On the integrability of transitive Lie algebroids

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Plan

1. Lie theory for groupoids
2. $\mathbf{Q}$-manifolds in the sense of R. Barre
3. Generalized integration of transitive Lie algebroids

Lie groupoids

groupoid = associative partial multiplication on a set ("of arrows") $\mathcal{G}$:

- set of unit elements ("objects") M
- object inclusion map $\mathbf{1}\colon M \hookrightarrow \mathcal{G}$, $x \mapsto \mathbf{1}_x$
- target/source maps $\boldsymbol{t}, \boldsymbol{s}\colon \mathcal{G} \to M$
- set of composable pairs $\mathcal{G} * \mathcal{G} := \{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} \mid \boldsymbol{s}(\gamma_1) = \boldsymbol{t}(\gamma_2)\}$
- multiplication map $\mathbf{m}\colon \mathcal{G} * \mathcal{G} \to \mathcal{G}$, $(\gamma_1, \gamma_2) \mapsto \gamma_1 \cdot \gamma_2$
- inversion map $\boldsymbol{\iota}\colon \mathcal{G} \to \mathcal{G}$

• The groupoid should be denoted $\mathcal{G} \overset{\boldsymbol{t}}{\underset{\boldsymbol{s}}{\rightrightarrows}} M$ but we keep it simple: $\mathcal{G} \rightrightarrows M$.

Lie groupoid: $\mathcal{G}$, M smooth manifolds; $\boldsymbol{t}, \boldsymbol{s}$ submersions; $\mathbf{1}, \mathbf{m}$ smooth
($\Rightarrow \mathcal{G} * \mathcal{G} \subseteq \mathcal{G} \times \mathcal{G}$ submanifold, $\boldsymbol{\iota}$ smooth, $\mathbf{1}_M \subseteq \mathcal{G}$ submanifold)

locally trivial Lie groupoid: $(\boldsymbol{t}, \boldsymbol{s})\colon \mathcal{G} \to M \times M$ surjective submersion

Example: the gauge groupoid of a principal bundle

For every principal bundle $\begin{tikzcd} P \arrow[d, "q"'] & P \times G \arrow[l] \\ M \end{tikzcd}$ one has

the gauge groupoid $\mathcal{G} := \mathfrak{gauge}(P) \rightrightarrows P/G$ where

$$\mathfrak{gauge}(P) := \{P_y \xleftarrow{\gamma} P_x \mid \gamma \text{ is a } G\text{-equivariant diffeomorphism}; x, y \in M\}$$

and $P_x := q^{-1}(x) \in P/G$ for all $x \in M$, with its structural maps

- object inclusion map $\mathbf{1}\colon M \to \mathcal{G}$, $\mathbf{1}_x \mapsto \mathrm{id}_{P_x}$
- target/source maps $\boldsymbol{t}, \boldsymbol{s}\colon \mathcal{G} \to M$, $\boldsymbol{t}(\gamma) = P_y$, $\boldsymbol{s}(\gamma) = P_x$
- multiplication map $\mathbf{m}\colon \mathcal{G} * \mathcal{G} \to \mathcal{G}$, $(\gamma, \zeta) \mapsto \gamma \circ \zeta$ if $P_x \xleftarrow{\zeta} P_z$
- inversion map $\boldsymbol{\iota}\colon \mathcal{G} \to \mathcal{G}$, $\gamma \mapsto \gamma^{-1}$

We write $\mathfrak{gauge}(P) \rightrightarrows M$ via the identification $P/G \equiv M$, $P_x \leftrightarrow x$.

Lie algebroids

Lie algebroid: commutative diagram of vector bundles
$$\begin{tikzcd} A \arrow[r, "\mathbf{a}"] \arrow[d, "q"'] & TM \arrow[dl] \\ M & \end{tikzcd}$$

with a Lie bracket on $\Gamma(A)$ satisfying:

- $[X, \varphi Y] = (\mathbf{a}X)(\varphi)Y + \varphi[X, Y]$ for all $X, Y \in \Gamma(A)$, $\varphi \in \mathcal{C}^\infty(M)$.

The *anchor* $\mathbf{a}$ gives a Lie algebra morphism $\Gamma(A) \to \Gamma(TM) =: \mathcal{X}(M)$.
The Lie algebroid is transitive if $\mathbf{a}$ is (fiberwise) surjective.

