Invariants of $\mathrm{codim} = 2$ contact submanifolds

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Overview

Subject: "**contact divisors**" = "codim $= 2$ contact submanifolds", $\Gamma \subset (M, \xi)$.

Goals for today:

1. Definitions and examples (braids in $\mathbb{R}^3$, links of $\mathbb{C}$ singularities).
2. Generalize classical invariants to general $\Gamma \subset \left(\mathbb{S}^{2n+1}, \xi_{std}\right)$.
3. Structural results for the "new" invariant.
4. Applications: Novel behavior for $2n+1 \geq 5$, Milnor numbers.

Warning: Lots of definitions...

But techniques almost all "classical" from Milnor's books...

1. "Characteristic classes" (with Stasheff)
2. "Singular points of complex hypersurfaces"

Contact manifolds

Contact manifold: (M^{2n+1}, ξ) with $\xi^{2n} \subset TM^{2n+1}$ maximally non-integrable:

$$\xi = \ker \alpha, \quad \alpha \in \Omega^1, \quad \alpha \wedge (d\alpha)^n > 0 \implies d\alpha|_\xi \text{ symplectic.}$$

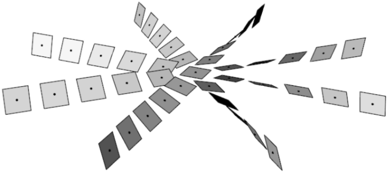
Ex: $n = 0$, $M = \sqcup_1^k \mathbb{S}^1$, $\xi = 0$.

Ex: "Standard $\mathbb{S}^{2n+1}$" is $(\mathbb{S}^{2n+1}, \xi_{std})$ where

$$\mathbb{S}^{2n+1} = \partial \mathbb{D}^{2n+2} \subset \mathbb{C}^{n+1}, \quad \xi_{std} = T\mathbb{S}^{2n+1} \cap JT\mathbb{S}^{2n+1} = \ker \left(\sum x_i dy_i - y_i dx_i \right).$$

Ex: "Standard $\mathbb{R}^3$" is $(\mathbb{R}^3, \xi_{std}) = (\mathbb{S}^3 \setminus pt, \xi_{std})$. Isomorphic to

$$\mathbb{R}^3 = \mathbb{R}_t \times \mathbb{C}, \quad \xi_{std} = \ker \left(dt + r^2 d\theta \right).$$



Divisors and examples in $(\mathbb{S}^3, \xi_{std})$

Contact divisor $\Gamma \subset (M, \xi)$: codim $= 2$, $\xi_\Gamma = T\Gamma \cap \xi$ contact on Γ.
Assume trivial normal bundle $\implies N(\Gamma) = \mathbb{D}^2 \times \Gamma, \quad \xi|_\Gamma = T\mathbb{D} \oplus \xi_\Gamma$.

$\Gamma \subset (M^3, \xi)$ is oriented link $\pitchfork^+ \xi$. Cf. [Etnyre]

$\Gamma \subset (\mathbb{R}^3_{t,r,\theta}, \ker(dt + r^2 d\theta))$ take braids around t-axis in $\mathbb{C}_{r,\theta}$ projection.

Proof $\Gamma \neq \Gamma'$: Take $\Sigma \subset \mathbb{S}^3$ Siefert surface $(\partial\Sigma = \Gamma)$.

Take $\mathfrak{s} : \Sigma \to \xi$ pointing into Γ along $\partial\Sigma$. Define

$$\ell(\Gamma) = \#\mathfrak{s}^{-1}(0) \in \mathbb{Z}$$

Depends only on contact isotopy class of Γ. For braids...

$$\ell(\Gamma) = \#(\text{strands}) - \#(+\text{ crossings}) + \#(-\text{ crossings}).$$

Now compute:

$$\ell(\Gamma) = 1, \quad \ell(\Gamma') = 3.$$

What just happened and where are we going?

