# On Chern-Losik class for codimension two foliations

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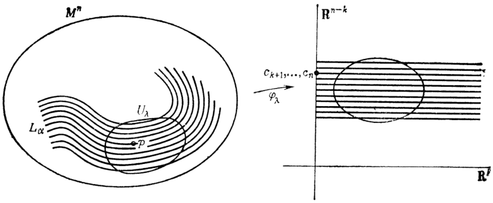
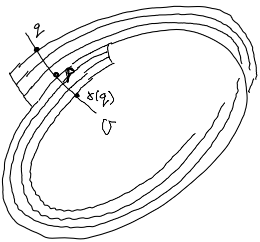
# Foliation

*Foliation of dimension $k$ on manifold $M^n$* is a partition of $M^n$ into non-itersecting subsets $L_\alpha, \alpha \in A$ (leaves) which is locally modeled by parallel leaves $x + \mathbb{R}^k$ in $\mathbb{R}^n = \mathbb{R}^{n-k} \times \mathbb{R}^k$.
Number $q = n - k$ is called *co-dimension* of foliation $\mathcal{F} = \{L_\alpha | \alpha \in A\}$.



# Holonomy Pseudogroup

Consider a transversal $U \subset M$ at point $p \in L_{\alpha_0} \subset M$. Let $\gamma \in \pi_1(L_{\alpha_0}, p)$. Every point $q \in U$ can be transported along the $\gamma$ using local decomposition $U \times \mathbb{R}^k$ to a uniquely defined point $\gamma^*(q)$.
All such $\gamma^* : U \to U$ (local diffeomorphism) generate holonomy pseudogrup $Hol_p(M/\mathcal{F})$.



# Godbillon-Vey-Losik class in codimension one

Let $U \subset M$ is a transversal to $\mathcal{F}$, $S_2(U)$ is a space of jets of order $\leq 2$ on $U$.
Let $S_2(U) = U \times \mathbb{R} \times \mathbb{R}$ with coordinates $x_0, x_1, x_2$: for $f : \mathbb{R} \to U$ we put

$$x_0 = f(0),\ x_1 = \ln|f'(0)|,\ x_2 = f''(0)f'(0)^{-2}.$$

For holonomy transformation $\varphi : U \to V$ we have the following transformation of jet coordinates $\tilde{\varphi} : S_2(U) \to S_2(V)$:

$$\begin{aligned}
\alpha_0 &= \varphi(x_0) \\
\alpha_1 &= x_1 + \ln|\varphi'(x_0)| \\
\alpha_2 &= \frac{x_2}{\varphi'(x_0)} + \frac{\varphi''(x_0)}{(\varphi'(x_0))^2}
\end{aligned}$$

# Godbillon-Vey-Losik class in codimension one

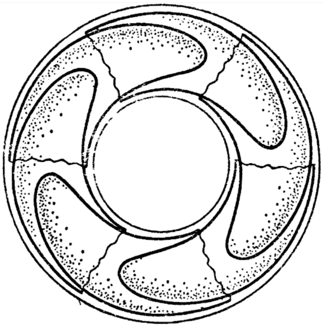
We have invariant 3-form:

$$\tilde{\varphi}^*(d\alpha_0 \wedge d\alpha_1 \wedge d\alpha_2) = dx_0 \wedge dx_1 \wedge dx_2$$

Godbillon-Vey-Losik class:

$$gvl = [-dx_0 \wedge dx_1 dx_2] \in H_2(S_2(M/\mathcal{F}))$$

# Godbillon-Vey-Losik class in codimension one



Reeb foliation has one compact leaf with holonomy generated by diffeomorphism

$$\varphi : \mathbb{R} \to \mathbb{R}, \ \varphi(0) = 0, \varphi'(0) = 1, \ \varphi^{(k)}(0) = 0, k = 2, 3, \ldots$$

Diffeomorphism $\varphi$ can be included in 1-parameter group of diffeomorphisms $\varphi_t : \mathbb{R}\mathbb{R}$ which is generated by vector field $V$ (Szekeres vector field).

$$V_\alpha(x) = \begin{cases} e^{-\frac{1}{|x|^\alpha}}, & \text{for } x \neq 0, \\ 0, & \text{for } x = 0. \end{cases}$$

