Seminar of the Institute of Mathematics CAS [Cohomology in Algebra, Geometry, Physics and Statistics]

The transfer and symplectic cobordism I, II

Malkhaz Bakuradze

Iv. Javakhishvili Tbilisi State University, Faculty of Exact and Natural Sciences,
A. Razmadze Math. Institute, Georgia

Prague, April 14 and 30, 2025

Introduction

- Definitions
- Statements
- Some Results

Unoriented Bordism Ring

The unoriented bordism ring, denoted by Ω_*, is a fundamental concept in differential topology that classifies manifolds up to bordism.

Two closed n-dimensional manifolds M and N are said to be *bordant* if their disjoint union $M \sqcup N$ is the boundary of an $(n+1)$-dimensional manifold W, i.e., $\partial W = M \sqcup N$.

The unoriented bordism ring Ω_* is a graded ring formed by the equivalence classes of closed manifolds under the unoriented bordism relation.

- The addition of two bordism classes $[M]$ and $[N]$ is given by the disjoint union of representative manifolds:

$$[M] + [N] = [M \sqcup N]$$

- The multiplication of two bordism classes $[M]$ and $[N]$ is given by the Cartesian product of representative manifolds:

$$[M] \cdot [N] = [M \times N]$$

- The grading of the ring is given by the dimension of the manifolds:

$$\Omega_* = \bigoplus_{n=0}^{\infty} \Omega_n$$

Thom's Theorem

A crucial result by René Thom describes the structure of the unoriented bordism ring, connecting it to the homotopy groups of the Thom spectrum MO:

$$\Omega_* \cong \pi_*(MO)$$

This connection allows us to use tools from stable homotopy theory to study the unoriented bordism ring.

Thom Spectrum MO

The Thom spectrum MO is constructed from a sequence of Thom spaces of universal vector bundles. Here's a more detailed explanation:

Grassmannian Manifolds and Universal Vector Bundles

- Let $O(n)$ denote the orthogonal group of rank n.
- The Grassmannian manifold $BO(n)$ is the classifying space for real vector bundles of rank n. In simpler terms, it's a space such that vector bundles of rank n over any space X are classified by maps from X to $BO(n)$.
- Over $BO(n)$, there exists a universal vector bundle γ_n of rank n.

Thom Space

- Given a vector bundle $E \to B$, its Thom space $Th(E)$ is constructed by taking the one-point compactification of each fiber of E and then collapsing the base space B to a single point.
- For the universal vector bundle $\gamma_n \to BO(n)$, its Thom space is denoted as $MO(n) = Th(\gamma_n)$.

- The Thom spectrum MO is a sequence of spaces $\{MO(n)\}_{n\geq 0}$ together with structure maps.
- The structure maps are defined using the inclusions $O(n) \hookrightarrow O(n+1)$, which induce maps between the Grassmannians and the Thom spaces.
- Specifically, the Thom spectrum MO is defined as follows:
  - The n-th space in the spectrum is $MO(n)$.
  - The structure maps are maps $\Sigma MO(n) \to MO(n+1)$, where Σ denotes the suspension. These maps are induced by adding a trivial line bundle to γ_n.

In essence, the Thom spectrum MO is a sequence of Thom spaces of universal vector bundles, assembled in a way that captures the stable information about real vector bundles. It plays a crucial role in Thom's theorem, which relates the homotopy groups of MO to the unoriented cobordism ring.

Pontryagin-Thom Construction

The Pontryagin-Thom construction, often referred to as the Pontryagin map, is a fundamental concept in differential topology that establishes a connection between framed submanifolds and homotopy theory.
A framed submanifold is a submanifold M embedded in a Euclidean space $\mathbb{R}^N$ with a trivialization of its normal bundle $\nu(M)$. This means that there exists a consistent choice of a basis for the vectors perpendicular to M at each point.

The Pontryagin-Thom construction provides a way to associate a map from the sphere S^N to the Thom space of the normal bundle to a framed submanifold M in $\mathbb{R}^N$.

1. Given a framed submanifold $M \subset \mathbb{R}^N$, we define a map $S^N \to S^N / \left(S^N \setminus U\right) \to Th(\nu(M))$, where U is a tubular neighborhood of M in S^N.
2. Conversely, given a map $S^N \to Th(\nu(M))$, we can construct a framed submanifold M as the preimage of a regular value.