We distinguished unknots $\Gamma, \Gamma' \subset (\mathbb{S}^3, \xi_{std})$ using invariant $\ell \in \mathbb{Z}$, showing

$$\ell(\Gamma') - \ell(\Gamma) = 2.$$

Today's topic: $\exists$ natural extension ℓ for general $\Gamma^{2n-1} \subset \left(\mathbb{S}^{2n+1}, \xi_{std}\right)$.

In higher dim, we will observe some weird behavior...

Ex: $\left(\mathbb{S}^3, \xi_{std}\right) \xrightarrow{\Gamma, \Gamma'} \left(\mathbb{S}^5, \xi_{std}\right), \quad \ell(\Gamma') - \ell(\Gamma) \in 24\mathbb{Z}.$

Modulus $(2, 24, ...)$ tends to grow quickly with $\dim \Gamma = 2n - 1$...

Ex: $\left(\mathbb{S}^{19}, \xi_{std}\right) \xrightarrow{\Gamma, \Gamma'} \left(\mathbb{S}^{21}, \xi_{std}\right), \quad \ell(\Gamma') - \ell(\Gamma) \in 95800320\mathbb{Z}.$

Both cases, $\ell(\Gamma') \neq \ell(\Gamma) \implies \Gamma, \Gamma' \subset \mathbb{S}^{2n+1}$ **distinct** as smooth divisors!

Formal contact manifolds (technical)

$\exists!$ contact structure on $\mathbb{S}^1$ ($\xi = 0$). For $\dim \Gamma > 1$ have to track ξ_Γ. Easier with...

Formal contact structure (FCS): "Homotopy version" of contact structure...

Subbundle $\xi_\Gamma^{2n-2} \subset T\Gamma$, $\omega \in \Omega^2(\Gamma)$ symplectic on ξ_Γ / homotopy
$\iff$ Subbundle $\xi_\Gamma^{2n-2} \subset T\Gamma$ with complex structure / homotopy.

Ex: FCS on $\mathbb{S}^3$ indexed by $\mathbb{Z}$.
But, there are $\infty + 1$ contact structures!

Ex: $\exists!$ FCS structure on $\mathbb{S}^5$.
But, there are ∞ contact structures! [Ustilovsky]

In general, FCS structure on $\mathbb{S}^{2n-1}$ classified by [Harris + Massey]

$$\pi_{2n-1} SO_{2n}/U_n = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & n = 0 \mod 4 \\ \mathbb{Z}/(n-1)!\mathbb{Z} & n = 1 \mod 4 \\ \mathbb{Z} & n = 2 \mod 4 \\ \mathbb{Z}/\frac{(n-1)!}{2}\mathbb{Z} & n = 3 \mod 4 \end{cases}$$

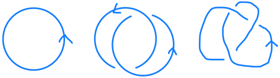
Going up in dimension by examples

Ex: Link of isolated singularity $f \in \mathbb{C}[z_0, \dots, z_n]$, $f(0) = 0$ is

$$\Gamma_f = \{f = 0\} \subset \left(\mathbb{S}^{2n+1}, \xi_{std}\right) \subset \mathbb{C}^n.$$

Complex tangencies of $T\Gamma_f$ give contact structure.

Ex: $f = z_0^p + z_1^q$ gives (p, q) torus link in $\mathbb{S}^3$. In $\mathbb{C}_{r,\theta}$ projection of $(\mathbb{R}^3, \xi_{std})$...



Ex: $f = z_0$ gives $\left(\mathbb{S}^3, \xi_{std}\right) \subset \left(\mathbb{S}^5, \xi_{std}\right)$.

Ex: $f = z_0^{2k+1} + \sum_1^3 z_i^2$ give $\mathbb{S}^5 \subset \mathbb{S}^7$. For k varying...
ξ_Γ are distinct contact structures but same FCS [Ustilovsky]

Ex: $f = z_0^{4k+1} + z_1^3 + \sum_2^5 z_i^2$ give Milnor's 7-spheres $\subset \mathbb{S}^9$.
Smooth structure determined by $k \bmod 28$ [cf. Hirzebruch survey].
All ξ_Γ distinct FCS [S. Morita].