**Theorem (B.-Galaev-Gumenyuk, 2022).** $\alpha \in \mathbb{N}$ is odd $\Rightarrow (M/\mathcal{R}_\alpha) \neq 0$

**Theorem (B.-Galaev-Gumenyuk, 2022).** $\alpha \in \mathbb{N}$ is even $\Rightarrow (M/\mathcal{R}_\alpha) = 0$

**Corollary.** If $\alpha \in \mathbb{N}$ is odd and $\beta \in \mathbb{N}$ is even, then the foliations $\mathcal{R}_\alpha$ and $\mathcal{R}_\beta$ are not diffeomorphic.

# Chern-Losik class in codimension one

Consider associated space $S_2'(U) = S_2(U)/GL(1)$, where $GL(1)$ acts by:

$$a \in GL(1) : (x_0, x_1, x_2) \to (x_0, ax_1, x_2).$$

We can consider coordinates $(x_0, x_2)$ on $S_2'(U)$ and get invariant differential form:

$$\tilde{\varphi}^*(d\alpha_2 \wedge d\alpha_0) = dx_2 \wedge dx_0.$$

for any diffeomorphism $\varphi : U \to V$ of transversal. Chern-Losik class:

$$cl_1 = [dx_2 \wedge dx_1] \in H_2(S_2'(M/\mathcal{F}))$$

**Theorem (B.-Galaev, 2022).** For all Reeb foliations $cl_1(M/\mathcal{F}) \neq 0$.

# Chern-Losik class in codimension one

**Losik example**: Let $f : S^1 \to S^1$ be diffeomorphism with two hyperbolic fixed points $p$ and $q$ (hyperbolicity means $|f'(p)| \neq 1$ and $|f'(q)| \neq 1$).

**Lemma**. Diffeomorphism $\tilde{f} : S_2'(S/f) \to S_2'(S/f)$ has two fixed points $\tilde{p}, \tilde{q}$, lying in the fibers over the $p$ and $q$.

Let $\gamma(t)$ connects $\tilde{p}$ and $\tilde{q}$ and consider 2-disk $D$ such that $\partial D = \{\gamma(t)\} \cup \{\tilde{f}(\gamma(t))\}$. Disk $D$ defines cycle in $S_2'(S^1/f)$ and

$$\int_D cl_1 = \frac{1}{2} \ln \frac{|f'(p)|}{|f'(q)|}$$

0.5cm

**Corollary (Losik, 1990)**. If $|f'(p)| \neq |f'(q)|$ then first Chern-Losik class of foliation with holonomy $f$ is non-trivial.

# Chern-Losik class for codimension two foliation

Consider
an example of codimension two foliation on solid torus: transversal $D$ is a two-dimensional disk (cross-section of solid torus). Holonomy action is represented by diffeomorphism $f : D \to D$.

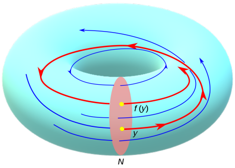



Figure: Holonomy of foliation $\mathcal{F}$.

We need
to study invariant classes in $H^*(S_2(D/f)/GL_2)$.

# Chern-Losik class for codimension two foliation

We consider the following coordinates in a jet space $S_2(D)$:

$$z^i = f^i(0,0), \quad z^i_j = \frac{\partial f^i}{\partial t^j}(0,0), \quad z^i_{jk} = \frac{\partial^2 f^i}{\partial t^j \partial t^k}(0,0),$$

for $f : \mathbb{R}^2 \to D$.
Now it is more convenient to transform standard coordinates to a new ones:

$$y^i = z^i, \quad y^i_j = z^i_j, \quad y^i_{jk} = v^p_j z^i_{pq} v^q_k,$$

where we assume that $(v^i_j)$ is an inverse matrix to $(z^i_j)$.
Action of $GL_2(\mathbb{R})$ on $S_2(D)$ is defined as follows

$$A : (y^i, y^i_j, y^i_{jk}) \longmapsto (y^i, y^i_p A^p_j, y^i_{jk}),$$

therefore we can consider $(y^i, y^i_{jk})$ as coordinates on the space $S_2(D)/GL_2(\mathbb{R}) = S'_2(D)$.