Ex.: For a Lie groupoid $\mathcal{G} \rightrightarrows M$, its Lie algebroid $\mathcal{AG} \to M$ is given by the pullback of vector bundles
$$\begin{tikzcd} \mathcal{AG} \arrow[r] \arrow[d] & \mathrm{Ker}\,(T\boldsymbol{s}) \subseteq T\mathcal{G} \arrow[d] \\ M \arrow[r, "\mathbf{1}"] & \mathcal{G} \end{tikzcd}$$
hence $(\mathcal{AG})_x = T_{\mathbf{1}_x}(\boldsymbol{s}^{-1}(x))$ and $\mathbf{a}_x = T_{\mathbf{1}_x}(\boldsymbol{t}|_{\boldsymbol{s}^{-1}(x)})\colon (\mathcal{AG})_x \to T_xM$.
The Lie bracket is defined via an embedding $\Gamma(A) \hookrightarrow \mathcal{X}(\mathcal{G})$, $X \mapsto \vec{X}$.

Non-integrable transitive Lie algebroid

Lie algebroid $A \to M$ is *integrable* $\Leftrightarrow$ ($\exists$ Lie groupoid $\mathcal{G} \rightrightarrows M$) $A \simeq \mathcal{A}\mathcal{G}$

Example:

- M smooth manifold
- $\omega \in \Omega^2(M)$, $\mathrm{d}\omega = 0$, with its group of periods $\mathrm{Per}(\omega) \subseteq \mathbb{R}$
- $A := TM \oplus (M \times \mathbb{R}) \to M$
- $\mathbf{a} \colon A \to TM$
- $\Gamma(A) := \mathcal{X}(M) \oplus \mathcal{C}^\infty(M)$ with the Lie bracket

$$[(X, f), (Y, g)]_\omega := ([X, Y], X(f) - Y(g) + \omega(X, Y))$$

for $X, Y \in \mathcal{X}(M)$, $f, g \in \mathcal{C}^\infty(M)$

$\rightsquigarrow$ Lie algebroid $A_\omega \to M$

$$\boxed{A_\omega \text{ integrable} \Leftrightarrow \mathrm{Per}(\omega) \subseteq \mathbb{R} \text{ closed subgroup} \Leftrightarrow (\exists r \in \mathbb{R})\ \mathrm{Per}(\omega) = r\mathbb{Z}}$$

$\mathbf{Q}$-manifolds in the sense of R. Barre (1973)

A **Q-chart** is a mapping $\pi\colon M \to S$, where M is a smooth manifold, satisfying:

(a) If $x, x' \in M$ with $\pi(x) = \pi(x')$, there are open sets $U, U' \subseteq M$ with $x \in U$, $x' \in U'$, and a diffeomorphism $h\colon U \to U'$ with $h(x) = x'$ and $\pi(h(z)) = \pi(z)$ for all $z \in U$.

(b) For every smooth manifold T and any smooth maps $f, h\colon T \to M$ with $\pi \circ f = \pi \circ h$, the set $\{t \in T \mid f(t) = h(t)\}$ is open in T.

If π is surjective then it is called a $\mathbf{Q}$-*atlas*. Two $\mathbf{Q}$-charts $\pi_j\colon M_j \to S$, $j = 1, 2$, are *equivalent* if their disjoint union $\pi_1 \sqcup \pi_2\colon M_1 \sqcup M_2 \to S$ is again a $\mathbf{Q}$-chart. A **Q-manifold** is a set S together with an equivalence class of $\mathbf{Q}$-atlases.

A mapping $f\colon S_1 \to S_2$ is **Q-smooth** if it has smooth lifts: for $s_j \in S_j$, $j = 1, 2$, with $f(s_1) = s_2$ there exist $\mathbf{Q}$-charts $\pi_j\colon M_j \to S_j$, a smooth mapping $\widehat{f}\colon M_1 \to M_2$, and points $m_j \in M_j$, $j = 1, 2$, with $\widehat{f}(m_1) = m_2$, $\pi_j(m_j) = s_j$ for $j = 1, 2$, and $f \circ \pi_1 = \pi_2 \circ \widehat{f}$.

Q-groups in the sense of R. Barre (1973)

A **Q**-*group* is a **Q**-manifold endowed with a group structure whose corresponding group operation and inversion are **Q**-smooth maps.