"Exotic" contact divisors can also be built using surgeries [A. 2024]

Fact: Every $\Gamma \subset \mathbb{S}^{2n+1}$ has a **Siefert hypersurface**: $\Sigma^{2n} \subset \mathbb{S}^{2n+1}, \partial\Sigma = \Gamma$.

Section $\mathfrak{s} : \Sigma \to \xi$ pointing into Γ along $\partial\Sigma$. Define

$$\ell(\Gamma) = \#\mathfrak{s}^{-1}(0) \in \mathbb{Z}.$$

Same as for $\Gamma \subset (\mathbb{S}^3, \xi_{std})$, but not previously defined/studied :)

Proof of invariance:

1. Every $\Sigma = \theta^{-1}$(regular value) of some $\mathbb{S}^{2n+1} \setminus \Gamma \xrightarrow{\theta} \mathbb{S}^1$.
2. θ is angle on $(\Gamma \times \mathbb{D}^2_{r,\theta}) \setminus (\Gamma \times \{0\})$.
3. Homotopy θ_s gives cobordism $\Sigma \rightsquigarrow \Sigma'$, trivial near $\partial\Sigma = \partial\Sigma'$.
4. $\Sigma \rightsquigarrow \Sigma'$ and $\mathfrak{s} \rightsquigarrow \mathfrak{s}' \implies$ oriented 1-cobordism $\mathfrak{s}^{-1}(0) \rightsquigarrow (\mathfrak{s}')^{-1}(0)$.
5. Signed count of points is cobordism invariant of $\mathfrak{s}^{-1}(0)$. QED.

Note: (3) says that signature $\sigma(\Gamma) := \sigma(\Sigma)$ is invariant of $\Gamma^{4m-1} \subset \mathbb{S}^{4m+1}$.

Ex: ℓ for links of singularities

$f \in \mathbb{C}[z_0, \cdots z_n]$, $f(0) = 0$ isolated singularity. Recall $\Gamma_f = \{f = 0\} \cap \mathbb{S}^{2n+1}$.

$\mathbb{S}^{2n+1} \setminus \Gamma_f \xrightarrow{\theta = f/|f|} \mathbb{S}^1$ is a fibration called the **Milnor fibration**.
Each fiber gives natural Siefert Σ for Γ_f called the **Milnor fiber**.
Σ has homotopy type of $\vee^{\mu_f}\mathbb{S}^n$. $\mu_f = \dim H_n(\Sigma)$ is the **Milnor number**.

Ex: Brieskorn-Pham singularity, used to make all our previous examples...

$$f = \sum_0^n z_i^{a_i} \implies \mu_f = (a_0 - 1)\cdots(a_n - 1).$$

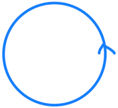
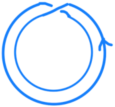
μ_f important to algebraic geometers and singularity theorists. Easily compute

$$\ell(\Gamma_f) = \chi(\Sigma) = 1 + (-1)^n \mu_f.$$

Intuition:

1. ℓ generalizes writhe for braids to higher dim.
2. ℓ is μ_f for general contact divisors.
3. Gives measurement of “contact topological complexity”.

Recall: Can modify a $\Gamma \subset (\mathbb{S}^3, \xi_{std})$ locally, keeping smooth knot type,

 $\Gamma \rightsquigarrow \Gamma'$ 

$$\ell(\Gamma') = \ell(\Gamma) + 2$$

Summary: In $\dim = 3$, $\exists$ lots of Γ which are:
- smoothly equivalent,
- distinct as contact divisors,
- "close" as measured by ℓ.

Rough Q: Does this phenomenon hold in higher dim?

First precise question...

Q: If $\Gamma, \Gamma' \subset \left(\mathbb{S}^{2n+1}, \xi_{std}\right)$ same FCS ξ_Γ, how big is $\ell(\Gamma') - \ell(\Gamma)$?

Structural properties: Definition of $\mathcal{C}^U, \mathcal{C}^{\mathbb{C}S}$

Q: If $\Gamma, \Gamma' \subset (\mathbb{S}^{2n+1}, \xi_{std})$ same FCS ξ_Γ, how big is $\ell(\Gamma') - \ell(\Gamma)$?