# Chern-Losik class for codimension two foliation

Any transversal diffeomorphism $g : D \to U$ can be lifted to diffeomorhism $\widetilde{g} : S_2'(D) \to S_2'(U)$. We have:

$$\eta^i = g^i(y^1, y^2), \quad \eta^i_{jk} = \frac{\partial y^p}{\partial \eta^j} \frac{\partial^2 g^i}{\partial y^p \partial y^q} \frac{\partial y^q}{\partial \eta^k} + \frac{\partial y^p}{\partial \eta^j} \frac{\partial \eta^i}{\partial y^s} \frac{\partial y^q}{\partial \eta^k} y^s_{pq}.$$

Now consider associated space $\widetilde{S}_2'(D)$ with coordinates $(y^i, y_i)$, where $y_i = y^j_{ji}$. On this space there exits differential form

$$\omega = dy_1 \wedge dy^1 + dy_2 \wedge dy^2,$$

which is invariant with respect to diffeomorphism $\widetilde{g} : \widetilde{S}_2'(D) \to \widetilde{S}_2'(D)$, generated by holonomy diffeomorphism $g : D \to D$.

Class $[\omega] \in H^*(\widetilde{S}_2'(D/f))$ is called *first Chern-Losik class*.
This construction can be easy extanded to foliations with arbitrary codimensions.

# Chern-Losik class for codimension two foliation

Class $[\omega]$ is trivial if and only if there exists $\widetilde{f}$–invariant form $\theta$, such that $d\theta = \omega$. Then

$$\theta = Ady_1 + Bdy_2 + Cdy^1 + Ddy^2.$$

We assume that all iterations of diffeomorphism $\{\widetilde{f}^n\}_{n\in\mathbb{Z}}$ can be included into a one-parameter group of diffeomorphisms $\{\widetilde{f}_t\}_{t\in\mathbb{R}}$. Then we can average form $\theta$:

$$\theta' = \int_{\xi}^{\xi+1} \widetilde{f}_t^*(\theta) dt.$$

Averaged form $\theta'$ does not depend of $\xi$, is $\widetilde{f}_t$–invariant and satisfies equation $d\theta' = d\theta = \omega$.

$\widetilde{f}_t$–invariance condition is equivalent to the equation:

$$L_{\widetilde{V}}\theta = 0, \tag{1}$$

where $\widetilde{V} = (V^1, V^2, V_1, V_2)$ is a vector field on $\widetilde{S}_2'(D)$, generated by $\{\widetilde{f}_t\}_{t\in\mathbb{R}}$, nd $L$ is a Lie derivative.

Equation $d\theta = dy_1 \wedge dy^1 + dy_2 \wedge dy^2$ can be rewritten:

$$\frac{\partial A}{\partial y^1} + 1 = \frac{\partial C}{\partial y_1};\ \frac{\partial B}{\partial y^2} + 1 = \frac{\partial D}{\partial y_2};\ \frac{\partial A}{\partial y_2} = \frac{\partial B}{\partial y_1};$$

$$\frac{\partial A}{\partial y^2} = \frac{\partial D}{\partial y_1};\ \frac{\partial B}{\partial y^1} = \frac{\partial C}{\partial y_2};\ \frac{\partial C}{\partial y^2} = \frac{\partial D}{\partial y^1}.(2)$$

# Chern-Losik class for codimension two foliation

equation (1) can be rewritten in the following form:

$$\widetilde{V}_1 \frac{\partial}{\partial y_1}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + \widetilde{V}_2 \frac{\partial}{\partial y_2}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + V^1 \frac{\partial}{\partial y^1}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} + V^2 \frac{\partial}{\partial y^2}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} +$$

$$\begin{pmatrix} -\frac{\partial V^1}{\partial y^1} & -\frac{\partial V^1}{\partial y^2} & 0 & 0 \\ -\frac{\partial V^2}{\partial y^1} & -\frac{\partial V^2}{\partial y^2} & 0 & 0 \\ \frac{\partial \widetilde{V}_1}{\partial y^1} & \frac{\partial \widetilde{V}_1}{\partial y^2} & \frac{\partial V^1}{\partial y^1} & \frac{\partial V^2}{\partial y^1} \\ \frac{\partial \widetilde{V}_2}{\partial y^1} & \frac{\partial \widetilde{V}_2}{\partial y^2} & \frac{\partial V^1}{\partial y^2} & \frac{\partial V^2}{\partial y^2} \end{pmatrix} \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} = 0.$$

Equations (14) can be resolved as $\theta = dF - y^1 dy_1 - y^2 dy_2$ for some function $F$, which allows to simplify above system of equations.