- The construction bridges the gap between the geometric world of framed submanifolds and the algebraic world of homotopy theory.
- It plays a crucial role in Thom's theorem, relating bordism groups to homotopy groups of Thom spaces.

Thom's Theorem on Unoriented Bordism

Thom's theorem provides a complete algebraic description of the unoriented bordism ring, Ω_*.
Thom's theorem states that the unoriented bordism ring Ω_* is isomorphic to a polynomial ring over the field $\mathbb{Z}/2$, with generators in degrees that are not of the form $2^k - 1$, i.e.,

$$\Omega_* \cong \mathbb{Z}/2[x_2, x_4, x_5, x_6, x_8, x_9, \dots]$$

where:

- $\mathbb{Z}/2$ is the field with two elements.
- x_i represents a generator in dimension i.
- The indices i run through all positive integers that are not of the form $2^k - 1$ (i.e., $1, 3, 7, 15, \dots$ are excluded).

- Polynomial Ring. Every unoriented bordism class can be uniquely expressed as a sum of products of these generators.
- Generators. The generators x_i correspond to specific manifolds that serve as the "building blocks" for all other unoriented manifolds.
- $\mathbb{Z}/2$ Coefficients. Each generator appears either 0 or 1 times in a given bordism class.
- Excluded Dimensions. Dimensions of the form $2^k - 1$ are excluded, reflecting the structure of Ω_* and its connection to Adams operations.

Becker-Gottlieb Transfer Map

The Becker-Gottlieb transfer is a map in stable homotopy theory associated with a fibration. For a fibration $p : E \to B$ with compact, smooth fiber F, the Becker-Gottlieb transfer is a map

$$\tau : B_+ \to E_+$$

in the stable homotopy category. Here, X_+ denotes the suspension spectrum of a space X.

Suspension Spectrum Given a pointed topological space X, the suspension spectrum of X, (also denoted by $\Sigma^\infty X$), is a spectrum constructed as follows:

- The n-th space in the spectrum is the n-fold reduced suspension of X, denoted by $\Sigma^n X$.
- The structure maps are the identity maps $\Sigma(\Sigma^n X) \to \Sigma^{n+1} X$.

In other words, the suspension spectrum of X is the sequence of spaces

$$X, \Sigma X, \Sigma^2 X, \Sigma^3 X, \dots$$

where each space is the suspension of the previous one.

Double Coset Formula for Compact Lie Groups

Let G be a compact Lie group, and let H and K be closed subgroups of G. The double coset formula describes the relationship between the transfer maps associated with these subgroups. The formula is given by:

$$(p_{K.G})^* \circ (\mathrm{tr}_{H,G}) = \sum_{g \in K \backslash G / H} \chi(X_g)(\mathrm{tr}_{K \cap gHg^{-1},K}) \circ (p_{K \cap g^{-1}Hg, g^{-1}Hg})^* \circ (c_g)^*$$

where:

- The sum is taken over the double cosets KgH of G with respect to K and H.
- X_g is the orbit type manifold component corresponding to the double coset representative g.
- $\chi(X_g)$ is the internal Euler characteristic of the orbit type manifold component X_g, given by $\chi(X_g) = \chi(\overline{X_g}) - \chi(\overline{X_g} \setminus X_g)$, where $\overline{X_g}$ is the closure of X_g.
- $c_g : BH \to B(gHg^{-1})$ is the map induced by conjugation by g.
- $p_{K\cap gHg^{-1},K} : B(K \cap gHg^{-1}) \to BK$ and $p_{K\cap g^{-1}Hg} : B(K \cap g^{-1}Hg) \to Bg^{-1}Hg$ are the maps induced by the inclusions.