Ω^U_{2n} ($\Omega^{\mathbb{C}S}_{2n}$) dim $= 2n$ stable complex and (stable complex spin) manifolds / bordism.

$$\mathcal{C}^U_{2n}, \mathcal{C}^{\mathbb{C}S}_{2n} \in \mathbb{Z}$$

defined as $\gcd\langle c_n, X\rangle$ for $X \in \Omega^U_{2n}$ ($\Omega^{\mathbb{C}S}_{2n}$) with all other Chern numbers 0.

$2n$	$\mathcal{C}^U_{2n}$ bound	$\mathcal{C}^{\mathbb{C}S}_{2n}$ bound
2	$2 = \langle c_1, \mathbb{CP}^1\rangle$	$\sim$
4	$12 = \langle c_2, 9(\mathbb{CP}^1)^2 - 8\mathbb{CP}^2\rangle$	$24 = \langle c_2, K3\rangle$
6	$2 = \langle c_3, \mathbb{S}^6\rangle$	$\sim$
8	720	$\sim$
10	48	$\sim$
12	30240	60480
14	720	$\sim$
16	1209600	$\sim$
18	80640	$\sim$
20	47900160	95800320

I have theoretical bounds matching computer calculations of $\mathcal{C}^U_{2n}, \mathcal{C}^{\mathbb{C}S}_{2n}$ for small n...
I can't prove the bounds are exact for $2n \gg 0$. **Help!**

Structural properties: First main theorem

Q: If $\Gamma, \Gamma' \subset (\mathbb{S}^{2n+1}, \xi_{std})$ same FCS ξ_Γ, how big is $\ell(\Gamma') - \ell(\Gamma)$?

Theorem

$\Gamma, \Gamma' \subset (\mathbb{S}^{2n+1}, \xi_{std})$ same FCS ξ_Γ, then $\ell(\Gamma) - \ell(\Gamma') \in \mathcal{C}^{U}_{2n}\mathbb{Z}$.
If $H^1(\Gamma, \mathbb{Z}/2) = 0$, then $\ell(\Gamma) - \ell(\Gamma') \in \mathcal{C}^{\mathbb{C}S}_{2n}\mathbb{Z}$.

Ex: $\mathbb{S}^1 \xrightarrow{\Gamma, \Gamma'} (\mathbb{S}^3, \xi_{std}), \quad \ell(\Gamma') - \ell(\Gamma) \in 2\mathbb{Z}$.

Ex: $(\mathbb{S}^3, \xi_{std}) \xrightarrow{\Gamma, \Gamma'} (\mathbb{S}^5, \xi_{std}), \quad \ell(\Gamma') - \ell(\Gamma) \in 24\mathbb{Z}$.

Ex: $(\mathbb{S}^{19}, \xi_{std}) \xrightarrow{\Gamma, \Gamma'} (\mathbb{S}^{21}, \xi_{std}), \quad \ell(\Gamma') - \ell(\Gamma) \in 95800320\mathbb{Z}$.

Rough answer to Q: For n large, $\ell(\Gamma') - \ell(\Gamma)$ has to be zero or big.

Application: For singularities, $\ell(\Gamma_f) = 1 + (-1)^n \mu_f \implies$

Corollary

If Γ_f, Γ_g links of singularities f, g, same FCS then $\mu_f - \mu_g \in \mathcal{C}^{U}_{2n}\mathbb{Z}$.
In "many" $n = 2$ cases and all $n > 2$ cases, $\mu_f - \mu_g \in \mathcal{C}^{\mathbb{C}S}_{2n}\mathbb{Z}$.

First main theorem: Proof sketch

Q: If $\Gamma, \Gamma' \subset \left(\mathbb{S}^{2n+1}, \xi_{std}\right)$ same FCS ξ_Γ, how big is $\ell(\Gamma') - \ell(\Gamma)$?

Theorem

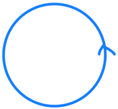
$\Gamma, \Gamma' \subset \left(\mathbb{S}^{2n+1}, \xi_{std}\right)$ same FCS ξ_Γ, then $\ell(\Gamma) - \ell(\Gamma') \in \mathcal{C}^U_{2n}\mathbb{Z}$.