# Chern-Losik class for codimension two foliation

It can be proved that above system of equations is equivalent to the following system for function $G = F - y_1 y^1 + y_2 y^2$:

$$V_1 \frac{\partial G}{\partial y_1} + V_2 \frac{\partial G}{\partial y_2} + V^1 \frac{\partial G}{\partial y^1} + V^2 \frac{\partial G}{\partial y^2} = -\frac{\partial V^1}{\partial y^1} - \frac{\partial V^2}{\partial y^2} + R, \qquad (3)$$

with some constant $R$.

**Theorem**

*An existence of smooth solution of equation (3) on $D \times \mathbb{R}^2$ is equivalent to triviality of first Chern-Losik class of foliation, generated by holonomy diffeomorphism $f$.*

# Example 1

Foliation on
the solid torus $T$. Central circle $S$ is a leaf with nontrivial holonomy pseudogroup. Space $T \setminus S$ is fibered into a concentric tori, on every torus consider Kronecker foliation. This construction defines codimension two foliation $\mathcal{F}_1$ on $T$.

Transversal
is a disk $D$, which is perpendicular to circle $S$. We consider foliation generated by vector field:

$$\mathbf{V} = \begin{cases} V^1(y^1, y^2) = f(r)y^2, \\ V^2(y^1, y^2) = -f(r)y^1, \end{cases}$$

where $f(r)$ is a smooth function of argument $r = \sqrt{(y^1)^2 + (y^2)^2}$.

**For any function $f$ first Chern-Losik class of foliation is trivial.**

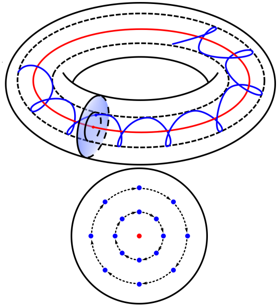


Figure: Example 1

# Example 1

Equation (3) takes form

$$f(r)y^2\frac{\partial G}{\partial y^1} - f(r)y^1\frac{\partial G}{\partial y^2} + \widetilde{V_1}\frac{\partial G}{\partial y_1} + \widetilde{V_2}\frac{\partial G}{\partial y_2} = R = \text{const},$$

where

$$\widetilde{V}_1 = f(r)y_2 + f'(r)\frac{y^1}{r}(y_2y^1 - y_1y^2),$$

$$\widetilde{V}_2 = -f(r)y_1 + f'(r)\frac{y^2}{r}(y_2y^1 - y_1y^2).$$

There are first integrals of the above equation:

$$\begin{cases} (y^1)^2 + (y^2)^2 = \alpha; \\ y_2y^1 - y_1y^2 = \beta; \\ \dfrac{f'(r)}{f(r)}\dfrac{y^1y^2}{2r} + \dfrac{f'(r)}{f(r)}\dfrac{r}{2}\arcsin\dfrac{y^1}{r} - \arcsin\dfrac{y^1}{r} = \gamma; \\ F - \dfrac{R}{f(r)}\left(\dfrac{y^1y^2}{2} + \dfrac{r}{2}\arcsin\dfrac{y^1}{r}\right) = \delta. \end{cases}$$

# Example 1

general solution of (3) is

$$G = \frac{R}{f(r)}\left(\frac{y^1 y^2}{2} + \frac{r}{2}\arcsin\frac{y^1}{r}\right) + D(\alpha, \beta, \gamma).$$

Then function

$$G = y_2 y^1 - y_1 y^2$$

for $R = 0$ and $D(\alpha, \beta, \gamma) = \beta$ is a smooth solution of equation (3).