Example: $G = S^3$, $K = H = S^1$

Let $G = S^3 = \{q \in \mathbb{H} \mid |q| = 1\}$, where $\mathbb{H}$ is the set of quaternions. Let $K = H = S^1 = \{e^{i\theta} \mid \theta \in [0, 2\pi)\} \subset S^3$.
The double coset decomposition is given by:

$$S^3 = \bigsqcup_{g \in S^1 \backslash S^3 / S^1} S^1 g S^1$$

where the union is over a set of representatives of the double cosets.
For S^3 and S^1, the double cosets $S^1 g S^1$ can be parameterized by a single angle $\phi \in [0, \pi/2]$. Thus, we can write:

$$S^3 = \bigcup_{0 \leq \phi \leq \pi/2} S^1(\cos(\phi) + j\sin(\phi))S^1$$

Let's consider the orbit type manifold components and their internal Euler characteristics:

- The double coset space $S^1 \backslash S^3 / S^1$ is parameterized by $\phi \in [0, \pi/2]$. We consider the cases $\phi = 0$, $\phi = \pi/2$, and $0 < \phi < \pi/2$ separately.
- For each $g = \cos(\phi) + j \sin(\phi)$, the intersection $S^1 \cap gS^1g^{-1}$ is:

- If $\phi = 0$, then $g = 1$, and $S^1 \cap gS^1g^{-1} = S^1 \cap S^1 = S^1$. The orbit type manifold component is a point, and the internal Euler characteristic $\chi(X_g) = 1$.
- If $\phi = \pi/2$, then $g = j$, and $gS^1g^{-1} = jS^1j^{-1} = S^1$. Thus, $S^1 \cap gS^1g^{-1} = S^1 \cap S^1 = S^1$. The orbit type manifold component is a point, and the internal Euler characteristic $\chi(X_g) = 1$.
- If $0 < \phi < \pi/2$, then $S^1 \cap g0g^{-1} = 0$. The orbit type manifold component is an interval, and the internal Euler characteristic $\chi(X_g) = -1$. This component gives zero.

So the double coset formula in this example becomes:

$$(p_{S^1,S^3})^* \circ (\mathrm{tr}_{S^1,S^3}) = 2 \cdot (\mathrm{tr}_{S^1,S^1}) \circ (p_{S^1,S^1})^*.$$

Thus for $a \in h^*(BS^1)$

$$(p_{S^1,S^3})^* \circ (\mathrm{tr}_{S^1,S^3})(a) = 2a$$

Complex Cobordism

- Complex cobordism deals with manifolds with a "stable almost complex structure."
- Cobordism classifies manifolds with stably complex tangent bundles up to equivalence. Two manifolds are cobordant if their disjoint union is the boundary of a higher-dimensional manifold.
- An almost complex structure on a real, even-dimensional manifold M is a linear transformation J of the tangent bundle TM such that $J^2 = -\mathrm{Id}$, where Id is the identity transformation. Essentially, it's a way to define "multiplication by i" (the imaginary unit) on the tangent spaces of the manifold. However, not all almost complex structures arise from complex coordinate systems. If they do, the structure is called a complex structure.

MU Complex Cobordism Spectrum

- The spectrum representing complex cobordism.
- Encodes information about complex manifolds.

Formal Group Laws

- Connection between complex cobordism and formal group laws.
- Quillen's theorem: The coefficient ring of MU is isomorphic to Lazard's universal ring.
- The formal group law of complex cobordism is the "universal" formal group law.

The structure of the complex cobordism ring MU_* is well-understood. It is a polynomial algebra over $\mathbb{Z}$ with generators in even degrees.

Theorem (Structure of MU_*):

$$MU_* \cong \mathbb{Z}[x_2, x_4, x_6, \ldots]$$

where x_{2i} are generators in degree $2i$.

Quaternionic(or Symplectic) Cobordism *MSp*

- Classifies manifolds with a stable almost quaternionic structure.
- Its coefficient ring, MSp_*, is where the difficulty explodes.
- While real and complex cobordism benefit from relatively simple algebraic structures and powerful tools, quaternionic cobordism is hampered by its intricate, non-polynomial structure with torsion and the lack of straightforward computational methods. This makes explicit calculations of MSp_* far more difficult and, in many ways, still a major challenge in algebraic topology.

Why is *MSp* so Hard?

- Lack of a Simple Algebraic Structure. The regularity that makes MO_* and MU_* computable simply doesn't exist for MSp_*. The presence of torsion and the non-polynomial nature of the ring make calculations extremely difficult.
- Complicated Formal Group Laws. The connection to formal group laws, which is so powerful in the complex case, becomes much more intricate in the quaternionic case. The relevant algebraic structures are far more challenging to work with.