If $H^1(\Gamma, \mathbb{Z}/2) = 0$, then $\ell(\Gamma) - \ell(\Gamma') \in \mathcal{C}^{\mathbb{C}S}_{2n}\mathbb{Z}$.

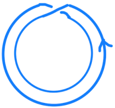
1. $\mathbb{R} \oplus T\mathbb{S}^{2n+1}|_\Sigma = \mathbb{R} \oplus \mathbb{R} \oplus \xi = \mathbb{C} \oplus \xi$. Σ for Γ is stable almost $\mathbb{C}$ manifold.
2. Glue Σ for Γ to $-\Sigma'$ for Γ' along boundaries $\rightsquigarrow \overline{\Sigma}$
3. Γ, Γ' FCS $\xi_\Gamma \implies \mathbb{R}^2 \oplus T\overline{\Sigma} \simeq \mathbb{C} \oplus \xi \implies \overline{\Sigma} \in \Omega^U_{2n}$.
4. Basic arithmetic $\implies c_I = c_{i_1} \cdots c_{i_k}, k \geq 2, \langle c_I, \overline{\Sigma} \rangle = 0$.
5. $\ell(\Gamma) - \ell(\Gamma') = \langle e(\xi), \overline{\Sigma} \rangle = \langle c_n, \overline{\Sigma} \rangle \in \mathcal{C}^U_{2n}\mathbb{Z}$. QED first statement.
6. If $H^1(\Gamma, \mathbb{Z}/2)$, Σ, Γ inherit spin structure inherited from $\mathbb{S}^{2n+1}$.
7. $\implies$ can glue spin structures $\implies \overline{\Sigma} \in \Omega^{\mathbb{C}S}_{2n} \implies \ell(\Gamma) - \ell(\Gamma') \in \mathcal{C}^U_{2n}\mathbb{Z}$.

Structural properties: Motivation

Recall: Can modify a $\Gamma \subset (\mathbb{S}^3, \xi_{std})$ locally, keeping smooth knot type,



$\Gamma \rightsquigarrow \Gamma'$



$\ell(\Gamma') = \ell(\Gamma) + 2$

Summary: In $\dim = 3$, $\exists$ lots of Γ which are:
- smoothly equivalent,
- distinct as contact divisors,
- "close" as measured by ℓ.

Rough Q: Does this phenomenon hold in higher dim?

Second precise question...

Q: Same FCS ξ_Γ, $\ell(\Gamma') \neq \ell(\Gamma)$. Can $\Gamma, \Gamma' \subset \mathbb{S}^{2n+1}$ be same **smoothly**?

Structural properties: Second main theorem

Q: Same FCS ξ_Γ, $\ell(\Gamma') \neq \ell(\Gamma)$. Can $\Gamma, \Gamma' \subset \mathbb{S}^{2n+1}$ be same **smoothly**?

Recall: Smooth $\Gamma^{4m-1} \subset \mathbb{S}^{4m+1}$, $\sigma(\Gamma) \in \mathbb{Z}$ is σ of Siefert hypersurface.

Theorem

For $\Gamma \in (\mathbb{S}^{4m+1}, \xi_{std})$, $\ell(\Gamma)$ depends only on FCS ξ_Γ and $\sigma(\Gamma)$. If Γ, Γ' same FCS ξ_Γ,

$$\ell(\Gamma) \neq \ell(\Gamma') \implies \sigma(\Gamma) \neq \sigma(\Gamma') \implies \textit{distinct } \textbf{\textit{smooth}} \textit{ divisors in } \mathbb{S}^{4m+1}.$$

Answer to Q: Theorem says **no** for $2n+1 = 4m+1$.
Answer is **yes** for $2n+1 = 4m-1$.... no time to explain today.

Second main theorem: Setup

Q: Same FCS ξ_Γ, $\ell(\Gamma') \neq \ell(\Gamma)$. Can $\Gamma, \Gamma' \subset \mathbb{S}^{2n+1}$ be same **smoothly**?