Example 1: $\mathbb{R}/\mathbb{Q}$ is a **Q**-group with the **Q**-atlas $\mathbb{R} \to \mathbb{R}/\mathbb{Q}$

Example 2: G Lie group with a normal immersed subgroup $N \hookrightarrow G$
$\implies G/N$ is a **Q**-group with the **Q**-submersion $\pi \colon G \to G/N$
The **Q**-submersion π is a **Q**-atlas iff N is *pseudo-discrete*, i.e.,
if $f \in \mathcal{C}^\infty(\mathbb{R}, G)$ and $f(\mathbb{R}) \subseteq N$ the f is necessarily constant.

Note: There exist connected, pseudo-discrete subgroups of $\mathbb{R}^n$ if $n \geq 2$.

$\mathbf{Q}$-principal bundles

Let G be a $\mathbf{Q}$-group and N be a smooth manifold. A $\mathbf{Q}$-principal bundle over N with structure group G is a $\mathbf{Q}$-manifold P endowed with a free right $\mathbf{Q}$-smooth action

$$P \times G \to P, \quad (p, g) \mapsto pg,$$

and a $\mathbf{Q}$-smooth mapping $\beta \colon P \to N$ satisfying:

- For every $p \in P$ we have $\beta^{-1}(\beta(p)) = pG$.
- For every point $n \in N$ there exist an open neighborhood $U \subseteq N$ and a G-equivariant $\mathbf{Q}$-diffeomorphism ψ for which the diagram

$$\begin{tikzcd} \beta^{-1}(U) \arrow[rr, "\psi"] \arrow[dr, "\beta|_{\beta^{-1}(U)}"'] & & U \times G \arrow[dl, "\mathrm{pr}_1"] \\ & U & \end{tikzcd}$$

  is commutative, where pr_1 is the first Cartesian projection.

Atiyah sequence of a $\mathbf{Q}$-principal bundle

Let $\beta\colon P \to N$ be a $\mathbf{Q}$-principal bundle with structural $\mathbf{Q}$-group G.
One has the commutative diagram

$$\begin{tikzcd} & TP \arrow[r, "\tau_P"] \arrow[d, "q_{TP}"'] & P \arrow[d, "\beta"] \\ \mathfrak{Atiyah}(P) := & (TP)/G \arrow[r, dotted, "\mathfrak{A}(\beta)"] & N \simeq P/G \end{tikzcd}$$

Also $\mathbf{a}_P\colon \mathfrak{Atiyah}(P) \to TN$, $a_P(vG) := (T\beta)(v)$, is well defined.
Finally, $\iota_P\colon P \times_{\mathrm{Ad}_G} \mathfrak{g} \to \mathfrak{Atiyah}(P)$ via the tangent map of $P \times G \to P$.

Atiyah sequence of a $\mathbf{Q}$-principal bundle (2)

Let $\beta\colon P \to N$ be a $\mathbf{Q}$-principal bundle with structural $\mathbf{Q}$-group G. The set $\mathfrak{Atiyah}(P) = (TP)/G$ is a smooth manifold satisfying:

1. The quotient map $q_{TP}\colon TP \to \mathfrak{Atiyah}(P)$ is a $\mathbf{Q}$-submersion.
2. The map $\mathfrak{A}(\beta)\colon \mathfrak{Atiyah}(P) \to N$ is a smooth vector bundle.
3. We have the short exact sequence of smooth vector bundles

$$0 \longrightarrow P \times_{\mathrm{Ad}_G} \mathfrak{g} \xrightarrow{\iota_P} \mathfrak{Atiyah}(P) \xrightarrow{\mathbf{a}_P} TN \longrightarrow 0.$$

4. The smooth vector bundle $\mathfrak{Atiyah}(P)$ has the structure of a transitive Lie algebroid with its anchor map $a_P\colon \mathfrak{Atiyah}(P) \to TN$.

Integration of abstract Atiyah sequences

An abstract Atiyah sequence is a short exact sequence of Lie algebroids $L \rightarrowtail^{\mathbf{j}} A \xrightarrow{\mathbf{a}} TN$ where A is a transitive Lie algebroid with its anchor $\mathbf{a}\colon A \to TN$ and L is a Lie algebra bundle over the manifold N.

