumed to be independent of time. The Λ-term can be either introduced "by hand" as a new fundamental constant or derived from the underlying theory of elementary particles; the last-mentioned case being more typical for the models of the early Universe.

Finally, the third term in equation (2) describes contribution of the spatial curvature into the temporal dynamics of the scale factor.

According to observations, the curvature should be close to zero. So, one possible option is to set $k = 0$ *a priori*. However, this looks not so good from the methodological point of view. A much more attractive option would be to construct such cosmological model where the curvature term becomes insignificant (as compared to other terms) in the course of time, starting from a generic set of the initial conditions; so that the approximately flat three-dimensional space is formed "dynamically" in a self-consistent way. In the subsequent sections, we shall study if such behavior is possible in various models of the early Universe.

2.1. The old, "pre-inflationary" model

Although the Λ-term was introduced into the equations of General Relativity already during its development in the late 1910's, it was commonly believed till the early 1980's that it is absent in the real world. Then, the general Friedmann equation (2) is reduced to

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3c^2}\,\rho_0 \left(\frac{R}{R_0}\right)^{-3(1+w)} - \, kc^2 \frac{1}{R^2}\,. \tag{3}$$

If we assume initially that $k = 0$, then the resulting equation can be easily integrated and gives

$$\tilde{R}(t) = R_0 \left(\frac{t+T}{T}\right)^{\frac{2}{3(w+1)}}, \tag{4}$$

where T is the age of the Universe, which is expressed through the previously-used cosmological parameters as

$$T = \frac{2}{3(w{+}1)} \left(\frac{3\,c^2}{8\pi G \rho_0}\right)^{1/2}. \tag{5}$$

From here on, solutions obtained under the assumption of zero curvature, $k = 0$, will be marked by a tilde.

In fact, it is clear even without the exact temporal dependence of the scale factor that the curvature term in equation (3) will decrease slower than the first term in the course of expansion of the Universe. Really, the curvature term decays as $1/R^2$, while the first term as $1/R^{3(1+w)}$, i.e., $1/R^3$ in the case of non-relativistic matter and $1/R^4$ in the case of radiation. Therefore, the curvature term will inevitably dominate after some time, and the flat three-dimensional space cannot be formed dynamically.

2.2. The standard (exponential) inflationary scenario

The main assumption of the inflationary scenario is that the density of ordinary matter in the early Universe can be ignored, but the Λ-term is of primary importance. In other words, the general Friedmann equation (2) is reduced to

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{c^2}{3}\,\Lambda \, - \, kc^2 \frac{1}{R^2}\,. \tag{6}$$

If the curvature term is initially ignored ($k = 0$), the integration of (6) is trivial and results in

$$\tilde{R}(t) = R_0 \exp\left(\frac{c\sqrt{\Lambda}}{\sqrt{3}}\,t\right), \tag{7}$$

where $\Lambda = \text{const}$, and only the increasing solution was taken into account.

Again, it is clear even without the exact temporal dependence of the solution of the total equation (6) that role of the curvature term will become negligible in the course of time as compared to the first term. Really, the curvature term decays as $1/R^2$, while the first term remains constant. So, as distinct from Sec. 2.1, solution $R(t)$ should tend to $\tilde{R}(t)$, corresponding to the flat three-dimensional space, at the sufficiently large time. Just this fact is one of the main advantages of the inflationary models.

2.3. The uncertainty-mediated inflationary model

Since the physical origin of the Λ-term remains unclear and there should be two absolutely different Λ-terms in the early Universe and nowadays, it may be reasonable to consider the cosmological model with a decaying Λ-term. A particular version of such theory was presented in our recent works [3, 4]. It is based on the qualitative idea by A. Coe [6] that the vacuum energy might be estimated from the quantum-mechanical uncertainty relation between the time and energy. Really, as was shown by L. I. Mandelstam and I. E. Tamm [7], the energy–time uncertainty relation can be used not only in the context of measurements but also for estimating the long-term evolution of quantum systems; for more details, see review [8]. Then, Λ-term appearing in equation (6) should be replaced by the function:

$$\Lambda(t) = \frac{4\pi C_{\rm UR}}{c\, l_{\rm P}} \frac{1}{t} \,, \tag{8}$$

where $l_{\rm P}=\sqrt{G\hbar/c^3}$ is the Planck length, and $C_{\rm UR} \geq 1$ (abbreviation "UR" means the uncertainty relation). So, a substitution of (8) into (6) gives the modified Friedmann equation:

$$\left(\frac{\dot{R}}{R} \right)^2 = \frac{4\pi C_{\rm UR}}{3\tau} \frac{1}{t} - kc^2 \frac{1}{R^2} \,, \tag{9}$$

where $\tau = l_{\rm P}/c = \sqrt{G\hbar/c^5}$ is the Planck time.