Theorem

For $\Gamma \in (\mathbb{S}^{4m+1}, \xi_{std})$, $\ell(\Gamma)$ depends only on almost complex structure ξ_Γ and the signature $\sigma(\Gamma)$ of the smooth divisor $\Gamma \subset \mathbb{S}^{4m+1}$. If Γ, Γ' same FCS ξ_Γ,

$$\ell(\Gamma) \neq \ell(\Gamma') \implies \sigma(\Gamma) \neq \sigma(\Gamma') \implies \text{distinct } \textbf{smooth} \text{ divisors in } \mathbb{S}^{2n+1}.$$

Proof uses new invariant of FCS on a Γ^{4m-1}: **Gompf-Morita invariant**

$$d_{4m-1}(\Gamma, \xi_\Gamma) \in \mathbb{Q}.$$

Generalizes Gompf's $d_3(\Gamma^3, \xi_\Gamma)$ and some computations of S. Morita.

(Γ, ξ_Γ) of dim $= 4m-1$. $c_k(\xi_\Gamma) = 0 \in H^{2k}(\Gamma, \mathbb{Q})\ \forall k > 0$.

Take Σ^{4m}, $\partial\Sigma = \Gamma$ and ξ^{4m} FCS on $\mathbb{R}_t \times \Sigma$. Looks like ξ along nbhd of Seifert

$$\xi = \mathbb{C}_{t,\tau} \oplus \xi_\Gamma \quad \text{along} \quad \mathbb{R}_t \times N(\partial\Sigma) = \mathbb{R}_t \times (-\epsilon, 0]_\tau \times \Gamma.$$

Consider L-genus $L_{4m}(\xi)$, from Hirzebruch's signature formula:

$$L_4 = \frac{1}{3}p_1, \quad L_8 = \frac{1}{45}(7p_2 - p_1^2), \quad L_{12} = \frac{1}{645}(62p_3 - 13p_1p_2 + 2p_1^3), \ldots$$

ξ complex, so can express top Pontryagin class using top Chern class...

$$p_m = 2(-1)^m c_{2m}(\xi) \stackrel{?}{=} 2(-1)^m e(\xi).$$

$e(\xi)$ ill defined since $\partial\Sigma \neq \emptyset$. Replace with relative version and define

$$p_m \rightsquigarrow 2(-1)^m e^{rel}(\xi, \partial_\tau), \quad L_{2m}(\xi) \rightsquigarrow L^{rel}_{4m}(\xi),$$
$$d_{4m-1}(\Gamma, \xi_\Gamma) := \langle \Sigma, L^{rel}_{4m}(\xi) \rangle - \sigma(\Sigma)$$

Hirzebruch + additivity of $\sigma \implies d_{4m-1}$ depends only on ξ_Γ!

Second main theorem: Proof

Theorem

For $\Gamma \in (\mathbb{S}^{4m+1}, \xi_{std})$, $\ell(\Gamma)$ depends only on FCS ξ_Γ and $\sigma(\Gamma)$. If Γ, Γ' same FCS ξ_Γ,

$$\ell(\Gamma) \neq \ell(\Gamma') \implies \sigma(\Gamma) \neq \sigma(\Gamma') \implies \textit{distinct } \textbf{\textit{smooth}} \textit{ divisors in } \mathbb{S}^{2n+1}.$$

Γ trivial normal bundle $\implies c_k(\xi_\Gamma) = 0 \implies d_{4m-1}(\Gamma, \xi_\Gamma)$ defined.