Low dimensional torsion part of the symplectic bordism

Definition

The Ray classes [10] $\phi_i \in MSp_{8i-3}$ are indecomposable torsion elements of order two in symplectic bordism ring. ϕ_i arise from the expansion of Euler (Conner-Floyd symplectic Pontryagin) class

$$e((\eta^1 - \mathbb{R}) \otimes_{\mathbb{R}} (\zeta - \mathbb{H})) = s \sum_{i \geq 1} \theta_i e^i(\zeta)$$

in $MSp^4(S^1 \wedge BSp(1))$, where s is the generator of $MSp^1(S^1) = \mathbb{Z}$, $\eta^1 \to S^1$ is the non-trivial real line bundle, $\zeta \to BSP(1)$ is the canonical $Sp(1)$ bundle and e is Euler class. The notation

$$\theta_{2i} = \phi_i$$

is used in the literature because $\theta_{2i+1} = 0$, for $i > 1$ by Roush [11]. We relabel $\theta_1 := \phi_0$.

Low dimensional free generators

Let $\zeta \to HP^\infty$ be the canonical quaternionic line bundle, and ζ_i be the pullback by the projection on i-th factor

$$HP^\infty \times HP^\infty \times HP^\infty \to HP^\infty.$$

Define elements a_{ijk} by the Pontryagin classes

$$P_1(\zeta_1 \otimes_{\mathbb{C}} \zeta_2 \otimes_{\mathbb{C}} \zeta_3) = \sum_{ijk} a_{ijk} x^i y^j z^k.$$

Symplectic bordism in Low dimensions

In low dimensions MSp^* was calculated by Ray [10]. In terms of ϕ_i and a_{ijk} above (see more details in [3]) we have the following picture in low dimensions

n	MSp_n	generators
0	$\mathbb{Z}$	1
1	0	ϕ_0
2	$\mathbb{Z}_2$	ϕ_0^2
3	0	
4	$\mathbb{Z}$	a_{011}
5	$\mathbb{Z}_2$	$\phi_1,$
6	$\mathbb{Z}_2$	$\phi_0\phi_1$
7	0	
8	$\mathbb{Z} \oplus \mathbb{Z}$	a_{012}, a_{111}

- By Vershinin [13] [Calculation of the symplectic cobordism ring in dimensions up to 32 and the nontriviality of the majority of ternary products of N. Ray's elements] the classes ϕ_i play an essential role in the torsion of the symplectic cobordism ring, i.e., most ternary products $\phi_i \phi_j \phi_k \neq 0$.
- By Ray and Gorbounov [8, 9] one has $\phi_0 \phi_i \phi_j = 0$ and $\phi_i^{2i+3} = 0$.

Some other Relations

Our one observation [3] is that multiplication by the elements ϕ_i, $i > 0$, carries most of the low-dimensional generators from the free part of MSp_{4n} to the ideal generated by the elements ϕ_0 and ϕ_1. In particular, one has

Theorem

For $i \geq 0$
i) $\phi_i a_{110} = \phi_i a_{012} = \phi_i a_{022} = \phi_i a_{014} = 0$;
ii) $\phi_i a_{111}$ and $\phi_i a_{122}$ belong to the ideal $\phi_0 MSp^*$;
iii) $\phi_i a_{211}$ belong to the ideal $\phi_0 MSp^* + \phi_1 MSp^*$.

Our main result

In ([2], Prop. 4.1) we proved the following

Theorem

i) All fourfold products of the Ray classes $\phi_i \phi_j \phi_k \phi_l$ *are zero;*
ii) The images of all double products $\phi_i \phi_j$ *in self-conjugate cobordism are zero.*

Motivation

- We review the proof of main Theorem above [4] as follows. We note that in some arguments in [2] the arguments of Remark 1.11, Lemma 1.12, and the proof of Proposition 1, (1.1) and (1.2), case $m = 5$ are inherited from the references. Still, these statements seem to be the consequences of Theorem 3.1 in [6]. However, all these points are used to derive the proof of Proposition 1 of [2], which we covered in Section 3. To do this, we first carry out some calculations with transfer in symplectic cobordism by using only double coset formula of [7]. For the reader's convenience, in Section 4 we briefly recall the proof of Theorem 3 by pointing to the sequence of necessary propositions of [2].