Solution of this equation at zero spatial curvature ($k = 0$) is the "quasi-exponential" function:

$$\tilde{R}(t) = R_0 \exp\left(\sqrt{\frac{16\pi C_{\rm UR}}{3}} \sqrt{\frac{t}{\tau}} \right) \tag{10}$$

if we consider only the increasing branch, *i.e.*, the expanding Universe. Discussion of its physical properties important for cosmology can be found in our papers [3, 4, 9, 10].

Next, let us return to the general ($k \neq 0$) Friedmann equation (9). To estimate the relative contributions of the energetic (first) and curvature (second) terms in the right-hand side, we can substitute there the simplified solution (10). Since

$$\lim_{t \to \infty} \frac{1/t}{1/\tilde{R}^2(t)} = \lim_{t \to \infty} \frac{\tilde{R}^2(t)}{t} = \infty \tag{11}$$

(as can be easily proved applying the L'Hôpital's rule two times), one should conclude that the role of the curvature term will be negligible at large time.[1] In other words, the situation is the same as in the standard (exponential) inflationary scenario.

3. Conclusion

Employing the simple estimates, we have shown that the problem of flatness was a serious conceptual drawback of the old cosmological model, governed by the ordinary matter and radiation. On the other hand, the flat three-dimensional space can be formed dynamically in the standard (exponential) inflationary scenario and—which is the novel result—in the inflationary model based on the quantum-mechanical uncertainty relation.

Since it was already shown before that this model resolves also the problem of causality in the early Universe [9, 10] and avoids the cosmological singularity, it can serve as a reasonable alternative to other inflationary scenarios considered currently in the literature, e.g., reviews [11, 12].

References

[1] Guth, A. H.: The inflationary Universe: A possible solution to the horizon and flatness problems. Phys. Rev. D **23** (1981), 347.

[2] Linde, A. D.: The inflationary Universe. Rep. Prog. Phys. **47** (1984), 925.

[3] Dumin, Y. V.: Cosmological inflation from the quantum-mechanical uncertainty relation. In: M. Křížek, Y. Dumin (Eds.), *Cosmology on Small Scales 2018, Dark matter problem and selected controversies in cosmology.* Inst. of Math., Prague, 2018, p. 136.

[4] Dumin, Y. V.: A unified model of Dark Energy based on the Mandelstam–Tamm uncertainty relation. Grav. Cosmol. **25** (2019), 169.

[5] Olive K. A., Peacock, J. A.: Big-Bang cosmology. In: Patrignani, C., et al. (Particle Data Group), *Review of Particle Physics.* Chin. Phys. C **40** (2016), 100001, p. 355.

[6] Coe, A.: *Observable Universe topology and Heisenberg uncertainty principle.* Preprint viXra:1710.0143, 2017.

[7] Mandelstam, L. I., Tamm, I. E.: The energy–time uncertainty relation in nonrelativistic quantum mechanics. Izv. Akad. Nauk SSSR (Ser. Fiz.) **9** (1945), 122, in Russian.

[1] Strictly speaking, this is not a rigorous mathematical proof but rather the "plausible reasoning", because we used here solution of the simplified equation, $\tilde{R}(t)$. A more rigorous proof will be published elsewhere.

[8] Deffner, S., Campbell, S.: Quantum speed limits: from Heisenberg's uncertainty principle to optimal quantum control. J. Phys. A: Math. Theor. **50** (2017), 453001.

[9] Dumin, Y. V.: The problem of causality in the uncertainty-mediated inflationary model. In: M. Křížek, Y. Dumin (Eds.), *Cosmology on Small Scales 2020, Excessive extrapolations and selected controversies in cosmology.* Inst. of Math., Prague, 2020, this book.

[10] Dumin, Y. V.: Cosmological inflation based on the uncertainty-mediated Dark Energy. Grav. Cosmol. **26** (2020), 259.

[11] Martin, J., Ringeval, C., Vennin, V.: Encyclopædia inflationaris. Phys. Dark Universe **5–6** (2014), 75.

[12] Ellis J., Wands, D.: Inflation. In: Patrignani, C., et al. (Particle Data Group), *Review of Particle Physics.* Chin. Phys. C **40** (2016), 100001, p. 367.