Compute d_{4m-1} using Siefert hypersurface $\Sigma \subset \mathbb{S}^{4m+1}$ gives...

$$p_i(T\mathbb{S}^{4m-1}) = 0 \implies L^{rel}_{4m-1} = 2\eta_m(-1)^m e^{rel}(\xi, \partial_\tau),$$

$$\underbrace{d_{4m-1}(\Gamma, \xi_\Gamma)}_{\text{inv. of } \xi_\Gamma} = 2\eta_m(-1)^m e^{rel}(\xi, \partial_s) - \sigma(\Sigma) = \underbrace{2\eta_m(-1)^m \ell(\Gamma)}_{\text{inv. of contact div.}} - \underbrace{\sigma(\Gamma)}_{\text{inv. of smooth div.}}$$

where $\eta_m \in \mathbb{Q}_{\neq 0}$ coefficient from L-genus. First statement QED. For second,

$$\sigma(\Gamma') - \sigma(\Gamma) = 2\eta_m(-1)^m(\ell(\Gamma') - \ell(\Gamma)).$$

Summary

$\ell(\Gamma)$ for contact divisor $\Gamma \subset (\mathbb{S}^{2n+1}, \xi_{std})$ generalizes $\ell(\Gamma^1)$ and μ_f.

Theorem

$\Gamma, \Gamma' \subset (\mathbb{S}^{2n+1}, \xi_{std})$ same FCS ξ_Γ, $\ell(\Gamma) - \ell(\Gamma') \in \mathcal{C}^U_{2n}\mathbb{Z}$.
$H^1(\Gamma, \mathbb{Z}/2) = 0$, then $\ell(\Gamma) - \ell(\Gamma') \in \mathcal{C}^{\mathbb{C}S}_{2n}\mathbb{Z}$.

Theorem

For $\Gamma \in (\mathbb{S}^{4m+1}, \xi_{std})$, $\ell(\Gamma)$ depends only on FCS ξ_Γ and $\sigma(\Gamma)$. If Γ, Γ' same FCS ξ_Γ,

$$\ell(\Gamma) \neq \ell(\Gamma') \implies \sigma(\Gamma) \neq \sigma(\Gamma') \implies \textit{distinct } \textbf{\textit{smooth}} \textit{ divisors in } \mathbb{S}^{2n+1}.$$

$\implies$ very different behavior of contact embeddings in low vs high dimensions!

Bonus for experts: Homotopy calculations + Bott periodicity give...

Theorem

Γ homotopy sphere, $\dim \Gamma \neq 7 \bmod 8$. Then $\ell(\Gamma)$ and smooth embedding classify formal embeddings $(\Gamma, \xi_\Gamma) \subset (\mathbb{S}^{2n+1}, \xi_{std})$.

Thanks for having me!

References:

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8. Milnor, "*Singular points of complex hypersurfaces*"
9. Milnor and Stasheff, "*Characteristic classes*"
10. S. Morita, "*A topological classification of complex structures...*"
11. Ustilovsky, "*Infinitely many contact structures on* $\mathbb{S}^{4m+1}$"

Appendix: Bounds on $\mathcal{C}^U, \mathcal{C}^{\mathbb{C}S}$

$$n = 0 \bmod 4 \implies \mathcal{C}^U_{2n} = \mathcal{C}^{\mathbb{C}S}_{2n} \mid \text{denom}\left(\frac{\mathcal{B}_{n/2}}{n!}\right)$$

$$n = 1 \bmod 4 \implies \mathcal{C}^U_{2n} = \mathcal{C}^{\mathbb{C}S}_{2n} \mid 2 \cdot (n-1)!$$

$$n = 2 \bmod 4 \implies \mathcal{C}^U_{2n} \mid \text{denom}\left(\frac{\mathcal{B}_{n/2}}{n!}\right), \quad \mathcal{C}^{\mathbb{C}S}_{2n} \mid \text{denom}\left(\frac{\mathcal{B}_{n/2}}{2 \cdot n!}\right)$$

$$n = 3 \bmod 4 \implies \mathcal{C}^U_{2n} = \mathcal{C}^{\mathbb{C}S}_{2n} \mid (n-1)!$$

$2n$	$\mathcal{C}^U_{2n}$ bound	$\mathcal{C}^{\mathbb{C}S}_{2n}$ bound
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4	$12 = \langle c_2, 9(\mathbb{CP}^1)^2 - 8\mathbb{CP}^2 \rangle$	$24 = \langle c_2, K3 \rangle$
6	$2 = \langle c_3, \mathbb{S}^6 \rangle$	$\sim$
8	720	$\sim$
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