Good question and starting point

Consider the bundle of classifying spaces

$$p : BS^1 \to BS^3$$

defined by inclusion of a circle $S^1 = U(1) = Spin(2)$ in 3-sphere $S^3 = Sp(1) = Spin(3)$.
For complex oriented cohomology theories h^* (including complex cobordism MU^*) p^* is monomorphism, which in terms of the Euler classes x and y of canonical bundles $\zeta \to BS^3$, $\xi \to BU(1)$ operates as $\zeta \to \xi \oplus \bar{\xi}$, i.e. we have monomorphism

$$\begin{aligned} p^* : h^*(BSpin(3)) &\to h^*(BSpin(2)), \\ h^*[[x]] &\to h^*[[y]], \;\; x \to y\bar{y}. \end{aligned}$$

What if h^* is the symplectic cobordism MSp^*? is p^* mono?

$$\begin{aligned} \text{Still} \quad & MSp^*(BS^3) = MSp^*[[x]], \\ \text{but} \quad & MSp^*(BS^1) \neq MSp^*[[y\bar{y}]]. \end{aligned}$$

The answer is no and immediate interest is the Kernel of p^* in symplectic cobordism.
K-theory hintes as follows. Consider the real line bundle $\theta \to S^1$ and consider the quaternionic line bundle

$$\theta \otimes_R \zeta \to S^1 \times BS^3.$$

Applying p^* we have over $S^1 \times BS^1$

$$\begin{aligned} \theta \otimes_{\mathbb{R}} (\xi \oplus \bar{\xi}) = & \\ & (1 \times p)^*(\theta \otimes_{\mathbb{R}} \zeta) = \theta \otimes_{\mathbb{R}} (r(\xi) \otimes_{\mathbb{R}} \mathbb{C}) \\ & (r(\xi) \otimes_{\mathbb{R}} (\theta \otimes_{\mathbb{R}} \mathbb{C}) = r(\xi) \otimes_{\mathbb{R}} \mathbb{C} \\ & = \xi \oplus \bar{\xi}. \end{aligned}$$

This is a hint of how the kernel of p^* is related to Ray's classes. We proceed as follows.

Idea

Here we follow the notations of [2]. The proof of Theorem is organized as follows.

The tensor square of the canonical $Sp(1)$-bundle $\zeta \to BSp(1)$ has a trivial summand

$$\zeta \otimes_H \zeta^* = \Lambda + 1,$$

where $\Lambda \to BSp(1)$ is the canonical $Spin(3)$-bundle.

Let N be the normalizer of the torus $S^1 = U(1)$ in $S^3 = Sp(1) = Spin(3)$. Clearly the bundle

$$p : BN \to BSp(1) = BSpin(3)$$

is the projective bundle of Λ. The quotient map $N/U(1) = Z/2$ induces the map

$$f : BN \to BZ/2,$$

the classifying map of the canonical real line bundle

$$\lambda \to BN, \ \ \lambda^{\otimes 2} = 1. \tag{1}$$

Because of projectivisation the pullback of Λ splits canonically over BN

$$p^*(\Lambda) = \lambda + \mu. \tag{2}$$

Then it turns out (we will turn to transfer map)

$$Tr^* f^* = 0 \tag{3}$$

in symplectic cobordism, where Tr is the transfer map of p.
Then recall above bundle relation hint and that by [9], [2] p.4394, Λ is MSp-orientable and the Thom class can be chosen in such a way that its restriction to the zero section is equal to

$$\tilde{e}(\Lambda) = \theta_1 + \sum_i \phi_i x^i, \;\; x = e(\zeta). \tag{4}$$

Theorem i) says that in $MSp^*(BSp(1)^4) = MSp^*[x_1, x_2, x_3, x_4]$ one has

$$e(\sum_{!}^{4} \Lambda_i) = 0.$$

Here Λ_i is the pullback of $\Lambda \to BSp(1)$ by projection on i-th factor. The idea of proof is to find relation in $MSp^*(BN \times BSp(1)^3)$ of the form

$$(p \times 1 \times 1 \times 1)^* e(\sum_{1}^{4} \Lambda_i) =$$

$$\sum_{m,n,p,q \geq 0} f^*(\gamma_{mnpq}) x_1^m x_2^n x_3^p x_4^q$$

and then apply Transfer map $Tr \times 1 \times 1 \times 1$ to get zero for the right side.