If $L' \rightarrowtail^{\mathbf{j}'} A' \xrightarrow{\mathbf{a}'} TN$ is another abstract Atiyah sequence over N, then a *morphism* of these abstract Atiyah sequences consists of two morphisms of Lie algebroids $\psi\colon L \to L'$ and $\varphi\colon A \to A'$ satisfying $\varphi \circ \mathbf{j} = \mathbf{j}' \circ \psi$.

Theorem

Every abstract Atiyah sequence over a connected, 2nd countable, smooth manifold is isomorphic to the Atiyah sequence of some $\mathbf{Q}$*-principal bundle.*

$\mathbf{Q}$-groupoids

A $\mathbf{Q}$-groupoid is a groupoid $\mathcal{G} \rightrightarrows M$ satisfying:

- $\mathcal{G}$ is a $\mathbf{Q}$-manifold and M is a smooth manifold.
- The maps $\boldsymbol{s}\colon \mathcal{G} \to M$, $\boldsymbol{\iota}\colon \mathcal{G} \to \mathcal{G}$, and $\mathbf{1}\colon M \to \mathcal{G}$ are $\mathbf{Q}$-smooth.
- The map $\boldsymbol{s} : \mathcal{G} \to M$ is a $\mathbf{Q}$-submersion.
- The multiplication $\mathbf{m} : \mathcal{G} * \mathcal{G} \to \mathcal{G}$ is $\mathbf{Q}$-smooth.

A $\mathbf{Q}$-groupoid morphism between $\mathcal{G} \rightrightarrows M$ and $\mathcal{H} \rightrightarrows N$ is a groupoid morphism $\Phi\colon \mathcal{G} \to \mathcal{H}$ over $\phi\colon M \to N$ for which Φ is a $\mathbf{Q}$-smooth map. (Since ϕ is the composition of the maps $M \xrightarrow{\mathbf{1}} \mathcal{G} \xrightarrow{\Phi} \mathcal{H} \xrightarrow{\boldsymbol{s}} N$, it follows that ϕ is a smooth map.)

$\mathbf{1}\colon M \to \mathcal{G}$ is a $\mathbf{Q}$-immersion but it may not be a '$\mathbf{Q}$-embedding'.

E.g., if G is a $\mathbf{Q}$-group, then the singleton subset $\{\mathbf{1}\} \subseteq G$ is an embedded $\mathbf{Q}$-submanifold if and only if G is a Lie group.

Lie functor for locally trivial $\mathbf{Q}$-groupoids

The Lie algebroid of a $\mathbf{Q}$-groupoid is defined just as for Lie groupoids. For a morphism of $\mathbf{Q}$-groupoids $\Phi\colon \mathcal{G} \to \mathcal{H}$ we have the commutative diagram

$$
\begin{tikzcd}
\mathcal{A}\mathcal{G} \arrow[rrr, dotted, "\mathcal{A}\Phi"] \arrow[dr, "\mathbf{1}_M^!"] \arrow[dddd] & & & \mathcal{A}\mathcal{H} \arrow[dl, "\mathbf{1}_N^!"'] \arrow[dddd] \\
& T^{\boldsymbol{s}}\mathcal{G} \arrow[r, dotted, "T\Phi"] \arrow[d, hook] & T^{\boldsymbol{s}}\mathcal{H} \arrow[d, hook] & \\
& T\mathcal{G} \arrow[r, "T\Phi"] \arrow[d, "\tau_{\mathcal{G}}"'] & T\mathcal{H} \arrow[d, "\tau_{\mathcal{H}}"] & \\
& \mathcal{G} \arrow[r, "\Phi"] & \mathcal{H} & \\
M \arrow[ur, "\mathbf{1}_M"] \arrow[rrr, "\phi"] & & & N \arrow[ul, "\mathbf{1}_N"']
\end{tikzcd}
$$

where $\mathcal{A}\Phi$ is a Lie algebroid morphism at least if $\mathcal{G}, \mathcal{H}$ are locally trivial.

Gauge groupoid of a $\mathbf{Q}$-principal bundle

Theorem

If G is a $\mathbf{Q}$-group and $\beta\colon P \to M$ is a $\mathbf{Q}$-principal bundle with structure group G, then $\mathfrak{gauge}(P) \rightrightarrows M$ is a locally trivial $\mathbf{Q}$-groupoid. There exist a locally trivial Lie groupoid $\widetilde{\mathcal{G}} \rightrightarrows \widetilde{M}$ and a $\mathbf{Q}$-groupoid morphism $\pi_{\mathcal{G}} : \widetilde{\mathcal{G}} \to \mathfrak{gauge}(P)$ which is a $\mathbf{Q}$-atlas.