Therefore first Chern-Losik class for foliation $\mathcal{F}_1$ is trivial for any function $f$ which defines "rotation" of leaves around the central circle.

# Example 2

Consider codimension
two foliation $\mathcal{F}_2$ on the solid torus $T$. Leaves run out of central circle and asymptotically converge to boundary two-dimensional torus. Leaves can be defined clearly:

$$L_0 = \{(2\cos\varphi, 2\sin\varphi, 0) : \varphi \in [0, 2\pi)\};$$
$$L_{r,\theta} = \{((2+\rho(r+\varphi)\cos\theta)\cos\varphi,$$
$$(2+\rho(r+\varphi)\cos\theta)\sin\varphi, \rho(r+\varphi)\sin\theta : \varphi \in \mathbb{R}\},$$

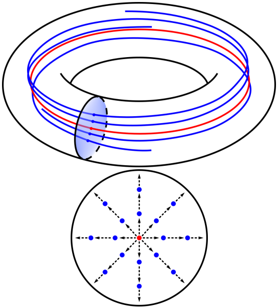


Figure: Example 2

Transversal
diffeomorphism is generated by vector field:

$$\mathbf{V}(X) = f(|X|)X,$$

where $X = (y^1, y^2)$, and $f(|X|)$ -is some smooth function of $r = \sqrt{(y^1)^2 + (y^2)^2}$.

# Example 2

Triviality of Chern-Losik class is equivalent to existence of smooth solution of the equation:

$$f(r)y^1\frac{\partial G}{\partial y^1}+f(r)y^2\frac{\partial G}{\partial y^2}+\widetilde{V_1}\frac{\partial G}{\partial y_1}+\widetilde{V_2}\frac{\partial G}{\partial y_2}=-2f(r)-f'(r)r+R \quad (4)$$

where

$$\widetilde{V}_1=f'(r)\frac{y^1}{r}(3-y_1y^1-y_2y^2)+f''(r)y^1-f(r)y_1$$
$$\widetilde{V}_2=f'(r)\frac{y^2}{r}(3-y_1y^1-y_2y^2)+f''(r)y^2-f(r)y_2.$$

Finding the first integrals of equation we conclude that general solution takes form

$$G=-\int_{r_0}^{r}\frac{2f(\rho)+f'(\rho)\rho-R}{f(\rho)\rho}d\rho+D\left(y^2/y^1,f(r)(2-y_1y^1-y_2y^2)+f'(r)r,\dots\right.$$

where $D:\mathbb{R}^3\to\mathbb{R}$ is a smooth function (remark that we need to replace $y^2/y^1$ by $y^1/y^2$ in the neighborhood of those points where $y^1=0$).

# Example 2

Is it possible to find smooth function $D$ and constant $R$, such that function $G$ is smooth/? We remark that equation (4) is invariant with respect to coordinate change

$$\begin{pmatrix} y^1 \\ y^2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \eta^1 \\ \eta^2 \end{pmatrix}, \quad \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}.$$

This change of variables maps solution to soution. Let $H_\alpha(D)$ is an image under the change of variables. It is easy to compute that

$$H_\alpha(D) = -\int_{r_0}^{d} \frac{2f(\rho) + f'(\rho)\rho - R}{f(\rho)\rho} d\rho +$$

$$\mathrm{D}\left(\frac{-\eta^1 \sin\alpha + \eta^2 \cos\alpha}{\eta^1 \cos\alpha + \eta^2 \sin\alpha}, f(d)(2 - \eta_1\eta^1 - \eta_2\eta^2) + f'(d)d, \eta_1\eta^2 - \eta_2\eta^1\right).$$

Then $\frac{1}{2\pi}\int_0^{2\pi} H_\alpha(D)d\alpha$ is a solution of equation (4), which does not depend of the first argument.

# Example 2

Therefore existence of smooth solution for (4) implies smothness of integral

$$\int_{r_0}^{r} \frac{2f(\rho)+f'(\rho)\rho - R}{f(\rho)\rho}\, d\rho.$$

**Theorem**

*Chern-Losik class of the foliation $\mathcal{F}_2$ is trivial if and only if $f(0) \neq 0$ and $f^{(k)} = 0$ for all $k \in \mathbb{N}$.*

Thank you for attention!