To start the proof let $\lambda \to \mathbb{Z}/2$ be as above. Then by ([2], Lemma 4.5)

$$\lambda \otimes_{\mathbb{R}} \sum_1^4 \Lambda_i \to B\mathbb{Z}/2 \times BSp(1)^4 \text{ is } MSp\text{-orientable}, \qquad (5)$$

$$\lambda \otimes_{\mathbb{R}} \sum_1^2 \Lambda_i \to B\mathbb{Z}/2 \times BSp(1)^2 \text{ is } SC\text{-orientable}. \qquad (6)$$

Because of (2) and (1) the pullback of (5) over

$$(f, p) \times 1 : BN \times BSp(1)^3 \to B\mathbb{Z}/2 \times BSp(1)^4$$

has a trivial summand and therefore zero MSp-orientation Euler class.

Thus ([2], Lemma 4.6) one has in $MSp^*(BN \times BSp(1)^3)$

$$0 = \prod_{s=1}^{4}(\theta_i + \sum_{r\geq 1}\phi_r x_s^{2r}) + \sum_{m,n,p,q\geq 0} f^*(\gamma_{mnpq})x_1^m x_2^n x_3^p x_4^q \tag{7}$$

$$= \sum_{i,j,k,l\geq 1}\phi_i\phi_j\phi_k\phi_l x_1^{2i}x_2^{2j}x_3^{2k}x_4^{2l} + \sum_{m,n,p,q\geq 0} f^*(\gamma_{mnpq})x_1^m x_2^n x_3^p x_4^q. \tag{8}$$

Here in (7) we use the relation $\theta_1\phi_i\phi_j = 0$ of [8].

Similarly the pullback of (6) over

$$(f, p) \times 1 : BN \times BSp(1) \to B\mathbb{Z}/2 \times BSp(1)^2.$$

has zero MSC-orientation Euler class. One has in $SC^*(BN \times BSp(1))$

$$0 = \sum_{i,j \geq 1} \phi_i \phi_j x_1^{2i} x_2^{2j} + \sum_{m,n \geq 0} f^*(\gamma_{mn}) x_1^m x_2^n. \tag{9}$$

Finally to complete the proof of Theorem 3 i) apply (8) and (3). Similarly apply (9) and (3) to complete the proof of Theorem 3 ii).

References I

- *M. F. Atiyah, R. Bott and A. Shapiro, : Clifford modules*, Topology, **3(suppl. 1)**(1964), 3-38.
- *M. Bakuradze, : The transfer and symplectic cobordism*, Trans. Amer. Math. Soc., **349, 11**(1997), 4385-4399.
- *M. Bakuradze, M Jibladze and V, Vershinin : Characteristic classes and transfer relations in cobordism*, Proc. Amer. Math. Soc., **131, 6**(2002), 1935–1942.
- *M. Bakuradze, : On vanishing of all fourfold products of the Ray classes in symplectic cobordism*, Proc. Amer. Math. Soc., **148, 9**(2020), 4107-4115.
- *J. C. Becker and D.H. Gottlieb : The transfer map and fibre bundles*, Topology **14** (1975), 1-12.

References II

V. M. Buchstaber, K. E. Feldman : *Index of equivariant vector field and addition theorems for the Pontryagin characteristic classes*, IZV. RAS. Math. Series, **64(2)**(2000), 3-28.

M. Feshbach : *The transfer and compact Lie groups*, Trans. Amer. Math. Soc. **251** (1979), 139-169.

V. G. Gorbunov : *Symplectic cobordism of projective spaces*, Math. Sbornik **181** (1990), 506–520

V. G Gorbunov and N. Ray : *Orientation of Spin(n) bundles and symplectic cobordism*, Publ. RIMS. Kyoto Univ. **28(1)**,(1992), 39-55.

N. Ray, : *Indecomposables in TorsMSp$_*$*, Topology, **10**(1971), 261-270.

F. W. Roush, : *On some torsion classes in symplectic bordism*, Preprint.

References III

- *J. Schray, C. A. Manogue, : Octonionic representations of Clifford algebras and triality,*(1994), ArXiv:hep-th/9407179v1 .
- *V. V. Vershinin : Calculation of the symplectic cobordism ring in dimensions up to 32 and the nontriviality of the majority of ternary products of N. Ray's elements (Russian),* Sibirsk. Mat. Zh. **24(1)** (1983), 50–62.

Thanks

This work was supported by Shota Rustaveli NSF grant FR-23-779