Theorem

Every locally trivial $\mathbf{Q}$-groupoid is isomorphic to the gauge groupoid of a $\mathbf{Q}$-principal bundle.

Ehresmann isomorphism for a principal bundle $P(M, G)$

If $\mathcal{G} := \mathfrak{gauge}(P) = (P \times P)/G \rightrightarrows M$, there is the *Ehresmann isomorphism* of Lie algebroids $\mathfrak{Ehr}(P) \colon (TP)/G \xrightarrow{\sim} \mathcal{A}\mathcal{G}$. In fact, the quotient map $\Psi \colon P \times P \to (P \times P)/G$ gives the commutative diagram

$$\begin{tikzcd} TP \arrow[r, hook, "E"] \arrow[d] & (TP) \times P \arrow[r, "\partial_1 \Psi"] \arrow[d] & T^{\mathbf{s}}\mathcal{G} \arrow[d] \\ P \arrow[r, hook, "\mathbf{1}"] & P \times P \arrow[r, "\Psi"] & \mathcal{G} \end{tikzcd}$$

Here $\mathbf{1} \colon P \to P \times P$, $p \mapsto (p, p)$, and $E \colon TP \to (TP) \times P$, $E(v) := (v, p)$ if $v \in T_p P$. The pair $((\partial_1 \Psi) \circ E, \Psi \circ \mathbf{1})$ is a fiberwise isomorphism onto the Lie algebroid $\mathcal{A}\mathcal{G} \to \mathbf{1}_{\mathcal{G}} \simeq P/G$. Since $\Psi(pg, qg) = \Psi(p, q)$ for all $p, q \in P$ and $g \in G$, we obtain

$$\begin{tikzcd} TP \arrow[r, "E \circ \partial_1 \Psi"] \arrow[d] & \mathcal{A}\mathcal{G} \\ (TP)/G \arrow[ur, dotted, "\mathfrak{Ehr}(P)"'] & \end{tikzcd}$$

The extended Ehresmann isomorphism

For $\mathbb{PB}$ the category of **Q**-principal bundles, $\mathbb{LA}$ the category of transitive Lie algebroids, $\mathbb{LG}$ the category of locally trivial **Q**-groupoids, we have

$$\begin{tikzcd}
\mathbb{PB} \arrow[r, "\mathfrak{gauge}"] \arrow[d, "\mathfrak{Atiyah}"'] & \mathbb{LG} \arrow[d, "\mathfrak{Lie}"] \\
\mathbb{LA} & \mathbb{LA}
\end{tikzcd}$$

- $\mathfrak{gauge}\colon \mathbb{PB} \to \mathbb{LG}$, $\mathfrak{gauge}(P) = (P \times P)/G$
- $\mathfrak{Lie}\colon \mathbb{LG} \to \mathbb{LA}$, $\mathfrak{Lie}(\mathcal{G}) := \mathcal{AG}$, is the Lie functor
- $\mathfrak{Atiyah}\colon \mathbb{PB} \to \mathbb{LA}$, $\mathfrak{Atiyah}(P) := (TP)/G$.

(Here P is a **Q**-principal bundle with its structure group G.)

Theorem

There is an isomorphism of functors $\mathfrak{Ehr}\colon \mathfrak{Atiyah} \to \mathfrak{Lie} \circ \mathfrak{gauge}$ *that agrees with the Ehresmann isomorphism for principal bundles.*

Generalized integration of transitive Lie algebroids

Corollary

Every transitive Lie algebroid over a connected, second countable, smooth manifold is isomorphic to the Lie algebroid of some locally trivial $\mathbf{Q}$*-groupoid.*

Proof. Let $q\colon A \to N$ be an arbitrary transitive Lie algebroid with its anchor $\mathbf{a}\colon A \to TN$.

Step 1: Integrate the abstract Atiyah sequence $L \overset{\mathbf{j}}{\rightarrowtail} A \overset{\mathbf{a}}{\twoheadrightarrow} TN$ to a principal $\mathbf{Q}$-groupoid $\beta\colon P \to N$.

Step 2: Via the extended Ehresmann isomorphism, the Lie algebroid $q\colon A \to N$ is then isomorphic to the Lie algebroid of $\mathfrak{gauge}(P) \rightrightarrows M$.